Detecting inclusions using electrical impedance tomography

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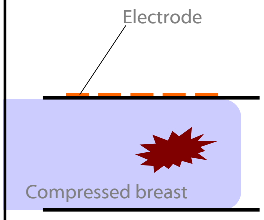
See *Probing for Electrical Inclusions with Complex Spherical Waves*, Comm. Pure Appl. Math. **60** (2007)

Application of EIT: Early detection of breast cancer



Cancerous tissue is up to four times more conductive than healthy tissue

[Jossinet 1998]

X-ray attenuation is almost equal in cancerous and healthy tissue

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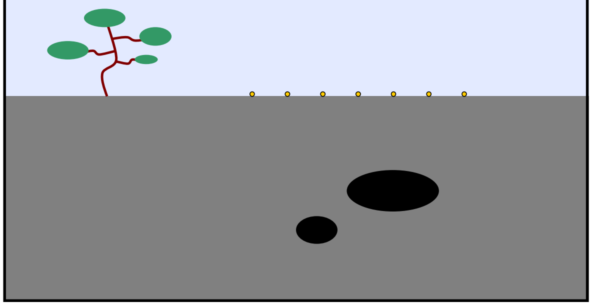
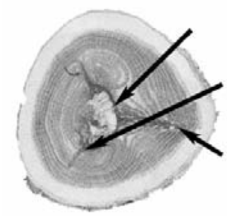
Application of EIT: Geological sensing of oil or metals



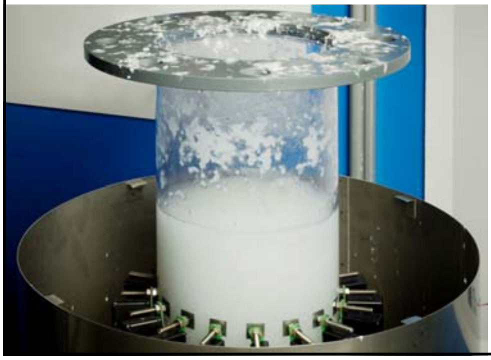
Application of EIT: Nondestructive testing



Finding cracks or defects in known background.

Photo from: Shigo, A.L., 1983. Tree Defects: A Photo Guide. USDA Forest Service, No. Cent. For. Exp. Sta., GTRNE-82.

Application of EIT: Industrial process monitoring

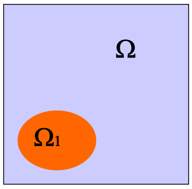


We consider conductivities having inclusions inside known smooth background

Here γ is a perturbation of $\gamma_0 \in C^\infty(\overline{\Omega})$:

$$\gamma(x) = \begin{cases} \gamma_1(x), & x \in \Omega_1, \\ \gamma_0(x), & x \in \Omega \setminus \overline{\Omega_1}, \end{cases}$$

with $\gamma_1 \in C^\infty(\overline{\Omega_1})$.



In addition, assume that $\gamma : \Omega \to \mathbb{R}$ satisfies

$$0 < c \le \gamma(x) \le C \text{ for all } x \in \Omega.$$

Current-to-voltage boundary measurements are modelled by Neumann-to-Dirichlet map

For $f \in H^{-1/2}(\partial\Omega)$ with

$$\int_{\partial\Omega} f dS = 0$$

consider the Neumann problem

$$\begin{cases} \nabla \cdot (\gamma(x)\nabla v) = 0 & \text{in} \quad \Omega, \\ \gamma \frac{\partial v}{\partial n} = f & \text{on} \quad \partial\Omega. \end{cases}$$

Fix $P_0 \in \partial\Omega$ and require $v(P_0) = 0$.

Define Neumann to Dirichlet map R_γ by

$$\boxed{R_\gamma f = v|_{\partial\Omega}.}$$

Note: physically f represents current through the boundary, and R_γ f represents the resulting voltage at the boundary.

We consider the following problem:
Given the ND map R_γ
(possibly only on part of boundary),
how to extract information on γ?

Theoretical history of the problem

1980 Calderón (inverse conductivity problem)
1985 Kohn & Vogelius (uniqueness)
1987 Friedman (detection of mines)
1988 Isakov (uniqueness)
1989 Friedman & Isakov (uniqueness)
1994 B. Barcelò , Fabes & Seo (uniqueness)
1995 Alessandrini, Isakov & Powell (local uniqueness)
1996 Alessandrini & Isakov (uniqueness)
1998 Ikehata (recovering shape of inclusion)
1998 Alessandrini & Rosset (size bounds)
2000 Ikehata (enclosure method)
2004 Alessandrini & Rosset (volume bounds)
2005 Alessandrini & Di Cristo (stability)
2007 Di Cristo (stability with local data)

Numerical history of the problem

2000 Ikehata & S (enclosure method)
2000 Brühl & Hanke (enclosure & sampling method)
2000 Kaipio, Kolehmainen, Somersalo & Vauhkonen
2001 Brühl
2001 Ito, Kunisch & Li (level set method)
2002 Ikehata & Ohe (enclosure method)
2002 Kwon, Seo & Yoon
2003 Erhard & Potthast (point source method)
2003 Huang, Lee, Schultz & Ceccio (bubbly flows)
2003 Kwon, Yoon, Seo, Woo & Cho
2004 Ikehata & S (Mittag-Leffler approach)
2004 Hyvönen

Instead of the conductivity equation, we study an equivalent Schrödinger equation

Define $q = \gamma^{-1/2}\Delta\gamma^{1/2}$ and $\tilde{u} = \gamma^{1/2}u$. Then

$$\boxed{\nabla \cdot \gamma\nabla u = 0 \quad \Longleftrightarrow \quad (-\Delta + q)\tilde{u} = 0.}$$

This can be seen by direct computation:

$$\begin{aligned} & (-\Delta + q)\tilde{u} \\ = \; & -\nabla \cdot \nabla(\gamma^{1/2}u) + q\tilde{u} \\ = \; & -u\Delta\gamma^{1/2} - \nabla\gamma^{1/2} \cdot \nabla u - \nabla \cdot (\gamma^{1/2}\nabla u) + q\gamma^{1/2}u \\ = \; & \ldots \\ = \; & -\gamma^{-1/2}\nabla \cdot \gamma\nabla u. \end{aligned}$$

Note: we use the strict positivity of γ.

The reconstruction method makes heavy use of exponentially growing solutions

Look for solutions of the Schrödinger equation

$$(-\Delta + q)\psi(\,\cdot\,, k) = 0,$$

where $k \in \mathbb{C}$ is a parameter and

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}.$$

More precisely: $\mu(x,k) := e^{-ikx}\psi(x,k)$ with

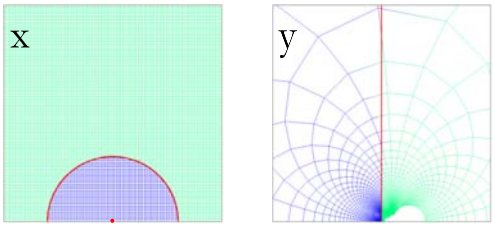
$$\mu(\,\cdot\,,k) - 1 \in W^{1,p}(\mathbb{R}^2),$$

where $p > 2$.

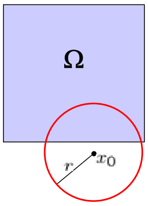
By Nachman [1996] we know that there is a unique solution ψ for every k if q has the form $q = \gamma^{-1/2}\Delta\gamma^{1/2}$.

Transformation to hyperbolic space

$$y_1 = \frac{x_1^2 + x_2^2 - R^2}{(x_1+R)^2 + x_2^2}, \qquad y_2 = \frac{2x_2R}{(x_1+R)^2+x_2^2}.$$



Main Theorem.
Take x_0 from outside the convex hull $\mathcal{C}(\Omega)$ and let $r > 0$. Choose $\epsilon > 0$ so that $x_0 \notin U_\epsilon := \{z \,|\, \mathrm{dist}(\mathcal{C}(\Omega), z) < \epsilon\}$.
Then there is $u_\tau(x) \in C^\infty(U_\epsilon)$ such that



(1) u_τ satisfies the equation
$\nabla \cdot (\gamma_0(x)\nabla u_\tau(x)) = 0 \quad \text{on} \quad \Omega.$

(2) Let $K_\pm$ be any compact sets such that

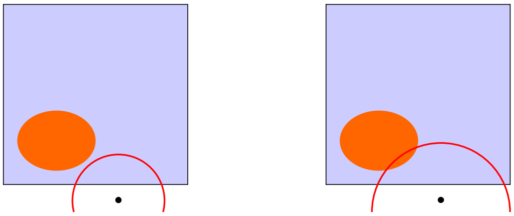
$$K_+ \subset \{x \in U_\epsilon \,;\, |x-x_0| < r\}, \quad K_- \subset \{x \in U_\epsilon \,;\, |x-x_0| > r\}.$$

Then there exist $\delta > 0$ and $T > 0$ such that

$$\int_{K_+} |u_\tau(x)|^2 dx \ge e^{\delta\tau}, \quad \sup_{x\in K_-} |u_\tau(x)| \le e^{-\delta\tau}, \quad \forall \tau > T.$$

Main theorem continues: detecting inclusions

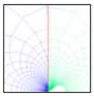
Let $f_\tau = \gamma \frac{\partial u_\tau}{\partial n}|_{\partial\Omega}$. Then we have two cases:



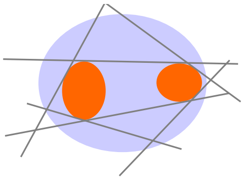
$0 \le \langle (R_0 - R)f_\tau, f_\tau \rangle_{\partial\Omega} < e^{-\delta\tau}$

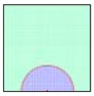
$\langle (R_0 - R)f_\tau, f_\tau \rangle_{\partial\Omega} > e^{\delta\tau}$

Our theorem can be seen as a generalization of the enclosure method of Ikehata

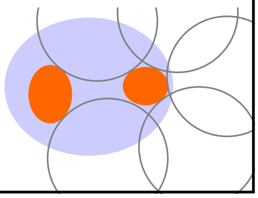


Probing with half-spaces



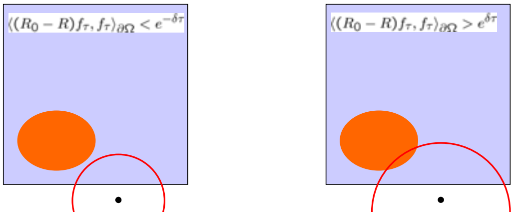


Probing with discs



Practical scheme for detecting inclusions

Let $f_\tau = \gamma \frac{\partial u_\tau}{\partial n}|_{\partial\Omega}$ and take $0 < \tau_1 < \tau_2$.
Main theorem then suggests the following:



$\langle (R_0 - R)f_{\tau_1}, f_{\tau_1} \rangle_{\partial\Omega}$
$> \langle (R_0 - R)f_{\tau_2}, f_{\tau_2} \rangle_{\partial\Omega}$

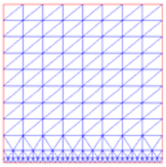
$\langle (R_0 - R)f_{\tau_1}, f_{\tau_1} \rangle_{\partial\Omega}$
$< \langle (R_0 - R)f_{\tau_2}, f_{\tau_2} \rangle_{\partial\Omega}$

We test our practical detection method in the case of constant background

We choose $\gamma_0 \equiv 1$, $\gamma_1 = 4$, $\tau_1 = 4$ and $\tau_2 = 5$.

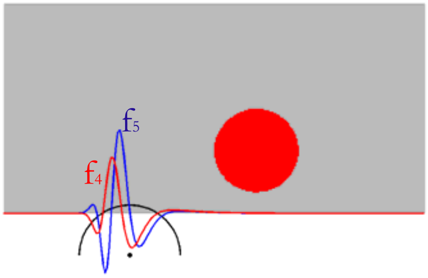
Let $f_\tau = \frac{\partial u_\tau}{\partial n}|_{\partial\Omega}$ with $u_\tau(x) = e^{-\tau(y_1 - iy_2)}$,

$$y_1 = \frac{x_1^2 + x_2^2 - R^2}{(x_1+R)^2 + x_2^2}, \quad y_2 = \frac{2x_2R}{(x_1+R)^2+x_2^2}.$$



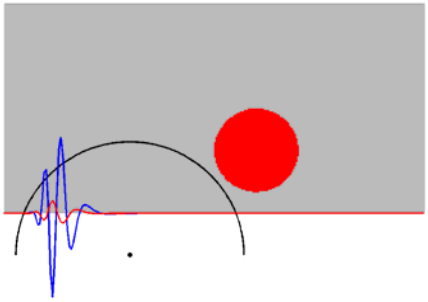
FEM $\rightarrow$

$$\langle (R_0 - R) f_4, f_4 \rangle_{\partial\Omega}$$
$$\langle (R_0 - R) f_5, f_5 \rangle_{\partial\Omega}$$



Here we see the Neumann data corresponding to the choice of probing disc.
They are used to compute

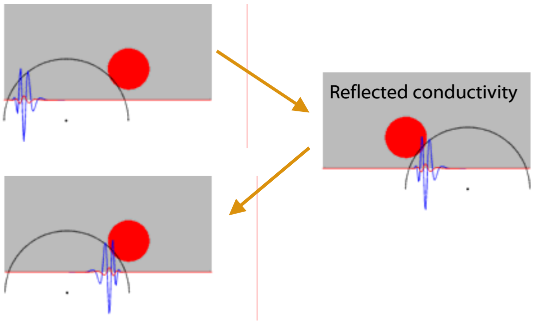
$$\langle (R_0 - R) f_4, f_4 \rangle_{\partial\Omega}$$
$$\langle (R_0 - R) f_5, f_5 \rangle_{\partial\Omega}$$



Due to the hyperbolic transformation,
the Neumann data concentrates to the left

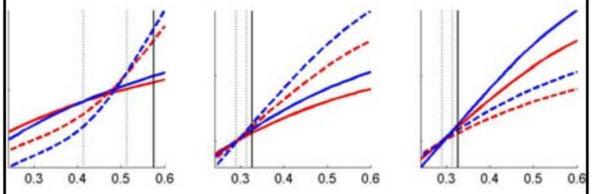
Problem:
Inclusions far away from the numerical support of Neumann data do not contribute much to measurements!

We use symmetry to add measurements



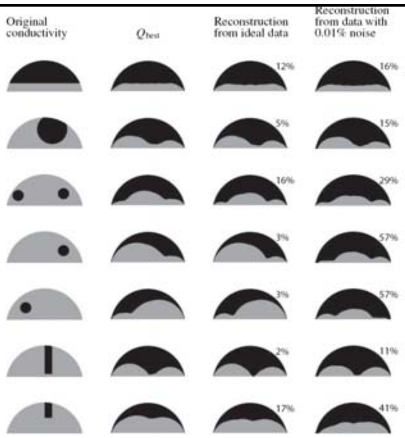
Reflected conductivity

Resulting inner products look like this



No correction:

Symmetry corrected:



Conclusion

We have developed a new kind of EIT algorithm
for locating inclusions using localized data

The algorithm has rigorous mathematical background

Our method is robust against measurement noise

Next step in 2D case: general boundary

Future work: 3D case