Numerical solution of the Cauchy problem with Faddeev's Green function

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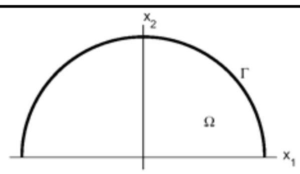


$$(-\Delta + V)u = 0 \qquad \text{in } \Omega,$$

$$u \in H^2(\Omega), \quad V(x) \in L^\infty(\Omega)$$

Cauchy problem: recover u from

$$(u|_\Gamma, \frac{\partial u}{\partial \nu}|_\Gamma)$$

Applications

- Recovering temperature distribution from surface temperature and heat flux
- Electrical impedance tomography
- Inverse scattering
- Analytic continuation

Earlier numerical work

- Klibanov and Santosa 1991 (based on Lattés and Lions 1969, Lavrentyev 1956)
- Kabanikhin and Karchevsky 1995
- Leitão 2000 (based on Maz'ya 1991)
- Háo and Lesnic 2000
- Berntsson and Eldén 2001
- Cheng, Hon and Yamamoto 2001

Solution with Faddeev's Green function

Ikehata [2001] proved that

$$u(y) = \lim_{\tau \to \infty} u_\tau(y)$$

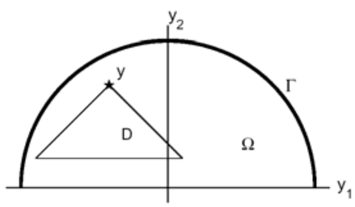
$$u_\tau(y) := \frac{2\tau^2 e^{-i\tau y_1}}{C_D} \int_\Gamma (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u) d\sigma(x)$$

Computation with suitable finite τ provides a regularised reconstruction method.

Exponentially growing solutions:

$$-\Delta v''_\tau + \tilde{V} v''_\tau = \chi_D e^{\tau(x_2-y_2)} e^{i\tau x_1} \text{ in } \mathbb{R}^2$$

$$v_\tau = v''_\tau|_\Omega$$



$$\int_D e^{\tau(x_2-y_2)} e^{i\tau x_1} u(x)dx = -\int_\Omega (\Delta v_\tau)u + \int_\Omega V v_\tau u$$

Green's formula $$= \int_{\partial\Omega} (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u)$$

$$= (\int_\Gamma + \int_{\partial\Omega\setminus\Gamma})(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u)$$

$$\int_D e^{\tau(x_2-y_2)} e^{i\tau x_1} u(x)dx \sim \frac{C_D}{2\tau^2} e^{i\tau y_1} u(y) \quad \text{as } \tau \to \infty$$

$$\int_{\partial\Omega\setminus\Gamma} (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u) \to 0 \quad \text{exponentially as } \tau \to \infty$$

Numerical implementation

$$u_\tau(y) = \frac{2\tau^2 e^{-i\tau y_1}}{C_D} \int_\Gamma (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u) d\sigma(x)$$

A. Integration on Γ
B. Choosing the triangle D=D(y)
C. Computing exponentially growing solutions
D. Computing normal derivatives of exponentially growing solutions
E. How to choose τ?

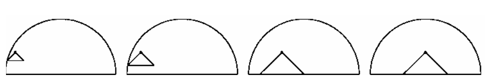
A. Integration on Γ

We choose a set of integration quadrature points and weights on Γ. Then any integral over Γ is approximated by the following sum:

$$\int_\Gamma f d\sigma \approx \sum_{k=1}^{K} w^{(k)} f(x^{(k)})$$

B. Choosing the triangle D

We take D to be the largest possible triangular patch such that D is a subset of Ω and the base of D is twice its height.



The choice is based on theoretical observations leading to the requirement "maximize $|C_D|$".

Note that always $|C_D|$<=2.

With the above choice we have $$C_D = 2$$

C. Computing exponentially growing solutions

•Piecewise smooth V: solve

$$w'_\tau(x) + g_\tau * (\tilde{V} w'_\tau) = g_\tau * \chi_D$$

with an adaptation of Vainikko's Lippmann-Schwinger equation solver [Mueller-S 2003]

•Case V=0: evaluate convolution

$$w'_\tau(x) = g_\tau * \chi_D$$

Faddeev's fundamental solution

The function

$$g_\tau(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\tau(\xi_1 - i\xi_2)} d\xi$$

satisfies

$$(-\Delta - 2i\tau(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}))g_\tau(x) = \delta(x)$$

Note the symmetry

$$g_\tau(x) = g_1(\tau x)$$

Faddeev's Green function

$$G_\zeta(x) = \frac{e^{i\zeta\cdot x}}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\zeta\cdot\xi} d\xi, \quad \zeta \in \mathbb{C}^2\backslash 0, \zeta\cdot\zeta = 0$$

$$-\Delta G_\zeta = \delta$$

$$G_{(\tau,-i\tau)}(x) = e^{i\tau x_1 - \tau x_2} g_\tau(x)$$

D. Computing normal derivatives of exponentially growing solutions

- Piecewise smooth V:

$$\frac{\partial v_\tau}{\partial \nu} = \frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi}\Big[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * (\tilde{V} w'_\tau - \chi_D)\Big]$$

- Case V=0:

$$\frac{\partial v_\tau}{\partial \nu} = -\frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi}\Big[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * \chi_D\Big]$$

Derivatives of Faddeev's fundamental solution

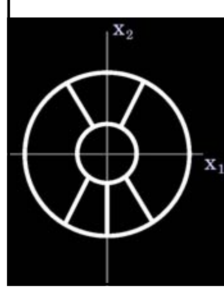
$$\frac{\partial g_\tau}{\partial x_1}(x) = -\frac{1}{4\pi\bar{x}} - \frac{e^{-i\tau(x+\bar{x})}}{4\pi x} - i\tau g_\tau(x)$$

$$\frac{\partial g_\tau}{\partial x_2}(x) = -\frac{1}{4\pi i\bar{x}} + \frac{e^{-i\tau(x+\bar{x})}}{4\pi i x} - \tau g_\tau(x)$$

The computation of exponentially growing solutions and their derivatives is essentially reduced to evaluating Faddeev's fundamental solution

$$g_1(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2(\xi_1 - i\xi_2)} d\xi$$

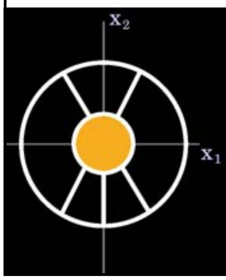
Numerical evaluation of Faddeev's Green function



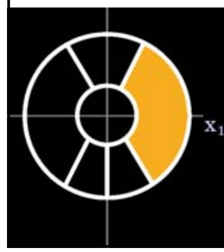
The x-plane is divided into 7 disjoint regions.

Computation of Faddeev's fundamental solution is done accordingly with one of 7 different algorithms.

$$g_1(x) = -\frac{e^{-ix}}{4\pi}(2\gamma + \log|x|^2 + \sum_{n=1}^{\infty}\frac{(ix)^n + (-i\bar{x})^n}{nn!})$$

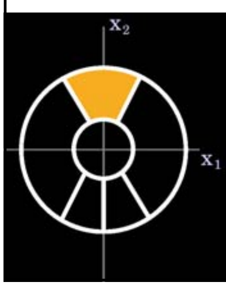


$$\begin{aligned} g_1(x) &= \frac{e^{-ix_1}}{2\pi}\text{Re}[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \\ &+ \frac{(N+1)!e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}}dt] \end{aligned}$$



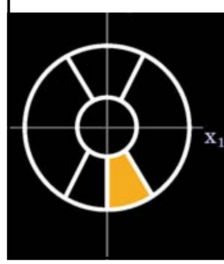
The integrand in the 1-D integral is exponentially decaying as t grows.

$$\begin{aligned} g_1(x) &= \frac{e^{-ix_1}}{2\pi}\text{Re}[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \\ &+ \frac{(1+i)(N+1)!e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-is(x_2+x_1)+s(x_2-x_1)}}{(s+is-i)^{N+2}}ds] \end{aligned}$$



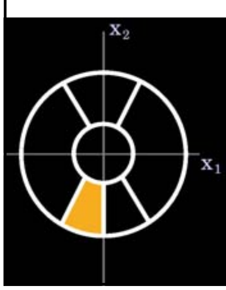
The integrand in the 1-D integral is exponentially decaying as t grows.

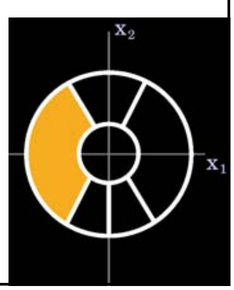
$$\begin{aligned} g_1(x) &= \frac{e^{-ix_1}}{2\pi}\text{Re}[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \\ &- \frac{i(N+1)!e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-x_2 s+ix_1 s}}{(-is-i)^{N+2}}ds] \end{aligned}$$



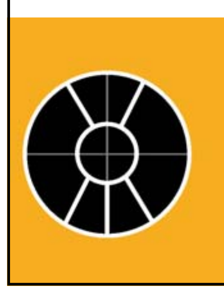
The integrand in the 1-D integral is exponentially decaying as t grows.

$$g_1(-x_1, x_2) = \overline{g_1(x_1, x_2)}$$





$$g_1(x) \approx \frac{e^{-ix_1}}{2\pi}\text{Re}\left[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}}\right]$$



This region is the complement of the disc with radius 25.

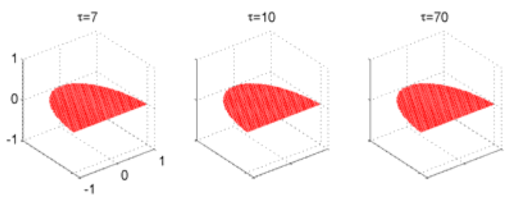
E. How to choose τ?

- The basic dilemma of this method is
  - If τ is too small, the recovered solution is not close to the true solution
  - If τ is too large, noise will be amplified
- Choice of τ depends on the Cauchy data, a priori bound on u in Ω and noise level δ
- We have given one theoretical choice of τ as function of noise level
- Numerical experiments suggest choosing τ depending on y

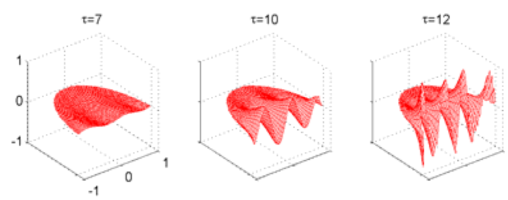
Numerical examples with V=0

1. Vanishing Cauchy data
2. Vanishing Cauchy data with noise
3. Oscillatory Cauchy data
4. Oscillatory Cauchy data with noise

Solution with non-noisy vanishing Cauchy data



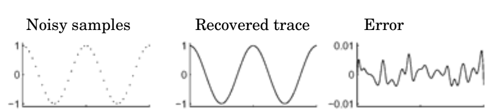
Solution with vanishing and noisy Cauchy data



Oscillatory Cauchy data

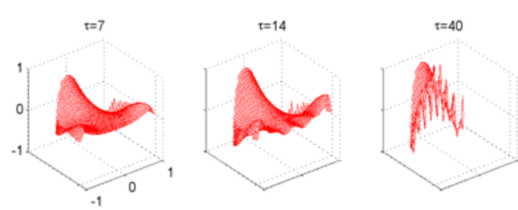
Choose the following harmonic function as the solution to be recovered:

$$u(y_1, y_2) = \mathrm{Re}(z^4), \qquad z = y_1 + iy_2$$

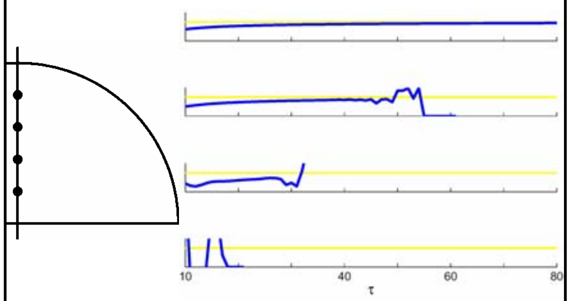


Standard deviation of Gaussian noise = 0.003

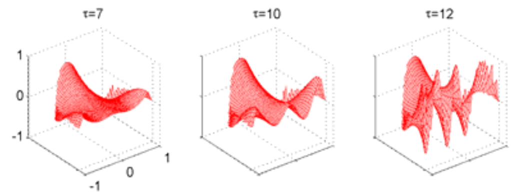
Solution with oscillatory Cauchy data without noise



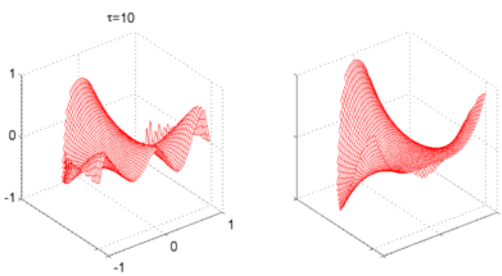
Convergence study



Solution with oscillatory and noisy Cauchy data



Comparison



Conclusion

- Computationally fast solution method: no need to solve direct problems
- Produces 3 types of error
  - With oscillatory Cauchy data, recovered solution has error in trace
  - Near almost-vertical parts of G, the triangle D is very small and lowers accuracy
  - Amplifies noise deep inside W
- Applicable to variable potential case

Future work

- Examples for Helmholtz equation and other nonzero V
- Implementation for three-dimensional problems, especially for flat Γ
- Application to measured data
- Extension with Isozaki's exponentially growing solutions?