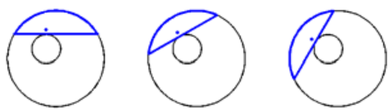
Solution of the inverse problem of cardiology using Faddeev's Green function



Samuli.Siltanen@iki.fi
Gunma University
Joint work with Masaru Ikehata

Finnish Inverse Days
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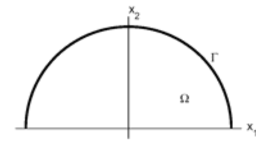
Geometry of our Cauchy problem is half-disc



$$(-\Delta + V)u = 0 \qquad \text{in } \Omega,$$

$$u \in H^2(\Omega), \quad V(x) \in L^\infty(\Omega)$$

Problem: Recover u from its Cauchy data

$$(u|_\Gamma, \frac{\partial u}{\partial \nu}|_\Gamma)$$

Applications of the Cauchy problem include

Recovering temperature distribution inside a physical body from surface temperature and heat flux

Recovering voltage potential on the heart from voltage measurements on the skin

Several numerical solution methods for the Cauchy problem have been presented

Klibanov and Santosa 1991
(based on Lavrentyev 1956, Lattés and Lions 1969)
Kabanikhin and Karchevsky 1995
Leitão 2000 (based on Maz'ya 1991)
Háo and Lesnic 2000 (V=0)
Berntsson and Eldén 2001 (V=0)
Cheng, Hon, Wei and Yamamoto 2001 (V=0)

We present a new non-iterative solution method for V≠0 that does not involve solution of boundary value problems.

We solve the Cauchy problem using Faddeev's Green function

Ikehata [2001] proved that

$$u(y) = \lim_{\tau \to \infty} u_\tau(y)$$

where

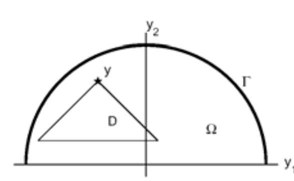
$$u_\tau(y) := \frac{2\tau^2 e^{-i\tau y_1}}{C_D} \int_\Gamma \left(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u\right) d\sigma(x)$$

Computation with finite τ provides a regularised reconstruction method.

The reconstruction method uses Faddeev's exponentially growing solutions

$$-\Delta v_\tau'' + \tilde{V} v_\tau'' = \chi_D e^{\tau(x_2 - y_2)} e^{i\tau x_1} \text{ in } \mathbb{R}^2$$

$$v_\tau = v_\tau''|_\Omega$$



Proof of the method is based on Green's formula

$$\begin{aligned}\int_D e^{\tau(x_2-y_2)} e^{i\tau x_1} u(x)dx &= -\int_\Omega (\Delta v_\tau)u + \int_\Omega V v_\tau u \\ &= \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u\right) \\ &= \left(\int_\Gamma + \int_{\partial\Omega\setminus\Gamma}\right)\left(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u\right)\end{aligned}$$

$$\int_D e^{\tau(x_2-y_2)} e^{i\tau x_1} u(x)dx \sim \frac{C_D}{2\tau^2} e^{i\tau y_1} u(y) \quad \text{as } \tau \to \infty$$

$$\int_{\partial\Omega\setminus\Gamma} \left(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u\right) \to 0 \quad \text{exponentially as } \tau \to \infty$$

Numerical implementation of the method is divided into 5 problems

$$u_\tau(y) = \frac{2\tau^2 e^{-i\tau y_1}}{C_D} \int_\Gamma \left(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u\right) d\sigma(x)$$

1. Integration on Γ
2. Choosing the triangle D=D(y)
3. How to choose τ?
4. Computing exponentially growing solutions v_τ
5. Computing normal derivatives of v_τ

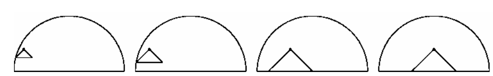
Implementation Step 1: integration on Γ

We choose a set of integration quadrature points and weights on Γ. Then the integral of *f(x)* over Γ is approximated by the following sum:

$$\int_\Gamma f d\sigma \approx \sum_{k=1}^{K} w^{(k)} f(x^{(k)})$$

Implementation Step 2: choosing the triangle D

We take D to be the largest possible triangular patch such that D belongs to Ω and the base of D is twice its height.



The choice is based on theory: large $|C_D|$ minimizes error.

We always have $|C_D|<=2$, and with this choice we have

$$C_D = 2$$

Implementation Step 3: Choosing the regularization parameter τ

If τ is too small, then recovered solution is not close to the true solution

If τ is too large, then noise will be amplified

Choice of τ depends on the Cauchy data, a priori bound on u in Ω and noise level; we do not have a general practical choice

Implementation Step 4: Computing exponentially growing solutions

Define

$$v_\tau(x_1, x_2) = e^{\tau(x_2-y_2)} e^{i\tau x_1} w'_\tau(x_1, x_2)$$

Case V=0: numerically evaluate convolution

$$w'_\tau(x) = g_\tau * \chi_D$$

Piecewise smooth V: solve the integral equation

$$w'_\tau(x) + g_\tau * (\tilde{V} w'_\tau) = g_\tau * \chi_D$$

Exponentially growing solutions are defined using Faddeev's fundamental solution

In the Lippmann-Schwinger –type equation

$$w'_\tau(x) + g_\tau * (\tilde{V} w'_\tau) = g_\tau * \chi_D$$

the function

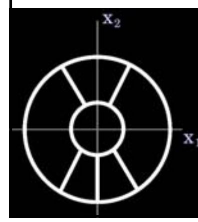
$$g_\tau(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\tau(\xi_1 - i\xi_2)} d\xi$$

satisfies

$$(-\Delta - 2i\tau(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}))g_\tau(x) = \delta(x)$$

Computation of Faddeev's fundamental solution g_1 is divided into 7 cases

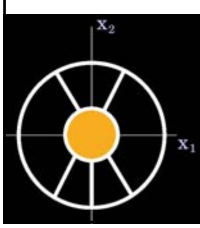
$$g_\tau(x) = g_1(\tau x)$$



The x-plane is divided into 7 disjoint regions, each leading to a different algorithm

Case 1: For x in the unit disc we compute $g_1(x)$ with a formula by [Boiti et al 1987]

$$g_1(x) = -\frac{e^{-ix}}{4\pi}(2\gamma + \log|x|^2 + \sum_{n=1}^{\infty}\frac{(ix)^n + (-i\bar{x})^n}{n n!})$$



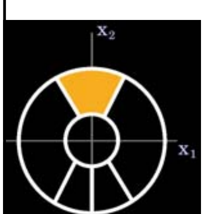
Here γ is Euler's constant

Case 2: We write $g_1(x)$ as a formula containing a rapidly converging one-dimensional integral

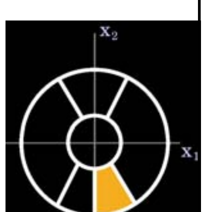
$$g_1(x) = \frac{e^{-ix_1}}{2\pi}\mathrm{Re}[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \frac{(N+1)!e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}}dt]$$

Cases 3 and 4: The integral in case 2 is modified using the residue theorem

$$-i\int_0^\infty \frac{e^{-x_2 s + i x_1 s}}{(-is-i)^{N+2}}ds$$

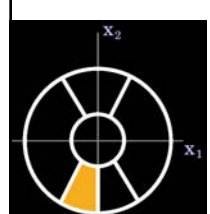


$$(1+i)\int_0^\infty \frac{e^{-is(x_2+x_1)+s(x_2-x_1)}}{(s+is-i)^{N+2}}ds$$



Cases 5 and 6 are reduced to cases 4 and 2 using the following symmetry:

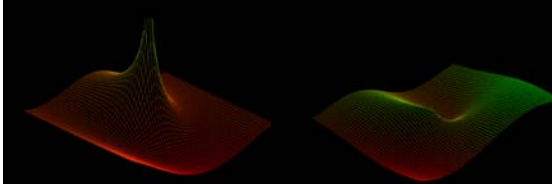
$$g_1(-x_1, x_2) = \overline{g_1(x_1, x_2)}$$



Case 7: for |x|>25 we ignore the integral in case 2 and use the truncated sum

$$g_1(x) \approx \frac{e^{-ix_1}}{2\pi} \mathrm{Re}\left[-e^{ix_1} \sum_{j=0}^{N} \frac{j!}{(ix)^{j+1}}\right]$$

We can now evaluate g_1



Following Vainikko, we solve the LS equation in a bounded domain using periodization

In the x-plane we have the LS equation

$$w'_\tau(x) = g_\tau * \chi_D - \int_{\mathrm{supp}(V)} g_\tau(x-y)V(y)w'_\tau(y)\,dy$$

We take a square S and solve the S-periodic equation

$$[I + g_\tau * (V\cdot\ )]w_\tau = (g_\tau * \chi_D)|_S =: f_\tau$$

The LS equation is also solved since it can be shown that

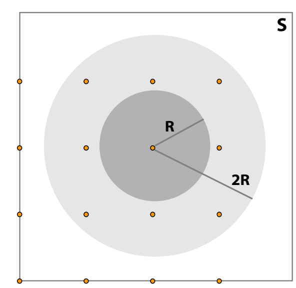
$$w'_\tau|_{\mathrm{supp}(V)} = w_\tau|_{\mathrm{supp}(V)}$$

We define a grid in the square S

The grid has (2^m x 2^m) points

Here m=2, in practice typically m=8.

This grid is suitable for the use of Fast Fourier Transform (FFT).



Vainikko's method is based on iterative solution of linear equations

We can solve the discretized equation

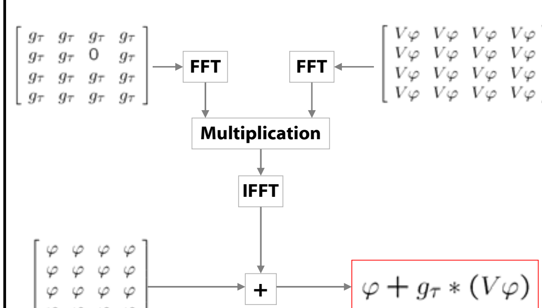
$$[I + g_\tau * (V\cdot\ )]w_\tau = f_\tau$$

using the iterative GMRES method.
We just need to implement the linear operator

$$\varphi + g_\tau * (V\varphi)$$

for a function ϕ given on the grid points.

The linear operator is implemented using Fast Fourier Transform (FFT)



Implementation Step 5: normal derivatives of exponentially growing solutions

Piecewise smooth V:

$$\frac{\partial v_\tau}{\partial \nu} = \frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi} \times \left[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * (\tilde{V} w'_\tau - \chi_D)\right]$$

Case V=0:

$$\frac{\partial v_\tau}{\partial \nu} = -\frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi} \times \left[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * \chi_D\right]$$

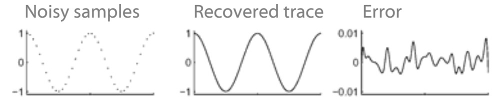
Example 1: The harmonic case V=0

Choose harmonic function u:

$$u(y_1, y_2) = \mathrm{Re}(z^4), \qquad z = y_1 + iy_2$$

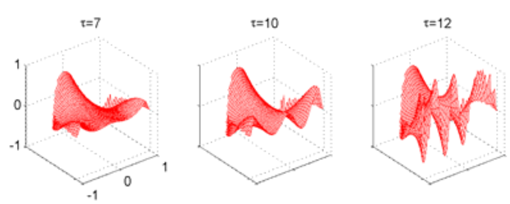
Produce computer simulated noisy Cauchy data:

Noisy samples Recovered trace Error



Standard deviation of Gaussian noise = 0.003

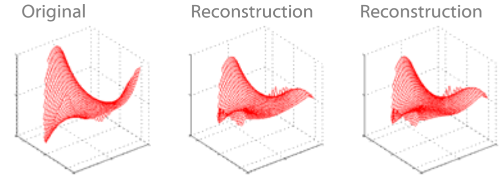
Example 1: Solution with noisy Cauchy data



Small τ gives good reconstruction deep inside Ω,
large τ gives good reconstruction near Γ

Solution with two noise levels illustrates the stability of our method

Original Reconstruction Reconstruction



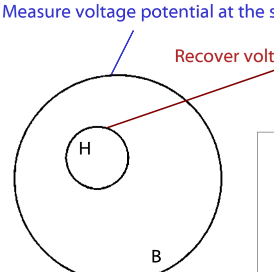
Noise = 0.003 Noise = 0.006

Here τ is chosen as function of y

We study the inverse potential problem of electrocardiography (ECG)

Measure voltage potential at the skin

Recover voltage potential at the heart

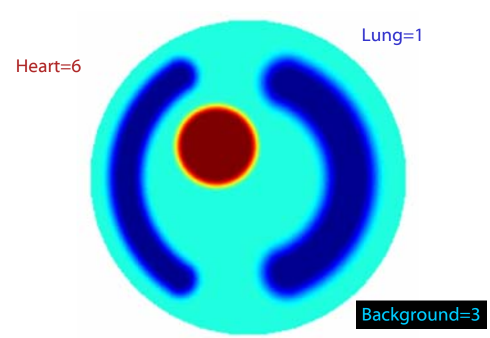


Conductivity equation:

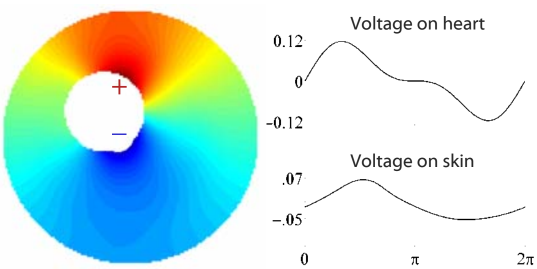
$$\nabla \cdot \gamma \nabla \tilde{u} = 0 \text{ in } B \setminus \overline{H}$$

$$\tilde{u}|_{\partial H} = f, \qquad \frac{\partial \tilde{u}}{\partial \nu}|_{\partial B} = 0$$

We construct a conductivity modelling a cross section of human chest



We compute voltage potential outside heart by Finite Element Method



The conductivity equation is transformed to the Schrödinger equation

Conductivity equation:

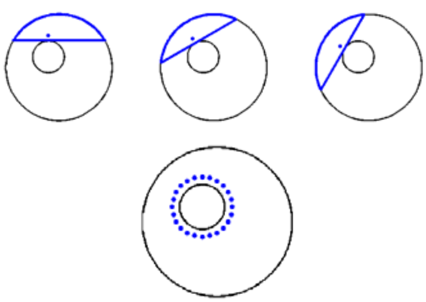
$$\nabla\cdot\gamma\nabla\tilde{u} = 0 \text{ in } B\backslash\overline{H}, \quad \tilde{u}|_{\partial H} = f, \quad \frac{\partial\tilde{u}}{\partial\nu}|_{\partial B} = 0$$

Define $u = \gamma^{1/2}\tilde{u}, \quad V(x) = \frac{\Delta\sqrt{\gamma(x)}}{\sqrt{\gamma(x)}}$

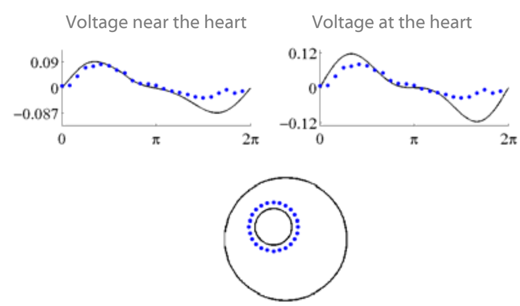
Then u satisfies the equation

$$(-\Delta+V)u = 0 \text{ in } B\backslash\overline{H}, \quad u|_{\partial B} = \sqrt{3}\tilde{u}|_{\partial B}, \quad \frac{\partial u}{\partial\nu}|_{\partial B} = 0$$

We recover voltage near the heart by rotating the canonical geometry



We reconstruct voltage near the heart from simulated ECG data with 2% noise



We presented a new numerical solution method for the Cauchy problem

Our method is fast (no direct problem solution)

In the inverse problem of cardiology,
our method has 20% average relative error
on the important surface of the heart

Future work: 3D problems