Fast two-grid d-bar solver for electrical impedance tomography

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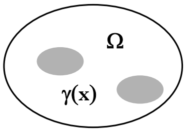
Applications of the d-bar equation

- Electrical impedance tomography
- Inverse scattering:
  reconstruction of eigenfunctions and potential from scattering data
- Nonlinear evolution equations:
  KP, DS, Novikov-Veselov
- Positron-emission tomography

Inverse conductivity problem



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega \\ u &= f \quad \text{on } \partial\Omega \end{aligned}$$

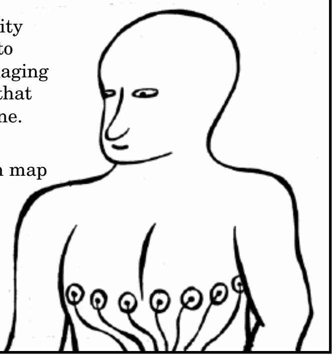
Calderón:

Given the Dirichlet-to-Neumann map, how to reconstruct the conductivity?

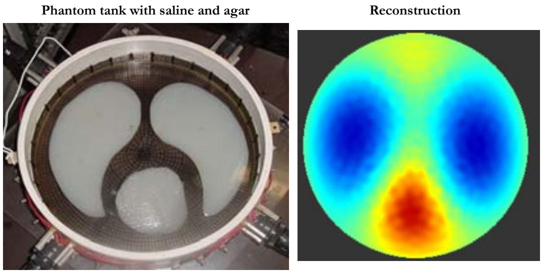
Electrical impedance tomography

2-D inverse conductivity problem corresponds to electric impedance imaging with the assumption that current flows in a plane.

Dirichlet-to-Neumann map represents voltage-to-current measurements.



Reconstruction of “heart and lungs” phantom



Phantom tank with saline and agar

Reconstruction

D-bar reconstruction method

Adrian Nachman solved the 2-D inverse conductivity problem in 1996. His approach has two steps:

$$\Lambda_\gamma \to \mathbf{t} \to \gamma$$

The intermediate object $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ is called the *scattering transform*.

The second step involves solution of a d-bar equation.

Regularized step
from measurement to t(k)

In practice the measured data is noisy. We only can compute a numerical approximation for t(k) for |k|<R:

$$\mathrm{t}_R^{\mathrm{exp}}(k) = \begin{cases} \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}d\sigma(x) & \text{for } |k| < R, \\ 0 \text{ otherwise.} \end{cases}$$

This truncation regularizes the inverse problem.

From t to γ

Solve the d-bar equation

$$\frac{\partial}{\partial \bar{k}}\mu(x,k) = \frac{\mathrm{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}$$

Then the conductivity is approximately given by

$$\gamma^{1/2}(x) = \lim_{k\to 0}\mu(x,k)$$

Generic d-bar equation

We consider numerical solution of

$$\frac{\partial}{\partial \bar{k}}v(k) = T(k)\overline{v(k)}$$

where $v : \mathbb{R}^2 \to \mathbb{C}$ satisfies $\lim_{|k|\to\infty} v(k) = 1$ and

$$\frac{\partial}{\partial \bar{k}} = \frac{1}{2}\left(\frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_2}\right)$$

Integral equation formulation

Convolving the dbar equation with the Green's function 1/(πk) yields

$$v(k) = 1 - \frac{1}{\pi}\int_{\mathbb{R}^2}\frac{T(k')}{k-k'}\overline{v(k')}dk_1'dk_2'$$

or in shorter form

$$v(k) = 1 - \frac{1}{\pi k} * (T(k)\overline{v(k)})$$

Solution method

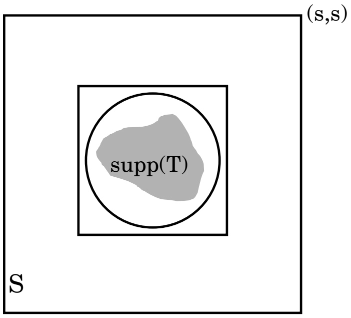
- Modification of G.Vainikko's [1997] Lippmann-Schwinger equation solver
- Based on periodization and the use of FFT
- Requires compactly supported T
- Modification needed because of nonlinearity of the dbar equation

Key observations

1. If v(k) is known for k in supp(T) then v is known for any k by
$$v(k) = 1 - \frac{1}{\pi k} * (T(k)\overline{v(k)})$$
2. If T is supported in D(0,r) and k belongs to D(0,r) then the integral in
$$v(k) = 1 - \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{T(k')}{k-k'}\overline{v(k')}dk_1'dk_2'$$
does not involve the values of the Green function outside D(0,2r).

Periodization

We consider functions f(k1,k2) that are 2s-periodic with respect to k1 and k2.



Periodic dbar equation

We consider the equation

$$w(k) = 1 - \int_{-s}^{s}\int_{-s}^{s} g(k-k')T(k')\overline{w(k')}dk_1'dk_2'$$

or in short form

$$[I + g * (T \cdot ^{-})]w = 1$$

where g is given by

$$g(k) = \frac{1}{\pi k} \quad \text{for } k \in S$$

Equivalence of equations

It is easily seen that unique solvability of equation
$$v(k) = 1 - \frac{1}{\pi k} * (T(k)\overline{v(k)})$$
in the plane is equivalent to the unique solvability of equation
$$[I + g * (T \cdot ^{-})]w = 1$$
in the space of periodic functions on S. Moreover,

$$v|_{\text{supp}(T)} = w|_{\text{supp}(T)}$$

Numerical solution of the periodic equation

We wish to solve

$$[I + g * (T \cdot ^{-})]w = 1$$

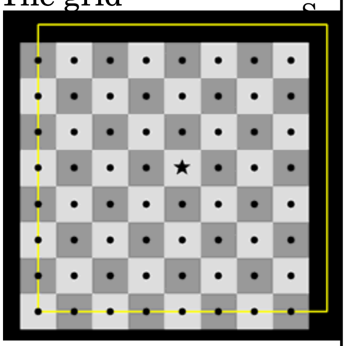
numerically. We introduce

- a grid on the square S,
- grid approximation of functions,
- discrete convolution operator,
- solution using GMRES,
- two-grid version of the algorithm

The grid

Choose an integer m, denote M=2^m and h=2s/M.

Here m=3.



$$\mathcal{G}_m = \{jh \,|\, j \in \mathbb{Z}_m^2\}$$
$$\mathbb{Z}_m^2 = \{j = (j_1, j_2) \in \mathbb{Z}^2 \,|\, -2^{m-1} \leq j_l < 2^{m-1}\}$$

Grid approximation of functions

The method works for piecewise continuous T.

$$\varphi_h(j) = \begin{cases} \varphi(jh), & \text{if} \quad B_{j,h} \cap \Gamma = \emptyset \\ h^{-2}\sum_{p=1}^{P_j} \varphi(x_{j,h}^{(p)})|B_{j,h}^{(p)}| & \text{otherwise} \end{cases}$$

Discretized Green's function

$$g_h(j) = \begin{cases} g(jh), & \text{for } j \in \mathbb{Z}_m^2 \setminus 0 \\ 0, & \text{for } j = 0 \end{cases}$$

Since the singularity at k=0 is integrable, the error caused by the approximation becomes small when the discretization is refined

Discrete convolution operator

The periodic convolution operator

$$(A\varphi)(k) = (g*\varphi)(k) = \int_{-s}^{s}\int_{-s}^{s} g(k-k')\varphi(k')dk_1'dk_2'$$

is discretized as multiplication of Fourier transforms:

$$A_h\varphi_h = \mathcal{F}^{-1}\Big(\mathcal{F}g_h \cdot \mathcal{F}\varphi_h\Big)$$

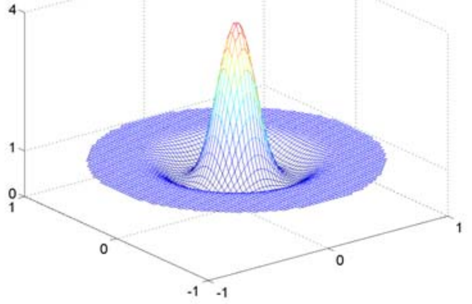
Application of the discrete operator can be implemented in O(M^2 log M) arithmetical operations using FFT

Solution

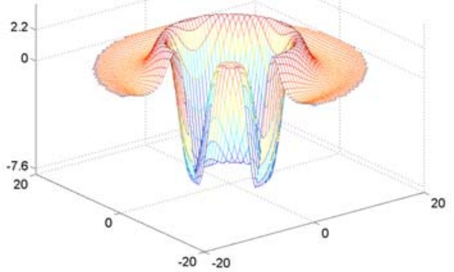
$$[I + A_h(T_h \cdot^{-})]w_h = 1_h$$

Use GMRES keeping real and imaginary parts separate.

Radial conductivity



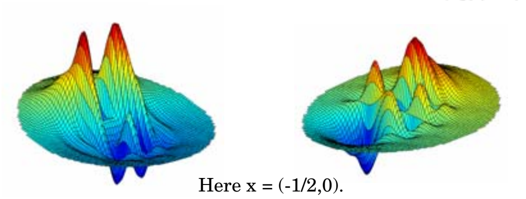
Scattering transform



Note: t(k) is not computed approximately from boundary data but instead accurately from its definition.

Solution of the d-bar equation

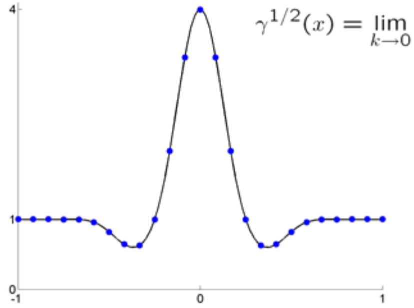
Real part of $\mu_R(x,\cdot)$ Imaginary part of $\mu_R(x,\cdot)$



Here x = (-1/2,0).
Solution is computed separately for each x.

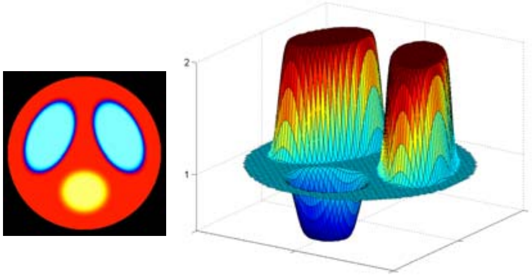
$$\frac{\partial}{\partial \bar{k}}\mu_R(x,k) = \frac{\mathbf{t}_R\quad(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}$$

Profile of reconstruction



Recall:
$$\gamma^{1/2}(x) = \lim_{k\to 0}\mu(x,k)$$

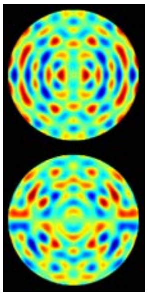
Simulated ”heart and lungs”



Truncated scattering transform

Conductivity

Scattering transform

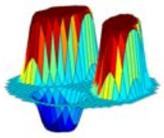


Real part

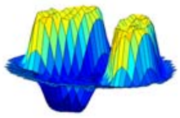
Imaginary part

Regularized reconstruction

True conductivity



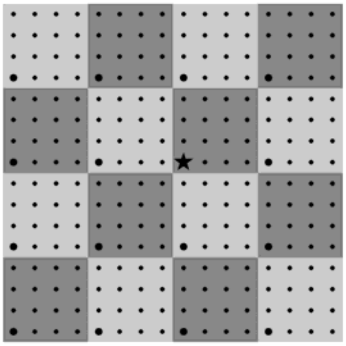
Reconstruction



Two-grid method

- It is possible to introduce another, finer grid and achieve extra resolution with small computational cost
- The periodic operator is inverted only on the coarse grid
- Roughly 50% reduction in memory usage and computation time compared to one-grid solution with 1024x1024 grid

Fine and coarse grid



Comparison of reconstructions

Conductivity

1-grid reconstruction

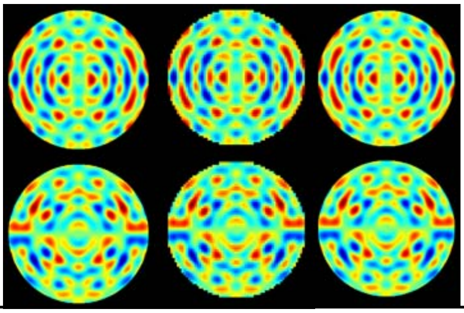
2-grid reconstruction

36%

33%

Relative sup-norm errors

Comparison of scattering transforms



True

1-grid (64x64)

2-grid (512x512)

Conclusion

- We have implemented a d-bar equation solver, or
  Fast Inverse Scattering Transform (FIST)
- Complexity of the method is $O(M^2 \log M)$
- Accuracy order of the method is $O(h)$
- Allows two-grid extension
- Future challenges:
  - speed-up by preconditioning
  - experiments with smooth truncation

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- Knudsen K, Mueller J L and Siltanen S, *Numerical solution method for the d-bar equation in the plane*, in preparation