

INVERSE PROBLEMS FOR TIME-DEPENDENT SINGULAR HEAT CONDUCTIVITIES — ONE-DIMENSIONAL CASE

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Abstract. An inverse boundary value problem for the heat equation is considered on the interval $(0, 1)$, where the heat conductivity $\gamma(t, x)$ is piecewise constant and the point of discontinuity depends on time: $\gamma(t, x) = k^2$ for $0 < x < s(t)$ and $\gamma(t, x) = 1$ for $s(t) < x < 1$. It is shown that k and $s(t)$ on the time interval $[0, T]$ are determined from the partial Dirichlet-to-Neumann map: $u(t, 1) \rightarrow \partial_x u(t, 1)$ for $0 < t < T$, where $u(t, x)$ is the solution to the heat equation, independently of the initial data $u(0, x)$.

Key words. Inverse problem, Dirichlet-to-Neumann map, heat probing, parabolic

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1. Introduction.

1.1. Inverse heat conductivity problem. Let $\Omega = (0, 1)$, and consider the following initial boundary value problem

$$\begin{cases} \partial_t u(t, x) - \partial_x (\gamma(t, x) \partial_x u(t, x)) = 0 & \text{in } (0, T) \times \Omega, \\ u(t, 0) = f_0(t), \quad u(t, 1) = f_1(t) & \text{for } 0 < t < T, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\gamma(t, x) \in L^\infty((0, T) \times \Omega)$ having the following properties : There exist a constant $k > 0$ and $s(t) \in C^2([0, T])$ such that

$$0 < \inf_{0 < t < T} s(t) \leq \sup_{0 < t < T} s(t) < 1, \quad (1.2)$$

$$\gamma(t, x) = \begin{cases} k^2, & \text{if } 0 < x < s(t), \\ 1, & \text{if } s(t) < x < 1. \end{cases} \quad (1.3)$$

Let $u(t, x)$ be the solution to the above equation. The Dirichlet-to-Neumann map, henceforth called the D-N map, $\Lambda(f_0, f_1, u_0)$ is then defined by

$$\Lambda(f_0, f_1, u_0) : (f_0, f_1, u_0) \rightarrow (-k^2 \partial_x u(t, 0), \partial_x u(t, 1)). \quad (1.4)$$

Physically, the region $D(t) = (0, s(t))$ in the domain Ω corresponds to some inclusion in the medium with heat conductivity different from the one in the background. The problem we address in this paper is to detect $s(t)$ and k with the aid of the D-N map Λ .

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As a matter of fact, we shall show that $\gamma(t, x)$ can be determined from the partial knowledge

$$\Lambda_{\text{partial}}(u_0) := \Lambda(0, f_1, u_0) : f_1(t) \rightarrow \partial_x u(t, 1)$$

without taking into account of the information of the initial data u_0 . Physically, this corresponds to maintaining zero temperature at the point $x = 0$ and temperature $f_1(t)$ at point $x = 1$ for the finite time $0 < t < T$, and measuring the resulting heat flux at the point $x = 1$. Theoretically, this infinite-precision measurement needs to be repeated infinitely many times to recover $D(t)$ and $\gamma(t, x)$ perfectly. However, approximate recovery should be possible from a finite number of finite-precision measurements similarly to [8, 6, 7], but this is outside the scope of the present paper.

1.2. Main theorem. Take a large parameter $\lambda > 0$, and put

$$h_{\text{fw}}(t, x; \lambda) = e^{\lambda^2 t + \lambda x}, \quad h_{\text{bw}}(t, x; \lambda) = e^{-\lambda^2 t + \lambda x}. \quad (1.5)$$

They solve the forward and the backward heat equation, respectively:

$$(\partial_t - \partial_x^2) h_{\text{fw}} = 0, \quad (\partial_t + \partial_x^2) h_{\text{bw}} = 0. \quad (1.6)$$

Let $u(t, x; \lambda)$ be the solution to (1.1) with $f_0 = 0$, $f_1 = h_{\text{fw}}(t, 1; \lambda)$, and define

$$I_{\text{ind}}(T; \lambda) = \int_0^T e^{\lambda \nu t} h_{\text{bw}}(t, 1; \lambda) \partial_x (u(t, x; \lambda) - h_{\text{fw}}(t, x; \lambda)) \Big|_{x=1} dt. \quad (1.7)$$

THEOREM 1.1. *Assume that $k \neq 1$. Fix ν such that*

$$\nu > \max\left(2, \left|1 - \frac{1}{k}\right|\right) \sup_{0 < t < T} |\dot{s}(t)|.$$

Then for $\lambda \rightarrow \infty$, we have

$$I_{\text{ind}}(T; \lambda) \simeq \frac{2(k-1)}{(\nu + 2\dot{s}(T))(k+1)} e^{\lambda \nu T + 2\lambda s(T) - \dot{s}(T)(1-s(T))}.$$

COROLLARY 1.2. *For any initial data $u_0(x) \in L^2(0, 1)$, one can determine k and $s(t)$, $0 < t < T$, from the partial Dirichlet-to-Neumann map*

$$\Lambda_{\text{partial}}(u_0) : f_1(t) \rightarrow \partial_x u(t, 1), \quad 0 < t < T.$$

The issues on uniqueness, stability and reconstruction of the inclusion-identification problem have been centered around the case in which $s(t)$ is independent of t . Bellout [1] proved the local uniqueness and stability. Elayyan and Isakov [5] proved the global uniqueness of the inverse problem using the localized Neumann-to-Dirichlet (N-D) map. In [3], Di Cristo and Vessella gave logarithmic stability estimates of the inclusion from the Dirichlet-to-Neumann map. Ikehata [9], and Ikehata-Kawashita [10] developed the probe method for the heat equation with time-independent inclusions. In [6], the case of time-independent inclusions was treated and a numerical computation result was given. The idea is based on the complex spherical wave given by Ide-Isozaki-Nakata-Siltanen-Uhlmann [8] for the elliptic case. The work of Daido, Kang and Nakamura [4] may be the closest to the present paper. They studied the

case of moving inclusions $D(t) = \{0 < a_0(t) < x < a_1(t) < 1\}$ using the probe method, which is based on the explicit form of the heat kernel, and Runge's approximation theorem, and proved that $a_1(t)$ is obtained from the whole knowledge of Neumann-to-Dirichlet map. Their initial data is assumed to be $0 : u_0(x) = 0$, and the computation of k was not done. As for the recent works on the inverse problem for the parabolic equation, see Bacchelli-Cristo-Sincich-Vessella [2] for the corrosion problem, and Vessella [14] and Kawakami-Tsuchiya [11] for the time-varying domain problem.

We use two main tools in this paper: the approximate solution of the heat equation to be constructed in §3, and the energy inequality in §4. Although the approximate solution is based on the standard construction of parametrics for the parabolic equation, a delicate choice is necessary for amplitude functions due to the discontinuity of the coefficient. The energy inequality is also a familiar tool, however, we need a careful choice of the auxiliary function to be multiplied by the equation. Our method can be extended to multi-dimensions, which will be discussed elsewhere.

Throughout the paper, we only deal with real-valued functions.

2. Existence theorem.

2.1. A Theorem of J. L. Lions. Let \mathcal{H} be a Hilbert space equipped with inner product (\cdot, \cdot) and norm $\|\cdot\|$. Suppose there exists another Hilbert space \mathcal{H}_1 with inner product $(\cdot, \cdot)_1$ and norm $\|\cdot\|_1$ such that \mathcal{H}_1 is a dense subset of \mathcal{H} and there exists a constant $C > 0$ such that

$$\|u\| \leq C\|u\|_1, \quad \forall u \in \mathcal{H}_1. \quad (2.1)$$

Then we have the following inclusion relations

$$\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_1^*. \quad (2.2)$$

For $t \in [0, T]$, let $a(t, \cdot, \cdot)$ be a quadratic form on $\mathcal{H}_1 \times \mathcal{H}_1$ such that

$$a(t, u, v) = \overline{a(t, v, u)}, \quad \forall u, v \in \mathcal{H}_1, \quad \forall t \in [0, T]. \quad (2.3)$$

We also assume that there exist constants $\delta > 0$, $C_0 > 0$ such that

$$|a(t, u, v)| \leq C_0\|u\|_1\|v\|_1, \quad \forall u, v \in \mathcal{H}_1, \quad \forall t \in [0, T], \quad (2.4)$$

$$a(t, u, u) \geq \delta\|u\|_1^2 - C_0\|u\|^2, \quad \forall u \in \mathcal{H}_1, \quad \forall t \in [0, T]. \quad (2.5)$$

The last assumption is :

$$\text{For any } u, v \in \mathcal{H}_1, [0, T] \ni t \mapsto a(t, u, v) \text{ is measurable.} \quad (2.6)$$

These assumptions imply that there exists a unique self-adjoint operator $A(t)$ such that $D(A(t)) \subset \mathcal{H}_1$ and

$$(A(t)u, v) = a(t, u, v), \quad \forall u \in D(A(t)), \quad \forall v \in \mathcal{H}_1. \quad (2.7)$$

With this operator $A(t)$, we consider the following evolution equation on \mathcal{H} :

$$\begin{cases} \partial_t u(t) + A(t)u(t) = f(t) & \text{in } (0, T), \\ u(0) = u_0 \in \mathcal{H}. \end{cases} \quad (2.8)$$

The theorem of J. L. Lions asserts as follows (see [12], [13]).

THEOREM 2.1. *Let $u_0 \in \mathcal{H}$ and $f \in L^2((0, T); \mathcal{H}_1^*)$. Then there exists a unique $u(t)$ having the following properties.*

(1) $u(t) \in C([0, T]; \mathcal{H}) \cap L^2((0, T); \mathcal{H}_1)$.

(2) $u(t)$ is \mathcal{H}_1^* -valued and absolutely continuous on $[0, T]$, $\partial_t u(t) \in L^2((0, T); \mathcal{H}_1^*)$, and $u(t)$ satisfies (2.8).

(3) $u(t)$ satisfies the following (in)equalities :

$$\frac{1}{2} \|u(t)\|^2 + \int_0^t a(s, u(s), u(s)) ds = \frac{1}{2} \|u_0\|^2 + \int_0^t (f(s), u(s)) ds, \quad (2.9)$$

$$\|u(t)\|^2 + \delta \int_0^t \|u(s)\|_1^2 ds \leq \|u_0\|^2 + \frac{1}{\delta} \int_0^t \|f(s)\|_{\mathcal{H}_1^*}^2 ds. \quad (2.10)$$

2.2. Heat equation. We take $\mathcal{H} = L^2((0, 1))$ and $\mathcal{H}_1 = H_0^1((0, 1)) =$ the Sobolev space of order 1 with 0 trace on the boundary $\partial\Omega = \{0, 1\}$. For $u, v \in H_0^1((0, 1))$, we put

$$\begin{aligned} a(t, u, v) &= (\gamma(t, x) \partial_x u, \partial_x v) \\ &= k^2 \int_0^{s(t)} \partial_x u(x) \overline{\partial_x v(x)} dx + \int_{s(t)}^1 \partial_x u(x) \overline{\partial_x v(x)} dx. \end{aligned} \quad (2.11)$$

Then the above assumptions (2.4) \sim (2.6) are satisfied, and the associated $A(t)$ is given by

$$D(A(t)) \ni u \iff \begin{cases} u \in H_0^1((0, 1)) \cap H^2((0, s(t))) \cap H^2((s(t), 1)), \\ \partial_x u(s(t) + 0) = k^2 \partial_x u(s(t) - 0), \end{cases} \quad (2.12)$$

$$A(t)u = \begin{cases} -k^2 \partial_x^2 u, & \text{on } (0, s(t)), \\ -\partial_x^2 u, & \text{on } (s(t), 1). \end{cases} \quad (2.13)$$

In the following sections, we also use the notation $A(t)$ to denote the *formal* differential operator (2.13).

3. Approximate solutions.

3.1. Ansatz. We shall construct an approximate solution of the equation (1.1) for large λ . As can be imagined easily, the first approximation will be

$$v_0(t, x; \lambda) = \begin{cases} h_{\text{tw}}(t, x; \lambda), & s(t) < x < 1, \\ e^{\lambda^2 t + \frac{\lambda}{k}(x-s(t))} e^{s(t)}, & 0 < x < s(t), \end{cases}$$

which satisfies $v_0(t, s(t) + 0; \lambda) = v_0(t, s(t) - 0; \lambda)$. Although the other conditions are not satisfied, this suggests the introduction of the factor $e^{\lambda(x-s(t))}$. In the following, we use the notation $\dot{u} = \partial_t u$, $u' = \partial_x u$. Our ansatz is

$$v(t, x; \lambda) = v_+(t, x; \lambda) \chi_+(t, x) + v_-(t, x; \lambda) \chi_-(t, x), \quad (3.1)$$

$$v_+(t, x; \lambda) = h_{\text{fw}}(t, x; \lambda) + h_{\text{fw}}(t, s(t); \lambda) \left[a_+(t; \lambda) \exp\{(\lambda + \dot{s}(t))(x - s(t))\} + b_+(t; \lambda) \exp\{-(\lambda + \dot{s}(t))(x - s(t))\} \right], \quad (3.2)$$

$$v_-(t, x; \lambda) = h_{\text{fw}}(t, s(t); \lambda) a_-(t, x; \lambda) \exp\left\{ \left(\frac{\lambda + \dot{s}(t)}{k} \right) (x - s(t)) \right\}, \quad (3.3)$$

where $\chi_+(t, x)$, $\chi_-(t, x)$ are the characteristic functions of the sets $\{x; s(t) < x\}$, $\{x; x < s(t)\}$, respectively. Note that $a_+(t; \lambda)$ and $b_+(t; \lambda)$ do not depend on x .

The intuition for this ansatz is as follows. The heat flow h_{fw} given on the boundary $x = 1$ is transmitted and reflected at the inner boundary $x = s(t)$, which gives rise to $a_- \exp(\frac{\lambda + \dot{s}}{k}(x - s))$ and $b_+ \exp(-(\lambda + \dot{s})(x - s))$. This latter is again reflected at the boundary $x = 1$ and produces $a_+ \exp((\lambda + \dot{s})(x - s))$.

It must be subject to the following conditions

$$\begin{cases} v_+(t, 1; \lambda) = h_{\text{fw}}(t, 1; \lambda), \\ v_+(t, s(t); \lambda) = v_-(t, s(t); \lambda), \\ v'_+(t, s(t); \lambda) = k^2 v'_-(t, s(t); \lambda). \end{cases} \quad (3.4)$$

Using the abbreviation

$$a_+ = a_+(t; \lambda), \quad b_+ = b_+(t; \lambda), \quad a_-(x) = a_-(t, x; \lambda), \quad s = s(t),$$

we can rewrite (3.4) as

$$\begin{cases} a_+ e^{(\lambda + \dot{s})(1-s)} + b_+ e^{-(\lambda + \dot{s})(1-s)} = 0, \\ 1 + a_+ + b_+ = a_-(s), \\ \lambda + (\lambda + \dot{s})(a_+ - b_+) = k^2 a'_-(s) + k(\lambda + \dot{s})a_-(s). \end{cases} \quad (3.5)$$

We put

$$\varphi = \varphi(t, x; \lambda) = (\lambda + \dot{s}(t))(x - s(t)), \quad (3.6)$$

$$\varphi_1 = \varphi_1(t; \lambda) = (\lambda + \dot{s}(t))(1 - s(t)). \quad (3.7)$$

By a direct computation, we have for $x > s(t)$

$$\begin{aligned} \frac{\dot{v}_+ - v_+''}{h_{\text{fw}}(t, s; \lambda)} &= \left[-2\lambda \dot{s} a_+ + \dot{a}_+ + (\ddot{s}(x - s) - 2\dot{s}^2) a_+ \right] e^\varphi \\ &\quad + \left[\dot{b}_+ - \ddot{s}(x - s) b_+ \right] e^{-\varphi}, \end{aligned} \quad (3.8)$$

and for $x < s(t)$

$$\begin{aligned} \frac{\dot{v}_- - k^2 v_-''}{h_{\text{fw}}(t, s; \lambda)} &= \left[-\lambda(2k a'_- + (1 + \frac{1}{k}) \dot{s} a_-) + \left(\frac{\ddot{s}(x - s)}{k} - (1 + \frac{1}{k}) \dot{s}^2 \right) a_- \right. \\ &\quad \left. + a_- - 2k \dot{s} a'_- - k^2 a_-'' \right] e^{\varphi/k}. \end{aligned} \quad (3.9)$$

3.2. Construction. By the 1st equation of (3.5), we have

$$a_+ = -b_+ e^{-2\varphi_1}. \quad (3.10)$$

By the 2nd equation of (3.5), we have

$$a_-(s) = 1 + b_+(1 - e^{-2\varphi_1}). \quad (3.11)$$

We take a_- to be the solution of the differential equation

$$a'_- + \frac{1}{2k} \left(1 + \frac{1}{k}\right) \dot{s} a_- = 0, \quad (3.12)$$

satisfying (3.11), i.e.

$$a_-(x) = e^{-\frac{1}{2k}(1+\frac{1}{k})\dot{s}(x-s)} (1 + b_+(1 - e^{-2\varphi_1})). \quad (3.13)$$

Plugging them to the third equation of (3.5), we get

$$b_+ = \frac{1-k}{1+k} (1 + e^{-2\varphi_1})^{-1}. \quad (3.14)$$

Note that $\sup s(t) < 1$ by our assumption. Hence $e^{-2\varphi_1}$ is exponentially decreasing in λ . Therefore, we have

$$b_+ = \frac{1-k}{1+k} + O(e^{-2\varphi_1}), \quad (3.15)$$

$$a_+ = \frac{k-1}{k+1} e^{-2\varphi_1} (1 + O(e^{-2\varphi_1})), \quad (3.16)$$

$$a_-(x) = \frac{2}{1+k} e^{-\frac{1}{2k}(1+\frac{1}{k})\dot{s}(x-s)} + O(e^{-2\varphi_1}), \quad (3.17)$$

and these expansions can be differentiated term by term. By (3.8) and (3.10), we have

$$\frac{\dot{v}_+ - v_+''}{h_{\text{rw}}(t, s; \lambda)} = \left(\ddot{s}(2-s-x)b_+ - \dot{b}_+ \right) e^{\varphi-2\varphi_1} + \left(\dot{b}_+ - \ddot{s}(x-s)b_+ \right) e^{-\varphi}. \quad (3.18)$$

In view of (3.18) and (3.9), we then have

$$|\dot{v}_+ - v_+''| \leq C h_{\text{rw}}(t, s; \lambda) e^{-\varphi}, \quad s < x < 1, \quad (3.19)$$

$$|\dot{v}_- - k^2 v_-''| \leq C h_{\text{rw}}(t, s; \lambda) e^{\varphi/k}, \quad 0 < x < s, \quad (3.20)$$

where the constant C is independent of $0 < t < T$ and $\lambda > 0$.

The above $v(t, x; \lambda)$ does not satisfy $v(t, 0; \lambda) = h_{\text{rw}}(t, 0; \lambda)$. We modify it in the following way: Let

$$s_0 = \inf_{0 < t < T} s(t). \quad (3.21)$$

By our assumption, $0 < s_0 < 1$. Pick $\chi(x) \in C^\infty(\mathbf{R})$ such that

$$\chi(x) = \begin{cases} 0, & x < s_0/4, \\ 1, & s_0/2 < x, \end{cases} \quad (3.22)$$

and put

$$w(t, x; \lambda) = v_+(t, x; \lambda) + \chi(x)v_-(t, x; \lambda). \quad (3.23)$$

LEMMA 3.1. *Let $w(t, x; \lambda)$ be defined by (3.23). Then w satisfies*

$$w(t, 1; \lambda) = h_{\text{rw}}(t, 1; \lambda), \quad w(t, 0; \lambda) = 0. \quad (3.24)$$

Moreover, we have

$$|\dot{w} - A(t)w| \leq Ce^{\lambda^2 t + \lambda s(t)} \begin{cases} e^{-\varphi}, & s(t) < x < 1, \\ e^{\varphi/k} + e^{-\delta_0 \lambda}, & 0 < x < s(t), \end{cases} \quad (3.25)$$

where $C, \delta_0 > 0$ are constants independent of $t, \lambda > 0$.

Proof. (3.24) follows directly from (3.23). Since

$$\dot{w} - A(t)w = \chi(x)(\dot{v} - A(t)v) + 2\chi'(x)v'_- + \chi''(x)v_-,$$

and $x < s_0/2$ on the support of $\chi'(x)$, we obtain (3.25). \square

4. Proof of the main theorem.

4.1. Estimates for boundary integrals. Let us first prepare an elementary lemma.

LEMMA 4.1. *Let $0 < \delta < 1$ and $I = [1 - \delta, 1]$. Suppose that $u(x) \in C^1(I)$ satisfies $u(1) = 0$. Then for any nonnegative function $a(x) \in C(I)$, we have*

$$\liminf_{\epsilon \rightarrow 0} \int_{1-\delta}^1 a(x) |u'(x)|^2 \frac{\epsilon}{(\epsilon + |u(x)|^2)^{3/2}} dx \geq a(1) |u'(1)|.$$

Proof. We have only to consider the case $u'(1) \neq 0$. Shrinking I if necessary, we can assume that $u(x) > 0$ on $[1 - \delta, 1)$ and adopt $u = u(x)$ as a new variable. Letting $c = u(1 - \delta)$ and $\tilde{a}(u) = a(x(u))$, we have

$$\int_{1-\delta}^1 a(x) \left| \frac{du}{dx} \right|^2 \frac{\epsilon}{(\epsilon + u^2)^{3/2}} dx = \int_0^c \tilde{a}(u) \left| \frac{dx}{du} \right|^{-1} \frac{\epsilon}{(\epsilon + u^2)^{3/2}} du.$$

By the change of variable $u = \sqrt{\epsilon}y$, this is equal to

$$\int_0^{c/\sqrt{\epsilon}} \tilde{a}(\sqrt{\epsilon}y) \left| \frac{dx}{du}(\sqrt{\epsilon}y) \right|^{-1} \frac{dy}{(1 + y^2)^{3/2}},$$

which tends to

$$\tilde{a}(0) \left| \frac{dx}{du}(0) \right|^{-1} \int_0^\infty \frac{dy}{(1 + y^2)^{3/2}} = a(1) \left| \frac{du}{dx}(1) \right|. \quad \square$$

We put

$$\begin{aligned}\mathcal{D}_- &= \{(t, x); 0 \leq t \leq T, 0 \leq x \leq s(t)\}, \\ \mathcal{D}_+ &= \{(t, x); 0 \leq t \leq T, s(t) \leq x \leq 1\},\end{aligned}\tag{4.1}$$

and also

$$[f]_{s(t)} = f(s(t) + 0) - f(s(t) - 0).\tag{4.2}$$

LEMMA 4.2. *Let $U = U(t, x)$ be a solution in the sense of Theorem 2.1 with $\mathcal{H}_1 = H_0^1((0, 1))$ to the equation*

$$\dot{U} + A(t)U = F \quad \text{in } (0, T) \times \Omega\tag{4.3}$$

satisfying

$$\begin{cases} U(s(t) + 0) = U(s(t) - 0), & 0 < t < T, \\ U'(s(t) + 0) = k^2 U'(s(t) - 0), & 0 < t < T, \\ U(t, 0) = U(t, 1) = 0, & 0 < t < T, \end{cases}\tag{4.4}$$

where $F = F(t, x) \in L^2((0, T) \times \Omega)$. Let $H = H(t, x) \in C([0, T] \times \bar{\Omega})$ be such that $H(t, x) \geq 0$ on $[0, T] \times \Omega$, and $H \in C^2(\mathcal{D}_+) \cap C^2(\mathcal{D}_-)$. We assume that there exists a constant K such that

$$\dot{H} + \gamma(t, x)H'' \leq -KH \quad \text{on } \mathcal{D}_- \cup \mathcal{D}_+.\tag{4.5}$$

We put

$$E(U, H; t) = \int_{\Omega} |U(t, x)|H(t, x) dx.\tag{4.6}$$

Then we have

$$\begin{aligned}E(U, H; T) &\leq e^{-KT}E(U, H; 0) + \iint_{(0, T) \times \Omega} e^{K(t-T)} |F(t, x)|H(t, x) dt dx \\ &\quad + \int_0^T e^{K(t-T)} |U(t, s(t))| [\gamma(t, \cdot)H'(t, \cdot)]_{s(t)} dt \\ &\quad - \int_0^T e^{K(t-T)} |U'(t, 1)|H(t, 1) dt.\end{aligned}\tag{4.7}$$

Proof. Let $\chi_\epsilon(x) = x(\epsilon + x^2)^{-1/2}$ for $\epsilon > 0$, and note the following properties :

$$x\chi_\epsilon(x) \rightarrow |x|, \quad \epsilon \rightarrow 0,\tag{4.8}$$

$$\chi'_\epsilon(x) > 0, \quad |x\chi'_\epsilon(x)| \leq 1/2, \quad x\chi'_\epsilon(x) \rightarrow 0, \quad \epsilon \rightarrow 0.\tag{4.9}$$

In fact, (4.9) follows from $\chi'_\epsilon(x) = \epsilon(\epsilon + x^2)^{-3/2}$, and the fact that, putting $x = \sqrt{\epsilon}y$, $|x\chi'_\epsilon(x)| = |y(1 + y^2)^{-3/2}| \leq 1/2$.

Integration by parts using (4.4) yields

$$\begin{aligned}&\int_0^1 (A(t)U) \chi_\epsilon(U) H dx - \int_0^1 U \chi_\epsilon(U) (A(t)H) dx \\ &= \int_0^1 \gamma |U'|^2 \chi'_\epsilon(U) H dx - U(s) \chi_\epsilon(U(s)) [\gamma H']_s - \int_0^1 \gamma U \chi'_\epsilon(U) U' H' dx.\end{aligned}\tag{4.10}$$

where $U = U(x) = U(t, x)$ and $H = H(t, x)$, and we have used $\chi_\epsilon(U(0)) = \chi_\epsilon(U(1)) = 0$, since $\chi_\epsilon(0) = 0$. We put

$$E_\epsilon(t) = \int_0^1 U(t, x) \chi_\epsilon(U(x, t)) H(t, x) dx.$$

Then we have

$$\begin{aligned} \dot{E}_\epsilon(t) &= - \int_0^1 (A(t)U) \chi_\epsilon(u) H dx + \int_0^1 U \chi_\epsilon(U) (A(t)H) dx \\ &\quad + \int_0^1 F \chi_\epsilon(U) H dx + \int_0^1 U \chi_\epsilon(U) (\dot{H} - A(t)H) dx \\ &\quad + \int_0^1 U \chi'_\epsilon(U) \dot{U} H dx. \end{aligned}$$

Plugging this with (4.10), we have

$$\begin{aligned} \dot{E}_\epsilon(t) &= - \int_0^1 \gamma |U'|^2 \chi'_\epsilon(U) H dx + U(s) \chi_\epsilon(U(s)) [\gamma H']_s \\ &\quad + \int_0^1 \gamma U \chi'_\epsilon(U) U' H' dx + \int_0^1 F \chi_\epsilon(U) H dx \\ &\quad + \int_0^1 U \chi_\epsilon(U) (\dot{H} - A(t)H) dx + \int_0^1 U \chi'_\epsilon(U) \dot{U} H dx. \end{aligned}$$

By (4.9), the integrals containing the term $U \chi'_\epsilon(U)$ vanish as $\epsilon \rightarrow 0$. Using (4.5), we then have

$$\begin{aligned} \dot{E}_\epsilon(t) + K E_\epsilon(t) &+ \int_0^1 \gamma |U'|^2 \chi'_\epsilon(U) H dx \\ &\leq U(s) \chi_\epsilon(U(s)) [\gamma H']_s + \int_0^1 F \chi_\epsilon(U) H dx + o(1). \end{aligned}$$

Integrating this inequality, we have

$$\begin{aligned} E_\epsilon(T) &+ \int_0^T \int_0^1 e^{K(t-T)} \gamma |U'|^2 \chi'_\epsilon(U) H dt dx \\ &\leq e^{-KT} E_\epsilon(0) + \int_0^T e^{K(t-T)} U(t, s(t)) \chi_\epsilon(U(t, s(t))) [\gamma H']_{s(t)} dt \\ &\quad + \int_0^T \int_0^1 e^{K(t-T)} F \chi_\epsilon(U) H dt dx + o(1). \end{aligned} \quad (4.11)$$

By Lemma 4.1, we have

$$H(t, 1) |U'(t, 1)| \leq \liminf_{\epsilon \rightarrow 0} \int_0^1 \gamma |U'(t, x)|^2 \chi'_\epsilon(U(t, x)) H dx.$$

Taking the inferior limit in (4.11) and noting (4.8), we obtain the lemma. \square

LEMMA 4.3. *Let U be as in Lemma 4.2, and take*

$$H_1(t, x) = e^{-\lambda^2 t + \theta_1(t, x)}, \quad (4.12)$$

$$\theta_1(t, x) = \frac{\lambda - \nu}{\sqrt{\gamma(x, t)}} (x - s(t)) + \lambda s(t) + \nu(1 - s(t)), \quad (4.13)$$

$$\nu > \sup_{0 < t < T} \left| \left(1 - \frac{1}{k}\right) \dot{s}(t) \right|. \quad (4.14)$$

Then, there exists a constant $C > 0$ such that if $\lambda \geq \max\left(\nu, \sup_{0 < t < T} |\dot{s}(t)|\right)$

$$\begin{aligned} & \int_0^T e^{\lambda \nu t} |U'(t, 1)| h_{\text{bw}}(t, 1; \lambda) dt \\ & \leq C \left(\int_0^1 |U(0, x)| H_1(0, x) dx + \iint_{(0, T) \times \Omega} e^{\lambda \nu t} |F(t, x)| H_1(t, x) dt dx \right). \end{aligned}$$

Proof. We consider the case $k > 1$. We have for $\lambda \geq \nu$

$$[\gamma H_1']_{s(t)} = (1 - k)(\lambda - \nu) H_1(t, s(t)) \leq 0. \quad (4.15)$$

Moreover,

$$\frac{\dot{H}_1 + \gamma H_1''}{H_1} = -\lambda \nu + (\nu - \lambda) \left(\nu - \left(1 - \frac{1}{\sqrt{\gamma}}\right) \dot{s} \right).$$

Therefore by (4.14),

$$\dot{H}_1 + \gamma H_1'' \leq -\lambda \nu H_1. \quad (4.16)$$

Note that this also holds for the case $0 < k < 1$. We can then apply Lemma 4.2 with $K = \lambda \nu$ to obtain

$$\begin{aligned} & E(U, H_1; T) + \int_0^T e^{\lambda \nu(t-T)} |U'(t, 1)| H_1(t, 1) dt \\ & \leq e^{-\lambda \nu T} E(U, H_1; 0) + \iint_{(0, T) \times \Omega} e^{\lambda \nu(t-T)} |F(t, x)| H_1(t, x) dt dx. \end{aligned}$$

Omitting the first term of the left-hand side, and noting that $H_1(t, 1) = h_{\text{bw}}(t, 1; \lambda)$, we have proven the lemma.

Let us next consider the case $0 < k < 1$. In this case

$$[\gamma H_1']_{s(t)} = (1 - k)(\lambda - \nu) H_1(t, s(t)) \geq 0.$$

Then we have by (4.7)

$$\begin{aligned} & \int_0^T e^{\lambda \nu t} |U'(t, 1)| H_1(t, 1) dt \\ & \leq E(U, H_1; 0) + \iint_{(0, T) \times \Omega} e^{\lambda \nu t} |F(t, x)| H_1(t, x) dt dx \\ & + (1 - k)(\lambda - \nu) \int_0^T e^{\lambda \nu t} |U(t, s(t))| H_1(t, s(t)) dt. \end{aligned} \quad (4.17)$$

To estimate the last term, we put

$$H_2(t, x) = e^{-\lambda^2 t + \theta_2(t, x)}, \quad (4.18)$$

$$\theta_2(t, x) = \begin{cases} \lambda s(t) + \nu(1 - s(t)), & x > s(t), \\ \frac{\lambda - \nu}{k}(x - s(t)) + \lambda s(t) + \nu(1 - s(t)), & x < s(t). \end{cases} \quad (4.19)$$

Then

$$[\gamma H_2']_{s(t)} = -k(\lambda - \nu)H_2(t, s(t)) < 0. \quad (4.20)$$

We also have

$$\frac{\dot{H}_2 + \gamma H_2''}{H_2} = \begin{cases} -\lambda\nu - (\lambda - \nu)(\lambda - \dot{s}(t)), & x > s(t), \\ -\lambda\nu - (\lambda - \nu)\left(\nu + \left(\frac{1}{k} - 1\right)\dot{s}(t)\right), & x < s(t). \end{cases}$$

Hence we have, by the assumption,

$$\dot{H}_2 + \gamma H_2'' \leq -\lambda\nu H_2.$$

We can then apply Lemma 4.2 to have

$$\begin{aligned} & k(\lambda - \nu) \int_0^T e^{\lambda\nu t} |U(t, s(t))| H_2(t, s(t)) dt \\ & \leq E(U, H_2; 0) + \iint_{(0, T) \times \Omega} e^{\lambda\nu t} |F(t, x)| H_2(t, x) dt dx. \end{aligned} \quad (4.21)$$

Since $H_1(t, s(t)) = H_2(t, s(t))$, we have using (4.17)

$$\begin{aligned} & \int_0^T e^{\lambda\nu t} |U'(t, 1)| H_1(t, 1) dt \\ & \leq E(U, H_1; 0) + \iint_{(0, T) \times \Omega} e^{\lambda\nu t} |F(t, x)| H_1(t, x) dt dx \\ & + \frac{1-k}{k} \left(E(U, H_2; 0) + \iint_{(0, T) \times \Omega} e^{\lambda\nu t} |F(t, x)| H_2(t, x) dt dx \right). \end{aligned} \quad (4.22)$$

Since $H_2(t, x) \leq H_1(t, x)$, we have completed the proof. \square

4.2. Proof of Theorem 1.1. Using $w(t, x; \lambda)$ from (3.23), we put

$$U(t, x) = u(t, x; \lambda) - w(t, x; \lambda). \quad (4.23)$$

Then we have using (3.15), (3.16), (3.17), $|w(0, x; \lambda)| \leq C e^{\lambda s(0)}$, hence

$$|U(0, x)| \leq |u_0(x)| + C e^{\lambda s(0)},$$

which implies

$$|U(0, x)| H_1(0, x) \leq (|u_0(x)| + C e^\lambda) e^{2\lambda}, \quad (4.24)$$

where we have used (4.13). Letting $F(t, x) = \dot{U} + A(t)U$, and in view of (3.25), we also have

$$|F(t, x)|H_1(t, x) \leq Ce^{2\lambda s(t)}, \quad (4.25)$$

$$|F(t, x)|e^{-\lambda^2 t} \leq Ce^{\lambda s(t)}. \quad (4.26)$$

Then in view of Lemma 4.3, we have

$$\begin{aligned} & \int_0^T e^{\lambda\nu t} |U'(t, 1)| h_{\text{bw}}(t, 1; \lambda) dt \\ & \leq C \left(\lambda e^{3\lambda} + \int_0^T e^{\lambda\nu t + 2\lambda s(t)} dt + \lambda \int_0^T e^{\lambda\nu t + \lambda s(t)} dt \right). \end{aligned}$$

If $\nu > 2 \sup_{0 < t < T} |\dot{s}(t)|$, we can make the change of variable $y = t + 2s(t)/\nu$ to see that

$$\begin{aligned} \int_0^T e^{\lambda\nu t + 2\lambda s(t)} dt &= \int_{2s(0)/\nu}^{T+2s(T)/\nu} e^{\lambda\nu y} \frac{dy}{1 + 2\dot{s}(t)/\nu} \\ &\leq C \int_{2s(0)/\nu}^{T+2s(T)/\nu} e^{\lambda\nu y} dy \leq C \frac{e^{\lambda\nu T + 2\lambda s(T)}}{\lambda\nu}, \end{aligned}$$

and similarly

$$\int_0^T e^{\lambda\nu t + \lambda s(t)} dt \leq C \frac{e^{\lambda\nu T + \lambda s(T)}}{\lambda\nu}.$$

This yields

$$\int_0^T e^{\lambda\nu t} h_{\text{bw}}(t, 1; \lambda) |U(t, s(t))| dt \leq \frac{C}{\lambda\nu} e^{\lambda\nu T + 2\lambda s(T)}. \quad (4.27)$$

On the other hand, by using (3.2), (3.15) and (3.16), we have

$$\begin{aligned} & h_{\text{bw}}(t, 1; \lambda) \partial_x (w(t, x; \lambda) - h_{\text{fw}}(t, x; \lambda)) \Big|_{x=1} \\ & \simeq \frac{2\lambda(k-1)}{k+1} e^{2\lambda s(t) - \dot{s}(t)(1-s(t))}. \end{aligned}$$

By integration by parts, we then have

$$\begin{aligned} & \int_0^T e^{\lambda\nu t} h_{\text{bw}}(t, 1; \lambda) \partial_x (w(t, x; \lambda) - h_{\text{fw}}(t, x; \lambda)) \Big|_{x=1} \\ & \simeq \frac{2(k-1)}{(\nu + 2\dot{s}(t))(k+1)} e^{\lambda\nu T + 2\lambda s(T) - \dot{s}(T)(1-s(T))}. \end{aligned} \quad (4.28)$$

This proves Theorem 1.1. \square

4.3. Proof of Corollary 1.2. By Theorem 1.1, we have for any $0 < t < T$

$$\frac{\log I_{\text{ind}}(t; \lambda)}{\lambda} - \nu t \rightarrow 2s(t), \quad \lambda \rightarrow \infty.$$

The knowledge of $I_{\text{ind}}(t; \lambda)$ then gives $s(t)$ and $\dot{s}(t)$. Again looking at the behavior of $I_{\text{ind}}(t; \lambda)$, we recover k . \square

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