Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

PROBING FOR INCLUSIONS IN HEAT CONDUCTIVE BODIES

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Abstract. This work deals with an inverse boundary value problem arising from the equation of heat conduction. Mathematical theory and algorithm is described in dimensions 1–3 for probing the discontinuous part of the conductivity from local temperature and heat flow measurements at the boundary. The approach is based on the use of complex spherical waves, and no knowledge is needed about the initial temperature distribution. In dimension two we show how conformal transformations can be used for probing deeper than is possible with discs. Results from numerical experiments in the one-dimensional case are reported, suggesting that the method is capable of recovering locations of discontinuities approximately from noisy data.

1. Introduction.

1.1. **Inverse heat conductivity problem.** Let Ω be a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, and consider the following boundary value problem

$$\begin{cases} \partial_t v - \nabla\cdot(\gamma(x)\nabla v) &= 0 \quad \text{in} \quad (0,T)\times\Omega, \\ v &= f \quad \text{on} \quad (0,T)\times\partial\Omega, \\ v\big|_{t=0} &= v_0 \quad \text{in} \quad \Omega \end{cases} \tag{1}$$

where $\gamma(x)\in L^\infty(\Omega)$ such that $\gamma(x) > c$ for a constant $c > 0$. Let v^f be the unique solution to the above equation. The time-dependent Dirichlet-to-Neumann (DN) map Λ is then defined by

$$\Lambda : f \to \gamma(x)\frac{\partial v^f}{\partial\nu}\Big|_{\partial\Omega}, \tag{2}$$

where ν is the outer unit normal to $\partial\Omega$.

2000 *Mathematics Subject Classification.* Primary: 35R30, 35K05; Secondary: 58J35.
Key words and phrases. Inverse problem, Heat equation, Interface reconstruction.
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Physically, we consider a heat-conducting body modelled by the set $\overline{\Omega}$ and the strictly positive heat conductivity distribution γ inside the body. The function v_0 is the initial temperature distribution in Ω over which we do not have control. We perform boundary measurements by applying the temperature $f(x,t)$ at the boundary $\partial\Omega$ during the time $0 < t < T$ and measuring the resulting heat flux $\gamma(x)\frac{\partial v^f}{\partial \nu}|_{\partial\Omega}$ through the boundary. The DN map Λ defined in (2) is an ideal model of all possible infinite-precision measurements of the above type.

We study the ill-posed inverse problem of detecting conductivity inclusions inside Ω from the *local* knowledge of Λ. Namely, we assume that only a part $\Gamma \subset \partial\Omega$ is available for measurements and take as data the restrictions $\gamma(x)\frac{\partial v^f}{\partial \nu}|_{\Gamma}$ of heat fluxes corresponding to functions f that are supported in Γ. An inclusion is defined as a subdomain $\Omega_1 \subset \Omega$ such that $\gamma(x)$ is perturbed on Ω_1 from some known $\gamma_0(x)$. Our aim is to find the location of $\partial\Omega_1$ from the local measurements. The typical situation is drawn in the figure 1.

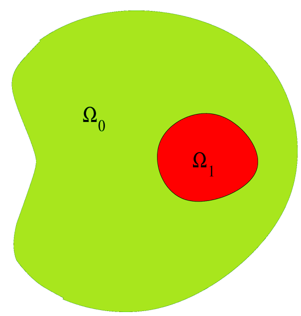

FIGURE 1.

The above inverse boundary value problem is related to nondestructive testing where one looks for anomalous materials inside a known material. One such example is monitoring a blast furnace used in ironmaking: the corroded thickness of the accreted refractory wall based on temperature and heat flux measurement on the accessible part of the furnace wall [11].

We remark that although Λ depends on both f and v_0, our method uses no information of the initial data v_0 to detect $\partial\Omega_1$.

1.2. **Main theorems.** Suppose we are given $\gamma_0(x) \in C^\infty(\overline{\Omega})$, $\gamma(x) \in L^\infty(\Omega)$ and an open subset $\Omega_1 \subset\subset \Omega$, i.e. $\overline{\Omega_1} \subset \Omega$, such that for some constant $C_0 > 0$,

$$C_0 < \gamma_0(x) < C_0^{-1}, \quad C_0 < \gamma(x) < C_0^{-1}, \quad \text{on} \quad \Omega,$$

and

$$\gamma(x) = \gamma_0(x), \quad \text{on} \quad \Omega_0 = \Omega \setminus \overline{\Omega_1}.$$

We put

$$\gamma_1(x) = \gamma(x) - \gamma_0(x).$$

Moreover, we shall assume that $\gamma_1(x) - \gamma_0(x)$ has a constant sign on Ω_1. Our main purpose is to study discontinuous perturbations, however, we allow $\gamma(x)$ to be continuous. Hence, we impose the following assumption.

(A) $\inf_{x\in K} |\gamma_1(x) - \gamma_0(x)| > 0$ *for any compact set* $K \subset \Omega_1$.

We define (formal) differential operators by

$$\mathcal{A}_0 = -\nabla \cdot (\gamma_0 \nabla), \quad \mathcal{A} = -\nabla \cdot (\gamma \nabla).$$

We extend $\gamma_0(x)$ smoothly outside Ω so that $\gamma_0(x) = 1$ for large $|x|$.

First, we consider the 1-dimensional problem. Namely, for the heat equation on (a, b), we try to detect $\mathrm{dist}(a, \partial\Omega_1)$ from the measurement at a.

Theorem 1.1. *Let $d = 1$, $\Omega = (a, b)$. Suppose $\Omega_1 = (a_1, b_1)$ with $a < a_1 < b_1 < b$. Take $x_0 \in \mathbb{R}$ arbitrarily. Then there exists a real fucntion $\varphi_\lambda(x) \in C^\infty(\mathbb{R})$ depending on a large parameter $\lambda > 0$ having the following properties.*
(1) It satisfies

$$(\mathcal{A}_0 + \lambda)\, \varphi_\lambda(x) = 0, \quad x \in (a, b), \quad \lambda >> 1. \tag{3}$$

(2) For any compact sets $K_- \subset (-\infty, x_0)$, and $K_+ \subset (x_0, \infty)$, there exist constants $C, \delta > 0$ such that for $\lambda > C$

$$|\varphi_\lambda(x)| \geq Ce^{\delta\sqrt{\lambda}}, \quad |\varphi'_\lambda(x)| \geq Ce^{\delta\sqrt{\lambda}}, \quad \forall x \in K_-,$$

$$|\varphi_\lambda(x)| \leq Ce^{-\delta\sqrt{\lambda}}, \quad |\varphi'_\lambda(x)| \leq Ce^{-\delta\sqrt{\lambda}}, \quad \forall x \in K_+.$$

(3) Take $0 < T_1 \leq T$ arbitrarily, and let

$$h_\lambda(x) = \varphi_\lambda(x)\big|_{\partial\Omega}, \quad f_\lambda(t, x) = e^{\lambda t} h_\lambda(x),$$

$$I(\lambda) = e^{-\lambda T_1} (\Lambda f_\lambda)(T_1, a) h_\lambda(a) - \gamma_0(a) \frac{dh_\lambda}{dx}(a) h_\lambda(a). \tag{4}$$

Then, if $\pm(\gamma_1 - \gamma_0) > 0$ on Ω_1, we have

$$\lim_{\lambda\to\infty} (2\sqrt{\lambda})^{-1} \log(\pm I(\lambda)) = -y(a_1), \tag{5}$$

where $y(x)$ is defined by

$$y(x) = \int_{x_0}^{x} \frac{dt}{\sqrt{\gamma_0(t)}}. \tag{6}$$

For higher dimensions, we need the following notations. By $\mathrm{dist}(x_0, A)$, we mean the distance between a point $x_0 \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$. Let $B(x_0, R) = \{x \in \mathbb{R}^d\,; |x - x_0| < R\}$. For dimensions $d \geq 2$, we take an open subset Γ on the boundary $\partial\Omega$, and try to recover the location of $\partial\Omega_1$ from a measurement on Γ. Take $x_0 \in \mathbb{R}^d$ arbitrarily. The roles of half-lines $(-\infty, x_0)$ and (x_0, ∞) are played by $B(x_0, R)$ and $B(x_0, R)^c$, respectively.

Theorem 1.2. *Suppose $d = 2, 3$. Take $B = B(x_0, R)$ satisfying*

(C-1) $\quad \emptyset \neq \overline{B} \cap \partial\Omega \subset \Gamma.$

Take $x^* \in \partial B \setminus \overline{\Omega}$ arbitrarily. For an open set $\mathcal{O} \subset \mathbb{R}^d$, let

$$a_B(\mathcal{O}) = \sup_{x\in\mathcal{O}} \frac{R^2 - |x - x_0|^2}{|x - x^*|^2}, \tag{7}$$

and fix constants T_1 and μ such that

$$0 < T_1 \leq T, \quad 0 < \mu < \frac{T_1}{\sqrt{2}\,(a_B(\Omega) - a_B(\Omega_1))}. \tag{8}$$

Then there exists $\varphi_\lambda(x) \in C^\infty(\mathbb{R}^d \setminus \{x^*\})$ depending on a large parameter $\lambda > 0$ having the following properties.
(1) It satisfies

$$(\mathcal{A}_0 + \lambda)\, \varphi_\lambda(x) = 0, \quad 0 < |x - x^*| < 2\sqrt{\lambda}R. \tag{9}$$

(2) For any compact sets $K_- \subset B(x_0, R) \setminus \{x^*\}$, and $K_+ \subset \overline{B(x_0, R)}^c$, there exist constants $C, \delta > 0$ such that for $\lambda > C$

$$|\varphi_\lambda(x)| \geq Ce^{\delta\lambda}, \quad |\nabla_x \varphi_\lambda(x)| \geq Ce^{\delta\lambda}, \quad \forall x \in K_-,$$

$$|\varphi_\lambda(x)| \leq Ce^{-\delta\lambda}, \quad |\nabla_x \varphi_\lambda(x)| \leq Ce^{-\delta\lambda}, \quad \forall x \in K_+.$$

(3) We put

$$h_\lambda(x) = \varphi_\lambda(x)\big|_{\partial\Omega}, \quad f_\lambda(t, x) = e^{\lambda t} h_\lambda(x),$$

$$I(\lambda) = e^{-\lambda T_1} \int_\Gamma (\Lambda f_\lambda)(T_1, x)\overline{h_\lambda(x)} d\sigma(x) - \int_\Gamma \gamma_0(x) \frac{\partial \varphi_\lambda(x)}{\partial \nu} \overline{h_\lambda(x)} d\sigma(x), \tag{10}$$

$d\sigma(x)$ being the induced measure on $\partial\Omega$. Then we have

$$\lim_{\lambda\to\infty} (\sqrt{2}\mu\lambda)^{-1} \log(\pm I(\lambda)) = a_B(\Omega_1),$$

accordingly as $\pm(\gamma_1 - \gamma_0) > 0$.

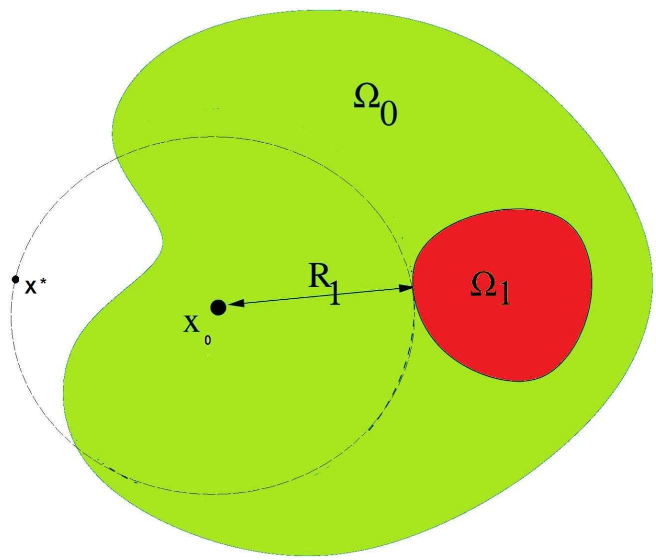

FIGURE 2.

In Theorems 1.1 and 1.2, $\varphi_\lambda(x)$ is constructed by using $\gamma_0(x)$ only. We put

$$R_1 = \text{dist}(x_0, \partial\Omega_1).$$

Let us remark that

$$\begin{array}{lllll} a_B(\Omega_1) > 0 & \iff & B(x_0, R) \cap \Omega_1 \neq \emptyset & \iff & R > R_1, \\ a_B(\Omega_1) < 0 & \iff & \overline{B(x_0, R)} \cap \overline{\Omega_1} = \emptyset & \iff & R < R_1. \end{array}$$

Then by Theorem 1.2, if $\pm(\gamma_1 - \gamma_0) > 0$ on Ω_1 and $R < R_1$, then $\pm I(\lambda)$ tends to 0 exponentially as $\lambda \to \infty$, and if $R > R_1$, then $\pm I(\lambda)$ tends to ∞ exponentially as $\lambda \to \infty$. This means, we can see whether $B(x_0, R)$ touches Ω_1 or not using only the knowledge of $\gamma_0(x)$.

Our results also hold for $4 \leq d \leq 6$. However, to fix the idea, we shall consider the case $1 \leq d \leq 3$.

1.3. **Detection algorithm.** Suppose for the sake of simplicity that $\gamma_0 \equiv 1$. The 1-dimensional case is rather easy. We have only to use (5) to compute a_1. The detection algorithm for $d = 2, 3$ is as follows.

1. Take an open set $\Gamma \subset \partial\Omega$, and x^* outside Ω, but close to Γ.
2. Draw a straight line l with end point x^*, take $x_0 \in l$, and the ball $B = B(x_0, R)$ such that $(\overline{B} \cap \partial\Omega) \subset \Gamma$.
3. For large $\lambda > 0$, compute $I(\lambda)$. The function

$$e^{\lambda t}|x - x^*|^{2-d}e^{-\zeta\cdot y(x)}, \quad y(x) = y^* + 2R\frac{x - x^*}{|x - x^*|^2}, \quad \zeta = \frac{\mu\lambda}{\sqrt{2}}(y^* + iy'_\perp),$$

$$y^* = \frac{x^* - x_0}{|x^* - x_0|}, \quad y^* \cdot y^*_\perp = 0, \quad |y^*_\perp| = 1$$

   will give an approximation of the probing data.
4. If $I(\lambda) \to 0$ as $\lambda \to \infty$ we infer that B does not intersect the inclusion.
5. If $I(\lambda) \to \infty$ (or $-\infty$) as $\lambda \to \infty$ we infer that B intersects the inclusion and $\gamma_1(x) > \gamma(x)$ (or $\gamma_1(x) < \gamma(x)$) on it.

1.4. **Transformation of exponentially growing solutions.** The main idea of the proof (for $d = 2, 3$) consists in using the probing data of the form $e^{\lambda t}\varphi_\lambda(x)$, where $\varphi_\lambda(x)$ is exponentially growing as $\lambda \to \infty$ in the ball $B(x_0, R)$ and exponentially decaying outside $B(x_0, R)$. Such a method was found and used in [4] for the case of the elliptic problem: $\nabla \cdot (\gamma(x)\nabla u) = 0$ by passing through the hyperbolic space. The essential feature of this idea is to use conformal transformation which maps a sphere to a plane. In this paper, we shall work entirely in the Euclidean space, and use inversion with respect to a sphere (this is also the case in [4]), which maps the ball $B(x_0, R)$ to the half-space $\{y \cdot y^* < 0\}$. We are then led to consider the equation of the form

$$(-\Delta + 2\zeta \cdot \nabla + q_\lambda)u = f,$$

where $\zeta \in \mathbb{C}^d$. Although in our case $q_\lambda(x)$ grows up linearly in λ, suitable change of variables enables us to reduce the construction of solutions to now standard method of Sylvester-Uhlmann [9]. Another important difference is that in [4], the point x_0, the center of the ball $B(x_0, R)$, is taken outside the covex hull of Ω, while in our method it should satisfy (C-1) and that $x^* \in \partial B \setminus \overline{\Omega}$. This makes it possible to probe regions deeper than that of [4].

It is interesting to find other conformal transformations mapping planes (lines) to some surfaces (curves) which are useful for the probing problem. Putting trivial transformations such as translation and orthogonal transformation aside, for $d = 3$, the spherical inversion is essentially the unique conformal transformation for the Euclidean Laplacian. It can be seen by embedding the problem into the hyperbolic space, as was done in [4]. However, in 2-dimensions, the complex function theory provides us with lots of conformal transformations. We shall give in this paper one of such examples and use it for the inclusion detection problem in §5.

1.5. **Literature review.** There are lots of works on inverse problem of heat conductivity. In a recent article of [6], the following heat equation is considered for domains $D \subset\subset \Omega$,

$$\begin{cases} u_t - \Delta u = 0, \quad x \in \Omega \subset D, \quad 0 < t < T, \\ \dfrac{\partial u}{\partial \nu} + \rho u = 0, \quad \text{on} \quad \partial D, \\ u(x,0) = 0, \quad \text{in} \quad \Omega \setminus \overline{D}, \end{cases}$$

ν being the unit normal on ∂D. Assuming that D and ρ are unknown, they recover the location and shape of D from the knowledge of $u(x,t)$ and $\partial u(x,t)/\partial\nu$ on $\partial\Omega$. The proof is based on the exponentially growing solution of the form $\exp(\tau x \cdot (\omega + i\omega^{\perp}))$ with $\omega, \omega^{\perp} \in S^2$, $\omega \perp \omega^{\perp}$. This is an extreme case of the problem treated in this paper by taking the heat conducting coefficient $\gamma(x) = 0$ on Ω_1.

In [2] the interior boundary curve of an arbitrary-shaped annulus is reconstructed from overdetermined Cauchy-data on the exterior boundary curve, by assuming $\gamma = 1$ and using the Newton method to a boundary integral equation approach. In [1] the shape of an inaccessible portion of the boundary is reconstructed by linearisation. In [3] the position of the boundary is reconstructed as a function of time, using the sideways heat equation. The proposed method of the present paper differs from these; it is a direct reconstruction method, and it finds inclusions in non-constant background conductivities. For other related results, see [11, 12, 8] and the survey article [10].

The paper is organized as follows. In §2 and §3, we shall constuct the probing data $\varphi_\lambda(x)$. Theorems 1.1 and 1.2 are proved in §4. The use of conformal mapping in 2-dimensions is explained in §5. In §6, we give some numerical results for the one dimensional case.

2. **1-dimensional trial function.** Take $x_0 \in \mathbb{R}$ arbitrarily, and make the change of variable by (6). Letting $\varphi_\lambda = \psi_1(y)e^{-\sqrt{\lambda}y}$, we transform the equation (3) into

$$\psi_1'' - (2\sqrt{\lambda} - p(y))\psi_1' + \lambda p(y)\psi_1 = 0, \quad ' = \frac{d}{dy},$$

where $p(y) = (2\sqrt{\gamma_0(x(y))})^{-1}\dfrac{d\gamma_0}{dx}(x(y)) = \dfrac{d}{dy}\log\left(\gamma_0(x(y))^{1/2}\right)$. Putting $P(y) = \gamma_0(x(y))^{-1/4}$, we then have $2\dfrac{d}{dy}P(y) = -P(y)p(y)$. Therefore, letting $\psi_1 = P(y)\psi_2$, we have

$$\psi_2'' - 2\sqrt{\lambda}\psi_2' - Q(y)\psi_2 = 0, \tag{11}$$

where $Q(y) = \dfrac{p'(y)}{2} + \dfrac{p(y)^2}{4} \in C_0^\infty(\mathbb{R})$. Putting $\psi_2 = 1 + \phi$, we then have

$$\phi'' - 2\sqrt{\lambda}\phi' - Q(y)\phi = Q. \tag{12}$$

We consider the integral operator

$$(Su)(y) = \frac{1}{2\sqrt{\lambda}}\int_{-\infty}^{y}(e^{-2\sqrt{\lambda}(t-y)} - 1)u(t)dt.$$

Take $A > 0$ sufficiently large. If $u \in C^2((-\infty, A))$ satisfies $u, u', u'' \in L^1((-\infty, A))$, we have

$$(Su'')(y) = u(y) + 2\sqrt{\lambda}(Su')(y).$$

Therefore, $\phi \in C^2((-\infty, A))$ is a solution to (12) satisfying $\phi, \phi' \in L^1((-\infty, A))$ if and only if ϕ is a bounded solution to

$$\phi = SQ\phi + SQ.$$

Since $Q(s)$ is compactly supported, we have

$$\|SQ\|_{\mathbf{B}(L^\infty(I);L^\infty(I))} \leq \frac{C}{\sqrt{\lambda}}, \quad I = (-\infty, A),$$

which implies the existence of a solution $\psi_2 \in C^2((-\infty, A))$ of (11) such that

$$|\partial_x^n(\psi_2(y) - 1)| \leq C(\sqrt{\lambda})^{-1+n}, \quad n = 0, 1, 2.$$

Putting $\varphi_\lambda(x) = P(y)e^{\sqrt{\lambda}y}\psi_2(y)$, we obtain the following Theorem.

Theorem 2.1. *There exists a solution $\varphi_\lambda(x)$ to the equation (3) such that*

$$\varphi_\lambda(x) = \gamma_0(x)^{-1/4}e^{-\sqrt{\lambda}y(x)}(1 + \phi_\lambda(x)), \tag{13}$$

where $y(x)$ is defined by (6) and $\phi_\lambda(x)$ satisfies

$$|\partial_x^n \phi_\lambda(x)| \leq C(\sqrt{\lambda})^{-1+n}, \quad n = 0, 1, 2, \quad a \leq x \leq b.$$

3. **Multi-dimensional trial function.** The construction of $\varphi_\lambda(x)$ is more complicated in multi-dimensions than in the one-dimensional case. We shall prove the following theorem.

Theorem 3.1. *Let $d = 2, 3$ and x_0, x^* be as in Theorem 1.2. Take $y^*, y^*_\perp \in \mathbb{R}^d$ such that*

$$y^* = \frac{x^* - x_0}{|x^* - x_0|}, \quad y^* \cdot y^*_\perp = 0, \quad |y^*_\perp| = 1, \tag{14}$$

and let $\zeta \in \mathbf{C}^d$ be a complex vector such that

$$\zeta = \frac{\mu\lambda}{\sqrt{2}}(y^* + iy^*_\perp), \tag{15}$$

where $\mu > 0$ is a constant satisfying the condition (8). Let $y = y(x)$ be the inversion defined by

$$y = y^* + 2R\frac{x - x^*}{|x - x^*|^2}. \tag{16}$$

Then for large $\lambda > 0$, there exists a solution $\varphi_\lambda(x)$ to the equation (9) in the region $0 < |x - x^*| < 2\sqrt{\lambda}R$ having the following form:

$$\varphi_\lambda(x) = |x - x^*|^{2-d}\gamma_0(x)^{-1/2}(1 + \phi_\lambda(x))e^{-\zeta\cdot y(x)}, \tag{17}$$

where for any $0 < \delta < 1$, $\phi_\lambda(x)$ satisfies

$$\|\phi_\lambda(x)\|_{L^\infty(\Omega)} \leq C_\delta\, \mu^{-1}\lambda^{-\delta/2}, \tag{18}$$

$$\|\nabla_x\phi_\lambda(x)\|_{L^\infty(\Omega)} \leq C_\delta\, \mu^{-1}\lambda^{(1-\delta)/2}$$

for a constant $C_\delta > 0$, which also depends on R and x_0 but is independent of μ and large λ. Moreover

$$\begin{aligned} \mathrm{Re}\,\zeta\cdot y(x) < 0 &\Longleftrightarrow |x - x_0| < R, \\ \mathrm{Re}\,\zeta\cdot y(x) = 0 &\Longleftrightarrow |x - x_0| = R, \\ \mathrm{Re}\,\zeta\cdot y(x) > 0 &\Longleftrightarrow |x - x_0| > R. \end{aligned} \tag{19}$$

3.1. **Schrödinger equation and spherical inversion.** To prove Theorem 3.1, we need two changes of (in)dependent variables. For a solution φ_λ to (9), we put $v = \sqrt{\gamma_0}\varphi_\lambda$. Then v satisfies

$$-\Delta v + \left(\frac{\Delta\sqrt{\gamma_0}}{\sqrt{\gamma_0}} + \frac{\lambda}{\gamma_0}\right) v = 0. \tag{20}$$

Next we pass to the inversion (16). Letting $y' = y - (y \cdot y^*)y^*$, and using $x^* = x_0 + Ry^*$, we have

$$y \cdot y^* = \frac{|x - x_0|^2 - R^2}{|x - x^*|^2}, \quad y' = \frac{2R(x - x_0)'}{|x - x^*|^2}. \tag{21}$$

The inverse map : $y \to x$ is given by

$$x = x^* + 2R\frac{y - y^*}{|y - y^*|^2}. \tag{22}$$

Direct computation yields the following lemma.

Lemma 3.2. *(1) The above transformation $x \to y$ maps*

the ball $\{|x - x_0| < R\}$ to the half-space $\{y \cdot y^* < 0\}$,
the sphere $\{|x - x_0| = R\}$ to the plane $\{y \cdot y^* = 0\}$,
the outer region $\{|x - x_0| > R\}$ to the half-space $\{y \cdot y^* > 0\}$,

where y^* is defined by (14).
(2) For any $f(y) \in C^\infty(\mathbb{R}^d)$,

$$4R^2\Delta_x f(y(x)) = |y - y^*|^{2+d}\Delta_y\left(|y - y^*|^{2-d}f(y)\right)\Big|_{y=y(x)}.$$

Here Δ_x means $\sum_{j=1}^d (\partial/\partial x_j)^2$.

For a solution $v(x)$ to (20), we put $w(y) = |y - y^*|^{2-d}v(x(y))$, where $x(y)$ is given by (22). Then by the above lemma, $w(y)$ satisfies

$$(-\Delta_y + q_\lambda(y))\, w(y) = 0, \tag{23}$$

$$q_\lambda(y) = \frac{4R^2}{|y - y^*|^4}\left(\frac{\Delta\sqrt{\gamma_0}}{\sqrt{\gamma_0}} + \frac{\lambda}{\gamma_0}\right)\Big|_{x=x(y)}.$$

3.2. **Proof of Theorem 3.1.** Taking ζ from (15), we look for a solution $w(y)$ of (23) of the form

$$w(y) = (1 + \phi(y))e^{-\zeta\cdot y}.$$

Since $\zeta^2 = 0$, ϕ satisfies

$$-\Delta\phi + 2\zeta\cdot\nabla\phi + q_\lambda\phi = -q_\lambda. \tag{24}$$

This is the equation treated in [9], with the difference that the potential q_λ grows up linearly in λ. The remedy is to make the following change of variables and parameter :

$$Y = \sqrt{\lambda}(y - y^*), \quad \eta = (\sqrt{\lambda})^{-1}\zeta.$$

Letting $\Phi(Y) = \phi(y)$, we then have

$$-\Delta_Y\Phi + 2\eta\cdot\nabla_Y\Phi + Q_\lambda\Phi = -Q_\lambda,$$

$$Q_\lambda(Y) = \frac{q_\lambda(y)}{\lambda} = \frac{4R^2}{|Y|^4}\left(\frac{\Delta\sqrt{\gamma_0}}{\lambda\sqrt{\gamma_0}} + \frac{1}{\gamma_0}\right)\Big|_{x=x(y)}.$$

The potential $Q_\lambda(Y)$ is now bounded in λ, however, it has a singularity at $Y = 0$. We take $\chi_\infty \in C^\infty(\mathbb{R}^d)$ such that $\chi_\infty(Y) = 1$ if $|Y| \geq 1/2$, $\chi_\infty(Y) = 0$ if $|Y| \leq 1/4$, and define $V_\lambda(Y)$ by

$$V_\lambda(Y) = \chi_\infty(Y) Q_\lambda(Y).$$

We prove that if $|\eta| = \mu\sqrt{\lambda}$ is sufficiently large, the equation

$$(-\Delta_Y + 2\eta \cdot \nabla_Y + V_\lambda)\Psi = -V_\lambda, \tag{25}$$

admits a unique solution $\Psi \in L^{2,\sigma}$, where the weighted L^2-space $L^{2,\sigma}$ is defined by

$$L^{2,\sigma} = \{u(Y)\,;\,(1+|Y|^2)^{\sigma/2} u \in L^2(\mathbb{R}^d)\}$$

with obvious norm.

The following lemma can be proved easily, and the proof is omitted.

Lemma 3.3. *If $\sigma < (6-d)/2$, V_λ satisfies the following inequalities:*

$$\|(1+|Y|)^4 V_\lambda\|_{L^\infty(\mathbb{R}^d)} \leq C,$$
$$\|V_\lambda\|_{L^{2,\sigma+1}} + \|\nabla_Y V_\lambda\|_{L^{2,\sigma+1}} \leq C,$$
$$\|(1+|Y|)^4 \nabla_Y V_\lambda\|_{L^\infty(\mathbb{R}^d)} \leq C,$$

where the constant C does not depend on μ or on $\lambda \geq 1$.

Then thanks to [9, Theorem 2.3] and Lemma 3.3, for all $\sigma \in (-1, 0)$, (25) admits a unique solution $\Psi \in L^{2,\sigma}$. Moreover, for $|\eta| > 1$,

$$\|\Psi\|_{L^{2,\sigma}} \leq C|\eta|^{-1}.$$

Letting $' = \partial/\partial Y_j$ and differentiating (25), we obtain

$$(-\Delta_Y + 2\eta \cdot \nabla_Y + V_\lambda)\Psi' = -V_\lambda'(1+\Psi).$$

Lemma 3.3 again implies that $\|V_\lambda'(1+\Psi)\|_{L^{2,\sigma+1}} \leq C$, hence

$$\|\nabla_Y \Psi\|_{L^{2,\sigma}} \leq C|\eta|^{-1}.$$

Let us recall the correspondence

$$|Y| > 1 \Longleftrightarrow |y - y^*| > \frac{1}{\sqrt{\lambda}} \Longleftrightarrow |x - x^*| < 2\sqrt{\lambda}R.$$

Therefore, for $\sigma \in (-1, 0)$, there exists a unique solution $\phi(y) \in L^{2,\sigma}$ of (24) in the region $|y - y^*| > 1/\sqrt{\lambda}$. Letting $\Omega' = \{y(x); x \in \Omega\}$, and noticing that $|\eta| = \mu\sqrt{\lambda}$, we then have

$$\begin{aligned}
\|\phi\|_{L^2(\Omega')} &\leq C_\sigma (1+\lambda)^{|\sigma|/2} \,\|(1+\lambda\,|y-y^*|^2)^{\sigma/2}\phi\|_{L^2(\Omega')} \\
&\leq C_\sigma (1+\lambda)^{|\sigma|/2}\,\lambda^{-d/2} \|\Psi(Y)\|_{L^{2,\sigma}} \\
&\leq C_\sigma \mu^{-1} \lambda^{(|\sigma|-d-1)/2}.
\end{aligned}$$

Similarly, since $\nabla_y \phi(y) = \sqrt{\lambda}\nabla_Y \Psi(Y)$, we obtain

$$\|\nabla_y \phi\|_{L^2(\Omega')} \leq C_\sigma \mu^{-1} \lambda^{(|\sigma|-d)/2},$$

and then, generally,

$$\|\phi\|_{H^s(\Omega')} \leq C_{\sigma,s} \mu^{-1} \lambda^{(|\sigma|-d+s-1)/2}, \quad s > 0.$$

Thanks to Sobolev's embeddings in $\mathbb{R}^d$, $d = 2, 3$, we obtain if $s > d/2$:

$$\|\phi\|_{L^\infty(\Omega')} \leq C_{\sigma,s} \mu^{-1} \lambda^{(|\sigma|-d+s-1)/2},$$

$$\|\nabla\phi\|_{L^\infty(\Omega')} \leq C_{\sigma,s} \mu^{-1} \lambda^{(|\sigma|-d+s)/2}.$$

Hence letting $\delta = 1 - |\sigma|$, if $\mu\sqrt{\lambda}$ is sufficiently large we obtain

$$\|\phi\|_{L^\infty(\Omega)} \le C_\delta\, \mu^{-1}\lambda^{-\delta/2},$$

$$\|\nabla\phi\|_{L^\infty(\Omega)} \le C_\delta\, \mu^{-1}\lambda^{(1-\delta)/2}.$$

The estimates (19) follow from Lemma 3.2. □

4. **Detection of inclusions.**

4.1. **Stationary D-N map.** We define the stationary Dirichlet-Neumann maps Λ_S^0 and $\Lambda_S \in B(H^{1/2}(\partial\Omega); H^{-1/2}(\partial\Omega))$ as follows:

$$\Lambda_S^0(h) = \gamma_0 \frac{\partial u_0}{\partial \nu}, \quad \Lambda_S(h) = \gamma \frac{\partial u}{\partial \nu},$$

where u_0 and u are solutions to

$$\begin{cases} (\mathcal{A}_0 + \lambda)\, u_0 &= 0 \quad \text{on} \quad \Omega, \\ u_0 &= h \quad \text{on} \quad \partial\Omega, \end{cases}$$

$$\begin{cases} (\mathcal{A} + \lambda)\, u &= 0 \quad \text{on} \quad \Omega, \\ u &= h \quad \text{on} \quad \partial\Omega. \end{cases} \tag{26}$$

Let $\langle\, ,\, \rangle_{\partial\Omega}$ be the inner product on $L^2(\partial\Omega)$ and put

$$J_S(h) = \langle (\Lambda_S - \lambda_S^0)h, h\rangle_{\partial\Omega}. \tag{27}$$

The following lemma was proven in [7] in the case $\lambda = 0$.

Lemma 4.1. *The following identities hold:*

$$J_S(h) = \int_\Omega \left\{ (\gamma - \gamma_0)|\nabla u_0|^2 - \gamma|\nabla(u - u_0)|^2 - \lambda|u - u_0|^2 \right\} dx, \tag{28}$$

$$J_S(h) = \int_\Omega \left\{ \frac{\gamma_0(\gamma - \gamma_0)}{\gamma}|\nabla u_0|^2 + \frac{1}{\gamma}|\gamma\nabla u - \gamma_0\nabla u_0|^2 + \lambda|u - u_0|^2 \right\} dx. \tag{29}$$

Proof. By Green's formula, we have for $g \in H^1(\Omega)$

$$\langle \Lambda_S^0 h, g\rangle_{\partial\Omega} = \int_\Omega (\gamma_0 \nabla u_0 \nabla\overline{g} + \lambda u_0 \overline{g})\, dx, \tag{30}$$

$$\langle \Lambda_S h, g\rangle_{\partial\Omega} = \int_\Omega (\gamma \nabla u \nabla\overline{g} + \lambda u \overline{g})\, dx. \tag{31}$$

Letting $g = u_0$ in (30) and $g = u$ in (31), we have

$$\langle \Lambda_S^0 h, h\rangle_{\partial\Omega} = \int_\Omega \left(\gamma_0|\nabla u_0|^2 + \lambda|u_0|^2\right) dx, \tag{32}$$

$$\langle \Lambda_S h, h\rangle_{\partial\Omega} = \int_\Omega \left(\gamma|\nabla u|^2 + \lambda|u|^2\right) dx, \tag{33}$$

Relation (30) with $g = u - u_0$ gives:

$$0 = \int_\Omega \{\gamma_0 \nabla u_0 \cdot \nabla(\overline{u} - \overline{u_0}) + \lambda u_0(\overline{u} - \overline{u_0})\} dx. \tag{34}$$

From (33) − (32) we have

$$\begin{aligned} J_S(h) &= \int_\Omega \left\{(\gamma-\gamma_0)|\nabla u|^2 + \gamma_0(|\nabla u|^2 - |\nabla u_0|^2) + \lambda(|u|^2 - |u_0|^2)\right\} dx \\ &= \int_\Omega \left\{(\gamma-\gamma_0)|\nabla u|^2 + \gamma_0|\nabla(u-u_0)|^2 \right. \\ &\quad +\gamma_0\left(\nabla(u-u_0)\cdot\nabla\overline{u_0} + \nabla u_0\cdot\nabla(\overline{u}-\overline{u_0})\right) \\ &\quad \left. +\lambda\left(|u-u_0|^2 + (u-u_0)\overline{u_0} + u_0(\overline{u}-\overline{u_0})\right)\right\} dx. \end{aligned}$$

Thanks to(34) we obtain

$$J_S(h) \;=\; \int_\Omega \left\{(\gamma-\gamma_0)|\nabla u|^2 + \gamma_0|\nabla(u-u_0)|^2 + \lambda|u-u_0|^2\right\} dx. \tag{35}$$

Now letting $\psi = \gamma\nabla u - \gamma_0\nabla u_0$, we take notice of the following equality

$$\begin{aligned} & (\gamma-\gamma_0)|\nabla u|^2 + \gamma_0|\nabla(u-u_0)|^2 \\ =\; & \gamma^{-2}\{(\gamma-\gamma_0)|\gamma_0\nabla u_0 + \psi|^2 + \gamma_0|(\gamma_0-\gamma)\nabla u_0 + \psi|^2\} \\ =\; & \gamma^{-1}\gamma_0(\gamma-\gamma_0)|\nabla u_0|^2 + \gamma^{-1}|\psi|^2. \end{aligned}$$

This and (35) imply (29). By exchanging $(\gamma,\, u,\, u-u_0)$ and $(\gamma_0,\, u_0,\, u_0-u)$ in (35), we can prove (28). □

The above lemma implies

$$\int_\Omega \frac{\gamma_0(\gamma-\gamma_0)}{\gamma}|\nabla u_0|^2 dx \le J_S(h) \le \int_\Omega (\gamma-\gamma_0)|\nabla u_0|^2 dx,$$

which yields the following corollary.

Corollary 1. *If $\gamma(x)-\gamma_0(x)$ has a constant sign, then $J_S(f)$ has the same sign and there exists a constant $C>0$ such that*

$$C\int_\Omega |\gamma-\gamma_0||\nabla u_0|^2 dx \le |J_S(h)| \le C^{-1}\int_\Omega |\gamma-\gamma_0||\nabla u_0|^2 dx. \tag{36}$$

4.2. **Uniform estimate for the parabolic equation.**

Lemma 4.2. *Let u_0 be a solution to the equation $(\mathcal{A}_0+\lambda)u_0 = 0$ in Ω. Construct solutions u to (26) with $h = u_0$, and $v(t)$ to (1) with $f = e^{\lambda t}u_0$. Then for any $0 < t \le T$ and $m \ge 0$, there exists a constant $C_{tm} > 0$ independent of λ such that*

$$\|v(t,\cdot) - e^{\lambda t}u(\cdot)\|_{H^m(\Omega\setminus\Omega_1)} \le C_{tm}\left(\|v_0\|_{L^2(\Omega)} + \|u_0\|_{H^1(\Omega)}\right).$$

Proof. Letting $w = v(t) - e^{\lambda t}u$, we have

$$\left\{\begin{array}{rcll} \partial_t w + \mathcal{A}w &=& 0 & \text{in} \quad (0,T)\times\Omega, \\ w(t) &=& 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ w(0) &=& v_0 - u & \text{on} \quad \Omega. \end{array}\right.$$

Thus, $w(t) = e^{-tA_D}(v_0-u)$, where A_D denotes the self-adjoint operator $\mathcal{A}$ with the homogeneous Dirichlet boundary condition. Therefore, for any $t>0$ and $m \ge 0$, $w(t) \in D((A_D)^m)$. By the regularity theorem for elliptic operators, we then have

$$\|w(t)\|_{H^{2m}(\Omega\setminus\Omega_1)} \le C_m(\|w(t)\|_{L^2(\Omega)} + \|(A_D)^m w(t)\|_{L^2(\Omega)}) \le C_{tm}\|v_0-u\|_{L^2(\Omega)}.$$

Let ϕ be the solution of

$$\left\{\begin{array}{rcll} \mathcal{A}\phi &=& 0 & \text{on} \quad \Omega, \\ \phi &=& u_0 & \text{on} \quad \partial\Omega. \end{array}\right.$$

Then $\|\phi\|_{H^1(\Omega)} \le C\|u_0\|_{H^{1/2}(\partial\Omega)}$ and $u-\phi$ satisfies

$$\begin{cases} (A+\lambda)(u-\phi) & = & -\lambda\phi \quad \text{on} \quad \Omega, \\ u-\phi & = & 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Since A_D is positive definite, $\|u-\phi\|_{L^2(\Omega)} \le C\lambda\|(A_D+\lambda)^{-1}\phi\|_{L^2(\Omega)} \le C\|\phi\|_{L^2(\Omega)} \le C\|u_0\|_{H^{1/2}(\partial\Omega)}$, where C is independent of $\lambda > 0$. Hence we obtain for $0 < t \le T$ and $m \ge 0$

$$\begin{aligned} \|w(t,\cdot)\|_{H^m(\Omega\setminus\Omega_1)} & \le C_t\|v_0 - u\|_{L^2(\Omega)} \le C_t\left(\|v_0\|_{L^2(\Omega)} + \|u_0\|_{H^{1/2}(\partial\Omega)}\right) \\ & \le C_t\left(\|v_0\|_{L^2(\Omega)} + \|u_0\|_{H^1(\Omega)}\right). \quad \square \end{aligned}$$

4.3. **Proof of Theorems 1.1 and 1.2.** For the sake of simplicity, we restrict the proof to the case $\gamma_1(x) > \gamma_0(x)$ on Ω_1. Let $y(x)$ be defined by (6) for $d=1$, and by (16) for $d=2,3$. Letting y^* be defined by (14), we put

$$y_1(x) = \begin{cases} y(x) & (d=1), \\ y(x)\cdot y^* & (d=2,3), \end{cases}$$

and for a bounded open set $\mathcal{O} \subset \mathbb{R}^d$

$$a(\mathcal{O}) = \sup\{-y_1(x)\,;\, x \in \mathcal{O}\}.$$

Note that $a(\mathcal{O}) = a_B(\mathcal{O})$ for $d=2,3$ by (7) and (21). We put

$$\kappa = \begin{cases} 2\sqrt{\lambda}, & \text{if} \quad d=1, \\ \sqrt{2}\mu\lambda, & \text{if} \quad d=2,3. \end{cases}$$

Let $\varphi_\lambda(x)$ be as in Theorem 2.1 or Theorem 3.1. Thanks to (13) and (17), we have

$$C\kappa^2 e^{-\kappa y_1(x)} \le |\nabla\varphi_\lambda(x)|^2 \le C^{-1}\kappa^2 e^{-\kappa y_1(x)}. \tag{37}$$

We then put

$$J(\mathcal{O},\kappa) = \int_{\mathcal{O}} e^{-\kappa\, y_1(x)}dx,$$

$$J_\gamma(\kappa) = \int_\Omega (\gamma-\gamma_0)|\nabla\varphi_\lambda(x)|^2 dx.$$

Lemma 4.3. *(1) For a non-empty bounded open set $\mathcal{O} \subset \mathbb{R}^d$, we have*

$$\lim_{\kappa\to\infty} \kappa^{-1}\log J(\mathcal{O},\kappa) = a(\mathcal{O}). \tag{38}$$

(2) We have

$$\lim_{r\to\infty} \kappa^{-1}\log J_\gamma(\kappa) = a(\Omega_1). \tag{39}$$

Proof. Letting $\mathcal{O}_\epsilon = \{x \in \mathcal{O}\,; a(\Omega) - \epsilon \le -y_1(x) \le a(\mathcal{O})\}$, we have

$$\int_{\mathcal{O}_\epsilon} e^{\kappa(a(\mathcal{O})-\epsilon)}dx \le \int_{\mathcal{O}} e^{-\kappa y_1(x)}dx \le \int_{\mathcal{O}} e^{\kappa a(\mathcal{O})}dx.$$

Taking the logarithm, and letting $\kappa \to \infty$, we get (38). Let $\mathcal{O}$ be a bounded open set such that $\overline{\mathcal{O}} \subset \Omega_1$. Then in view of (37), we have

$$C\kappa^2 J(\mathcal{O},\kappa) \le J_\gamma(\kappa) \le C^{-1}\kappa^2 J(\Omega_1,\kappa),$$

which implies by (38)

$$a(\mathcal{O}) \le \liminf_{\kappa\to\infty} \kappa^{-1}\log J_\gamma(\kappa) \le \limsup_{\kappa\to\infty} \kappa^{-1}\log J_\gamma(\kappa) \le a(\Omega_1).$$

Since $\mathcal{O} \subset\subset \Omega_1$ can be taken arbitrarily, we get (39). $\square$

We put

$$h_\lambda(x) = \varphi_\lambda(x)\Big|_{\partial\Omega}.$$

By Corollary 4.2 and (39), we have

Lemma 4.4. *Let $J_S(f)$ be defined by (27). Then we have*

$$\lim_{\kappa\to\infty} \kappa^{-1} \log J_S(h_\lambda) = a(\Omega_1).$$

Before entering into the probing problem, we need to study geometric properties of the map : $x \to y_1(x)$. The next lemma follows from a direct computation using $R = |x_0 - x^*|$ and (21).

Lemma 4.5. *Let $d \geq 2$. For $c \neq 1$, let*

$$x_c = \frac{x_0 - cx^*}{1-c}, \quad R_c = \frac{R}{|1-c|}.$$

Then we have

$$\left\{x \in \mathbb{R}^d\, ;\, -y_1(x) = c\right\} = B(x_c, R_c).$$

Note that $B(x_c, R_c)$ is the sphere passing through x^* with center on the line $l = \{x^* + t(x_0 - x^*)\, ;\, t \in \mathbb{R}\}$. When c grows up, so does its radius, and if $c = 0$, it is the sphere $B(x_0, R)$.

Now, by definition, there exists $\overline{x} \in \overline{\Omega_1}$ such that $-y_1(\overline{x}) = a(\Omega_1)$. Then Ω_1 lies outside $B(x_c, R_c)$ with $c = a(\Omega_1)$. Assume that the open set $\Gamma \subset \partial\Omega$ satisfies the condition (C-1). Then by a suitable choice of $c < a(\Omega_1)$, we see that $\Omega_2 = \Omega \setminus \overline{B(x_c, R_c)}$ satisfies

(C-2) $\quad \partial\Omega \setminus \Gamma \subset \partial\Omega_2, \quad \overline{\Omega} \cap \overline{B(x,R)} = \emptyset, \quad a(\Omega_2) < a(\Omega_1).$

The following lemma gives a relation between the time-dependent measurement $I(\lambda)$ defined by (4) or (10) and the stationary one $J_S(h_\lambda)$.

Lemma 4.6. *Assume the condition $(C-1)$ for $d = 2, 3$. Then there exists $\delta > 0$ such that the following estimate holds:*

$$\left|\frac{I(\lambda) - J_S(h_\lambda)}{J_S(h_\lambda)}\right| \leq e^{-\delta\kappa}, \quad \forall \lambda > 1/\delta.$$

Proof. Let $f_\lambda(t,x) = e^{\lambda t}\varphi_\lambda(x)$, and $v(t,x)$ the solution of (1) with $f = f_\lambda$. Let u_λ be the solution of (26) with $h = h_\lambda$. Letting $w = v - e^{\lambda t}u_\lambda$, we then have

$$I(\lambda) = J_S(h_\lambda) + R_\Gamma(\lambda) + R_S(\lambda),$$

$$\begin{aligned} R_\Gamma(\lambda) &= e^{-\lambda T_1} \int_\Gamma \gamma \frac{\partial w(T_1, \cdot)}{\partial \nu}\, \overline{h_\lambda} d\sigma, \\ R_S(\lambda) &= -\int_{\partial\Omega\setminus\Gamma} (\Lambda_S - \Lambda_S^0) h_\lambda\, \overline{h_\lambda} d\sigma, \end{aligned}$$

Then, thanks to lemma 4.2, we obtain:

$$\begin{aligned} |R_\Gamma| &\leq Ce^{-\lambda T_1} \|\frac{\partial w(T_1,\cdot)}{\partial\nu}\|_{H^{1/2}(\partial\Omega)}\, \|h_\lambda\|_{H^{1/2}(\partial\Omega)} \\ &\leq Ce^{-\lambda T_1} \|w\|_{H^2(\Omega\setminus\Omega_1)}\, \|\varphi_\lambda\|_{H^1(\Omega)} \\ &\leq Ce^{-\lambda T_1} (\|v_0\|^2 + \|\varphi_\lambda\|^2_{H^1(\Omega)}) \end{aligned}$$

By (37), we have $\|\varphi_\lambda\|^2_{H^1(\Omega)} \le C\kappa^2 e^{\kappa a(\Omega)}$. Hence

$$|R_\Gamma| \le C\kappa^2 e^{-\lambda T_1 + \kappa a(\Omega)}. \tag{40}$$

Let us estimate R_S. If $d = 1$ we set $\Omega_2 = (a_2, b) \subset \Omega$ with $a_1 < a_2 < b$. Recall that $\Omega_1 = (a_1, b_1)$. If $d = 2, 3$, we take Ω_2 satisfying the condition $(C-2)$. Then we have

$$\begin{aligned} |R_S| &\le C(\|u_\lambda\|_{H^1(\Omega_2)} + \|\varphi_\lambda\|_{H^1(\Omega_2)})\|\varphi_\lambda\|_{H^1(\Omega_2)} \\ &\le C(\|\tilde{w}\|_{H^1(\Omega_2)} + 2\|\varphi_\lambda\|_{H^1(\Omega_2)})\|\varphi_\lambda\|_{H^1(\Omega_2)}, \end{aligned} \tag{41}$$

with $\tilde{w} = u_\lambda - \varphi_\lambda$. Using (35) we have $\|\tilde{w}\|_{H^1(\Omega)} \le \sqrt{J_S(h_\lambda)}$. Hence, by (36), we obtain

$$|R_S| \le C\big(\sqrt{J_S(h_\lambda)} + \|\varphi_\lambda\|_{H^1(\Omega_2)}\big)\|\varphi_\lambda\|_{H^1(\Omega_2)}.$$

By (37), (41) and

$$\|\varphi_\lambda\|_{H^1(\Omega_2)} \le C\kappa e^{\kappa a(\Omega_2)/2}.$$

Lemma 4.5 implies that for any $\epsilon > 0$, there exists $\kappa_0 > 0$ such that

$$e^{\kappa(a(\Omega_1)-\epsilon)} \le J_S(h_\lambda) \le e^{\kappa(a(\Omega_1)+\epsilon)}, \quad \forall \kappa > \kappa_0. \tag{42}$$

In view of (40), (41) and (42), we have

$$\begin{aligned} \left|\frac{I(\lambda) - J_S(h_\lambda)}{J_S(h_\lambda)}\right| &\le \frac{|R_\Gamma(\lambda)|}{J_S(h_\lambda)} + \frac{|R_S(\lambda)|}{J_S(h_\lambda)} \\ &\le C\kappa^2\left(e^{-\lambda T_1 + \kappa(a(\Omega) - a(\Omega_1) + \epsilon)} + e^{\kappa(a(\Omega_2) - a(\Omega_1) + 2\epsilon)}\right). \end{aligned} \tag{43}$$

Now we use the geometric configuration of Ω_1 and Ω_2. In the case $d = 1$, $y(x)$ is monotone increasing. Hence

$$a(\Omega_2) = \sup_{x\in\Omega_2}(-y_1(x)) = -y(a_2) < -y(a_1) = \sup_{x\in\Omega_1}(-y_1(x)) = a(\Omega_1).$$

In the case $d > 1$, we have $a(\Omega_2) < a(\Omega_1)$ by $(C-2)$. By the condition (8), we have $-\lambda T_1 + \kappa(a(\Omega) - a(\Omega_1)) < -\delta\kappa$ for some $\delta > 0$. Therefore choosing ϵ small enough in (43), we prove the lemma. □

Now Theorems 1.1 and 1.2 follow from Lemma 4.7 and (42).

5. **2-dimensional conformal map.** In this section, we identify $x = (x_1, x_2) \in \mathbb{R}^2$ with a complex number $z = x_1 + ix_2 \in \mathbb{C}$. Consider a univalent holomorphic function $F(z)$ defined in a neighborhood of Ω, and the conformal map : $z \to w = y_1 + iy_2 = F(z)$. As in §3, we look for a solution of (9) in the form $\varphi_\lambda(x) = (1 + \phi(y))e^{-\zeta\cdot F(x)}$, where ϕ satisfies (24). Note that $\zeta \cdot F(z)$ is identified with $\frac{\mu\lambda}{\sqrt{2}}F(z)$. Assume that $0 \notin \overline{\Omega}$ and let us choose $F(z) = z^{-n} - \rho^{-n}$ with $\rho > 0$ and $n = 1, 2, \ldots$. The case $n = 1$ corresponds to the inversion we have studied in §3, where R corresponds to $2/\rho$.

Consider the curve $\operatorname{Re} F(z) = 0$ that separates the set of x where $\varphi_\lambda(x)$ has an exponential growth from the set of x where $\varphi_\lambda(x)$ has an exponential decay as $\lambda \to \infty$. Writing $z = re^{i\alpha}$, we have that $\operatorname{Re} F(z) = 0 \iff r = \rho(\cos(n\alpha))^{1/n}$. It appears that if Ω is convex, then $n = 3$ is a better choice than $n = 1, 2$ since the curve $\operatorname{Re} F(z) = 0$ goes more deeply inside Ω (cf. figure 3).

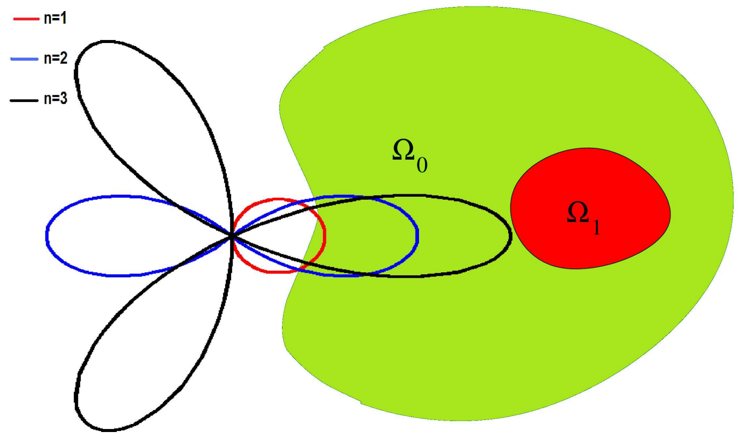

FIGURE 3.

6. **Numerical tests in the one-dimensional case.** We take $\gamma_0 = 1$ which implies

$$\begin{aligned} h_\lambda(x) &= e^{-\sqrt{\lambda}(x-x_0)} \\ f_\lambda(t,x) &= e^{\lambda t} h_\lambda(x) \\ I(T_1,\lambda) &= [e^{-\lambda T_1}(\Lambda f_\lambda)(T_1,a) - \frac{\partial h_\lambda}{\partial x}(a)] h_\lambda(a) \qquad (44) \\ \lim_{\lambda\to\infty} \frac{\log(I(T_1,\lambda))}{2\sqrt{\lambda}} &= x_0 - a_1. \qquad (45) \end{aligned}$$

The equation (45) is calculated with a finite λ, so we have the approximative reconstruction equation

$$a_1 \approx x_0 - \frac{\log(I(T_1,\lambda))}{2\sqrt{\lambda}}, \qquad (46)$$

which is the more accurate the larger the parameter λ is. In practice larger values here implies very large values of f_λ on the boundary, so numerical errors will be a problem. However for initial testing purposes we proceed by using $\lambda = 1.0, 1.5, 2.0, \ldots, 20.0$. Also we choose $a = 0, b = 1, x_0 = 0$ and $T = 1$.

Four test cases of the following heat conductivity are considered:

$$\begin{aligned} \gamma &= \gamma_1, \quad a_1 \le x \le b_1, \\ \gamma &= \gamma_0, \quad \text{otherwise}, \end{aligned}$$

where the values of a_1, b_1 and γ_1 are given in table 6. These test cases are pictured in figure 4. The solutions v in (1) are calculated with the finite element method (FEM) so that there are $N_x = 200$ points of $x \in [0,1]$ and $N_t = 100$ points of $t \in [0,1]$. We consider three different initial temperature distributions:

$$\begin{aligned} v(x,0) &= 1, \\ v(x,0) &= 2\sin(\pi x), \\ v(x,0) &= 4|\sin(4\pi x)|. \end{aligned}$$

In figure 5 we display the solution function $v(x,t), v(x,0) = 1, \lambda = 2$ calculated by FEM for the test case 1, smaller numbers of N_x and N_t are only used for printing purposes.

	a_1	b_1	γ_1
Case 1	0.1	0.3	0.5
Case 2	0.1	0.3	5
Case 3	0.4	0.6	0.5
Case 4	0.4	0.6	5

TABLE 1. Parameters a_1, b_1 and γ_1 for the test cases

To simulate measurement noise we define the vector

$$L_n = L \cdot (1 + c \cdot r), \tag{47}$$

where L is the vector containing the values of $\Lambda f_\lambda(T_1, 0) = \gamma(x) \frac{\partial v^{f_\lambda}}{\partial \nu}\Big|_{x=0}$ at the FEM mesh points and r is a normally distributed random vector (white noise) of the same size and the parameter c is adjusted separately for each λ so that the relative noise is approximately two percent:

$$\frac{||L_n - L||}{||L||} \approx 0.02. \tag{48}$$

The logarithm of the indicator function (44) is calculated for each $t \in [0,1]$ and $\lambda = 1.0, 1.5, 2.0, \ldots, 20.0$. This information can be presented in several ways. In figure 6 one can see how the logarithm of the indicator function, noisy and non-noisy, behaves as a function of T_1, for $\lambda = 1$ and $\lambda = 10$, in test case 1. We conclude that the value $\lambda = 1$ is too small to get accurate indicator function values, and that $\lambda = 10$ provides good values throughout the time interval $[T/2, T]$. In figure 7 we have a similar graph for $\lambda = 10$, but for each initial temperature distribution. We conclude that the initial distribution $v(x,0)$ does not change the indicator function values in the interval $[T/2, T]$ and thus it will not affect the reconstructions. In figure 8 one can see how the logarithm of the indicator function, noisy and non-noisy, behaves as a function of λ, for several choices of T_1, in test case 1. Notice that in these graphs the analytical value based on (46) is also shown.

Based on the observation that the values of the indicator function are good between the time interval $[T/2, T]$ we use the following indicator function value:

$$I^\star(\lambda) = \frac{1}{N_t^\star} \sum_{T_1 = T/2}^{T} I(T_1, \lambda), \tag{49}$$

where $I(T_1, \lambda)$ is defined by (44) and $N_t^\star$ is the number of FEM mesh points $t \in [0,1]$ between the time values $T/2$ and T. The value $I^\star(\lambda)$ is then inserted into (46), from which the reconstruction a_1 is calculated with $\lambda = 1.0, 1.5, 2.0, \ldots, 20.0$. In figure 9 the reconstructed a_1, noisy and non-noisy, is shown as a function of λ, for each test case.

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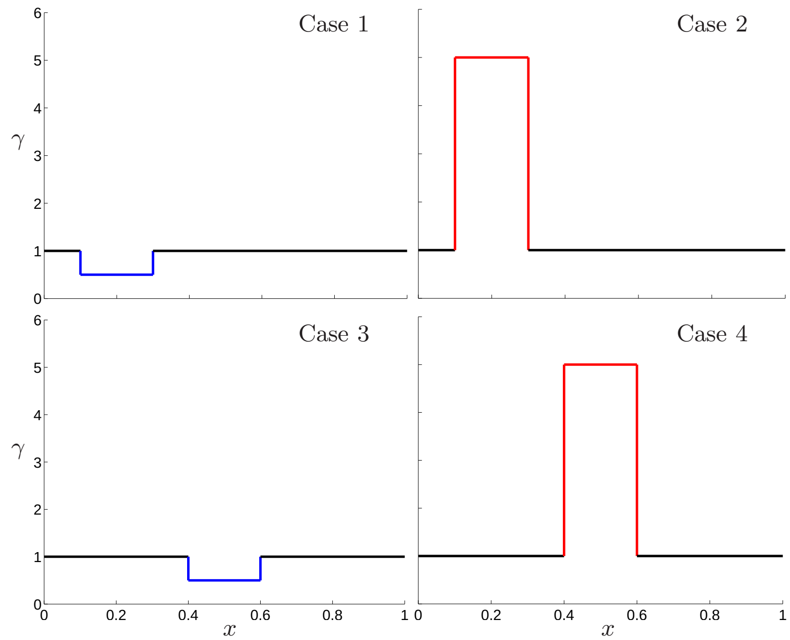

FIGURE 4. The test case heat conductivities

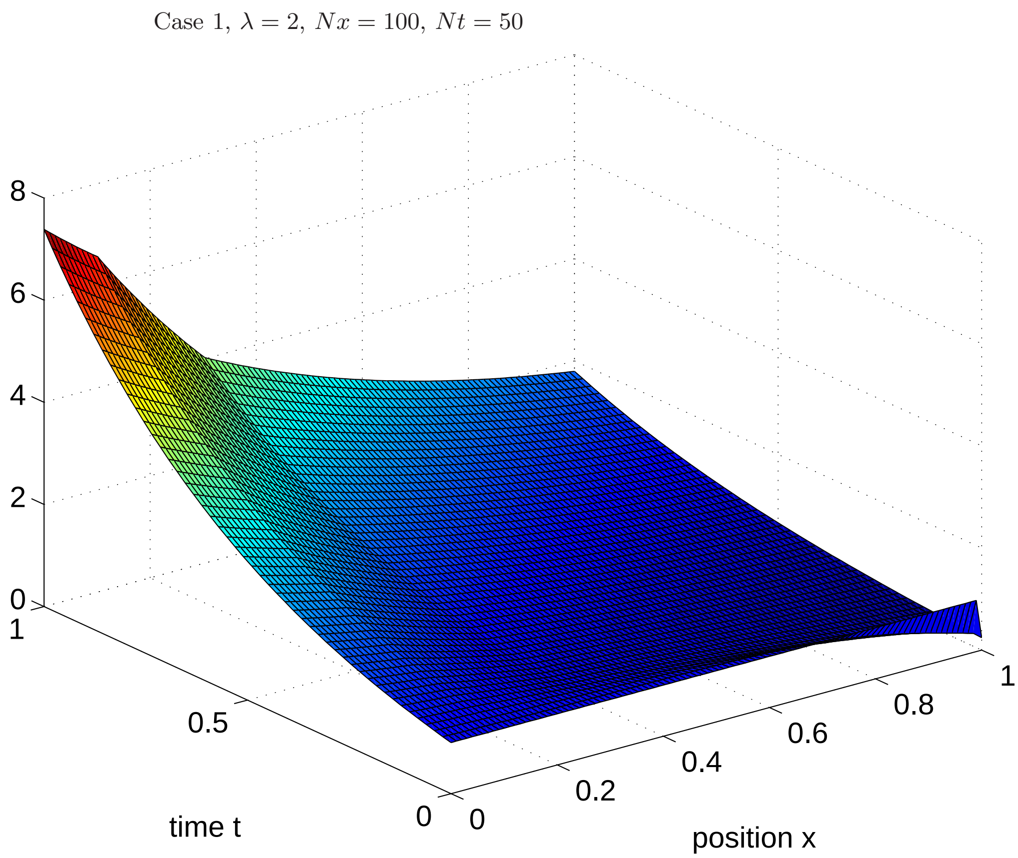

FIGURE 5. The solution function $v(x,t), \lambda = 2$ calculated by FEM for test case 1, $v(x,0) = 1$.

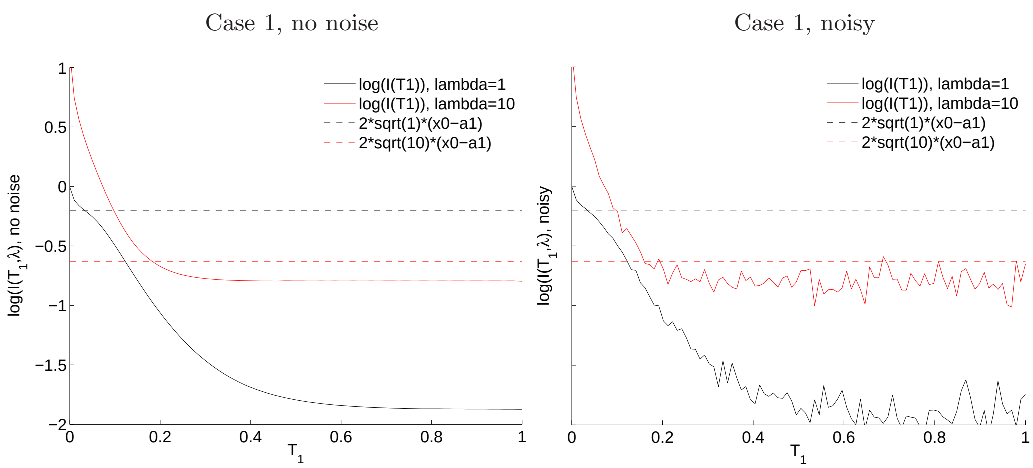

FIGURE 6. The logarithm of the indicator function, non-noisy on the left and noisy on the right, as a function of T_1 for $\lambda = 1$ and $\lambda = 10$ in test case 1.

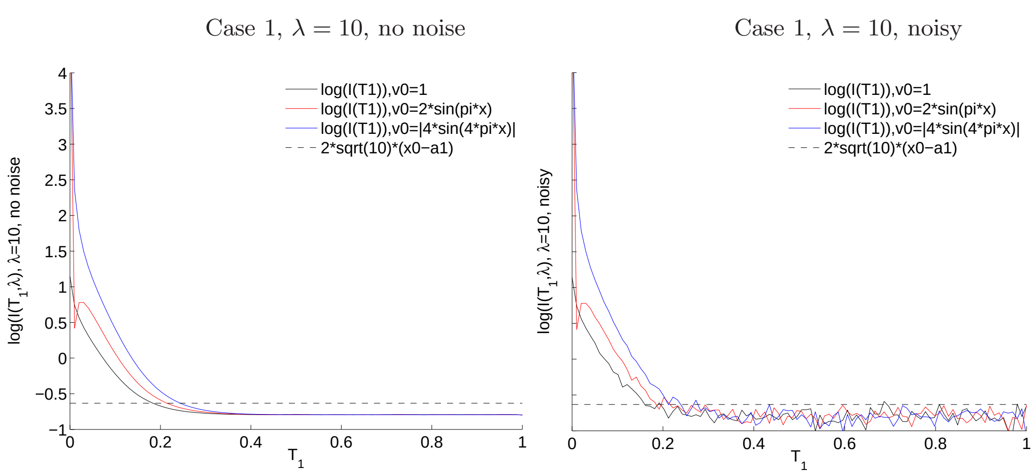

FIGURE 7. The logarithm of the indicator function, non-noisy on the left and noisy on the right, as a function of T_1 for $\lambda = 10$ in test case 1, with three different initial data $v(x,0) = 1, v(x,0) = 2\sin(\pi x)$ and $v(x,0) = 4|\sin(4\pi x)|$. Note that the effect of the initial data vanishes as T_1 grows.

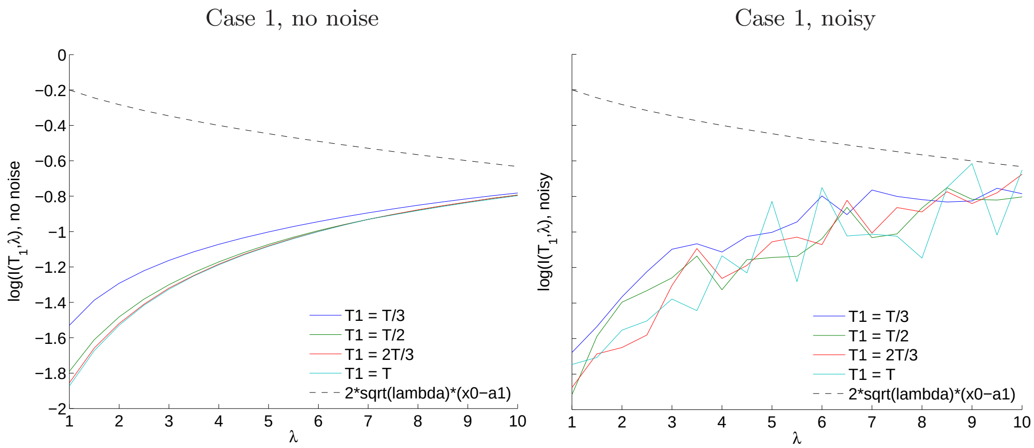

FIGURE 8. The logarithm of the indicator function, non-noisy on the left and noisy on the right, as a function of λ for several choices of T_1 in test case 1.

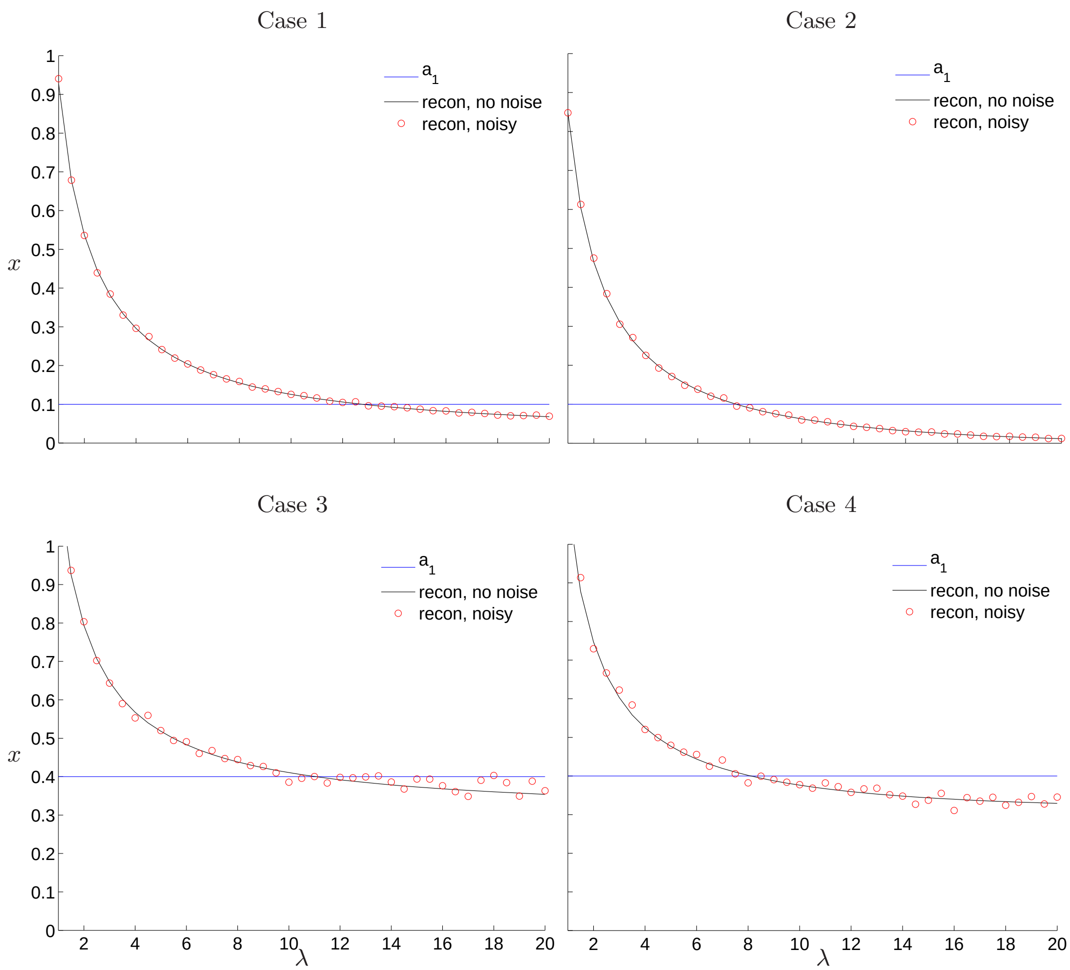

FIGURE 9. The reconstructed a_1, noisy and non-noisy, as a function of λ.