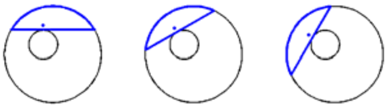
Numerical solution of the inverse problem of ECG using Faddeev's Green function



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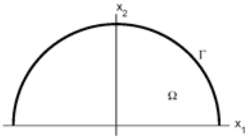
Geometry of the Cauchy problem considered in this work is half-disc



$$(-\Delta + V)u = 0 \qquad \text{in } \Omega,$$

$$u \in H^2(\Omega), \quad V(x) \in L^\infty(\Omega)$$

Problem: Recover u from its Cauchy data

$$(u|_\Gamma, \frac{\partial u}{\partial \nu}|_\Gamma)$$

Applications of the Cauchy problem include

Recovering temperature distribution inside a physical body from surface temperature and heat flux

Recovering voltage potential on the heart from voltage measurements on the skin

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Several numerical solution methods for the Cauchy problem have been presented

Klibanov and Santosa 1991
  (based on Lattés and Lions 1969, Lavrentyev 1956)
Kabanikhin and Karchevsky 1995
Leitão 2000 (based on Maz'ya 1991)
Háo and Lesnic 2000 (V=0)
Berntsson and Eldén 2001 (V=0)
Cheng, Hon, Wei and Yamamoto 2001 (V=0)

We present a new non-iterative solution method for V≠0 that does not involve solution of boundary value problems.

We solve the Cauchy problem using Faddeev's Green function

Ikehata [2001] proved that

$$u(y) = \lim_{\tau \to \infty} u_\tau(y)$$

where

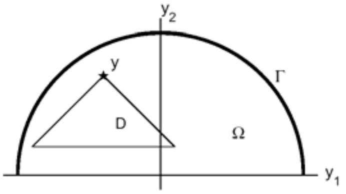
$$u_\tau(y) := \frac{2\tau^2 e^{-i\tau y_1}}{C_D} \int_\Gamma (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u) d\sigma(x)$$

Computation with finite τ provides a regularised reconstruction method.

The reconstruction method uses Faddeev's exponentially growing solutions

$$-\Delta v''_\tau + \tilde{V} v''_\tau = \chi_D e^{\tau(x_2 - y_2)} e^{i\tau x_1} \text{ in } \mathbb{R}^2$$

$$v_\tau = v''_\tau|_\Omega$$



The proof is based on Green's formula

$$\begin{aligned}\int_D e^{\tau(x_2-y_2)}e^{i\tau x_1}u(x)dx &= -\int_\Omega (\Delta v_\tau)u + \int_\Omega V v_\tau u \\ &= \int_{\partial\Omega}(\frac{\partial u}{\partial\nu}v_\tau - \frac{\partial v_\tau}{\partial\nu}u) \\ &= (\int_\Gamma + \int_{\partial\Omega\setminus\Gamma})(\frac{\partial u}{\partial\nu}v_\tau - \frac{\partial v_\tau}{\partial\nu}u)\end{aligned}$$

$$\int_D e^{\tau(x_2-y_2)}e^{i\tau x_1}u(x)dx \sim \frac{C_D}{2\tau^2}e^{i\tau y_1}u(y) \quad \text{as } \tau\to\infty$$

$$\int_{\partial\Omega\setminus\Gamma}(\frac{\partial u}{\partial\nu}v_\tau - \frac{\partial v_\tau}{\partial\nu}u) \to 0 \quad \text{exponentially as } \tau\to\infty$$

Numerical implementation of the method is divided into 5 problems

$$u_\tau(y) = \frac{2\tau^2 e^{-i\tau y_1}}{C_D}\int_\Gamma(\frac{\partial u}{\partial\nu}v_\tau - \frac{\partial v_\tau}{\partial\nu}u)d\sigma(x)$$

1. Integration on Γ
2. Choosing the triangle D=D(y)
3. How to choose τ?
4. Computing exponentially growing solutions
5. Computing normal derivatives of exponentially growing solutions

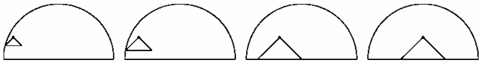
Implementation Step 1: integration on Γ

We choose a set of integration quadrature points and weights on Γ. Then the integral of *f(x)* over Γ is approximated by the following sum:

$$\int_\Gamma f d\sigma \approx \sum_{k=1}^{K} w^{(k)} f(x^{(k)})$$

Implementation Step 2: choosing the triangle D

We take D to be the largest possible triangular patch such that D belongs to Ω and the base of D is twice its height.



The choice is based on theory: large |CD| minimizes error.

We always have |CD|<=2, and with this choice we have $C_D = 2$

Implementation Step 3: Choosing the regularization parameter τ

If τ is too small, then recovered solution is not close to the true solution

If τ is too large, then noise will be amplified

Choice of τ depends on the Cauchy data, a priori bound on u in Ω and noise level; we do not have a general practical choice

Implementation Step 4: Computing exponentially growing solutions

Define

$$v_\tau(x_1,x_2) = e^{\tau(x_2-y_2)}e^{i\tau x_1}w'_\tau(x_1,x_2)$$

Case V=0: numerically evaluate convolution

$$w'_\tau(x) = g_\tau * \chi_D$$

Piecewise smooth V: solve

$$w'_\tau(x) + g_\tau * (\tilde{V}w'_\tau) = g_\tau * \chi_D$$

with an adaptation of Vainikko's Lippmann-Schwinger equation solver [Mueller-S 2003] based on FFT and GMRES

Exponentially growing solutions are defined using Faddeev's fundamental solution

In the Lippmann-Schwinger –type equation

$$w'_\tau(x) + g_\tau * (\tilde V w'_\tau) = g_\tau * \chi_D$$

The function

$$g_\tau(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\tau(\xi_1 - i\xi_2)} d\xi$$

satisfies

$$(-\Delta - 2i\tau(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}))g_\tau(x) = \delta(x)$$

Solution of the Lippmann-Schwinger –type equation is based on two key observations

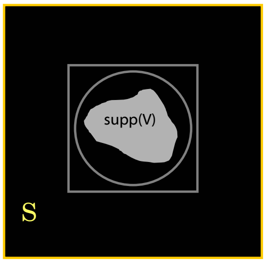
$$w'_\tau(x) = (g_\tau * \chi_D)(x) - \int_{\text{supp}(q)} g_\tau(x-y)\tilde V(y) w'_\tau(y)\,dy$$

1. It is enough to know w'(x) for x in supp(V)
2. If V is supported in D(0,r) and x belongs to D(0,r) then the integral does not involve the values of fundamental solution outside D(0,2r)

The L-S type equation is replaced by an equivalent periodic equation

$$[I + \tilde g_\tau * (\tilde V \cdot\ )]\tilde w'_\tau = \tilde g_\tau * \chi_D$$

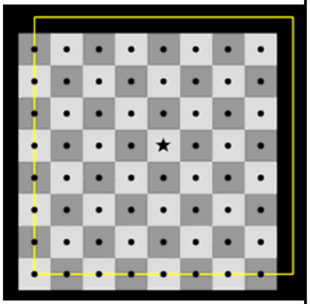
We consider functions that are 2s-periodic in x1 and x2.



Periodization is based on this type of grid

Choose an integer m, denote M=2^m and h=2s/M.

Here m=3.



$$\begin{aligned} \mathcal{G}_m &= \{jh \mid j \in \mathbb{Z}_m^2\} \\ \mathbb{Z}_m^2 &= \{j = (j_1, j_2) \in \mathbb{Z}^2 \mid -2^{m-1} \le j_l < 2^{m-1}\} \end{aligned}$$

Functions are represented by their values at grid points

Continuous functions:

$$\varphi_h(j) = \varphi(jh)$$

Fundamental solution:

$$\tilde g_\tau(j) = \begin{cases} g_\tau(jh), & \text{for } j \in \mathbb{Z}_m^2 \setminus 0, \\ 0, & \text{for } j = 0, \end{cases}$$

Since the singularity at x=0 is integrable, the error becomes small when the discretization is refined

Numerical solution is based on using FFT for convolution and applying GMRES

$$[I + \tilde g_\tau * (\tilde V \cdot\ )]\tilde w'_\tau = \tilde g_\tau * \chi_D$$

The convolution operator is implemented with FFT

GMRES is iterative: the linear operator need not be written explicitly as a matrix

The complexity of the method is O($M^2 \log M$)

Implementation Step 5: normal derivatives of exponentially growing solutions

Piecewise smooth V:

$$\frac{\partial v_\tau}{\partial \nu} = \frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi} \times \left[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * (\tilde{V} w'_\tau - \chi_D)\right]$$

Case V=0:

$$\frac{\partial v_\tau}{\partial \nu} = -\frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi} \times \left[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * \chi_D\right]$$

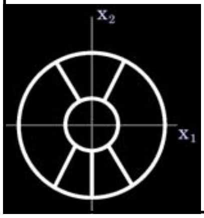
Computing exponentially growing solutions and their derivatives is reduced to evaluating

$$g_1(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2(\xi_1 - i\xi_2)} d\xi$$

This is due to the symmetry

$$g_\tau(x) = g_1(\tau x)$$

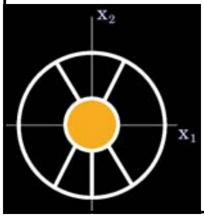
Computation of Faddeev's fundamental solution is divided into 7 cases



The x-plane is divided into 7 disjoint regions, each leading to a different algorithm

Case 1: For x in the unit disc we use a formula by [Boiti et al 1987]

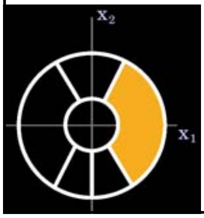
$$g_1(x) = -\frac{e^{-ix}}{4\pi}(2\gamma + \log|x|^2 + \sum_{n=1}^{\infty}\frac{(ix)^n + (-i\bar{x})^n}{nn!})$$



Here γ is Euler's constant

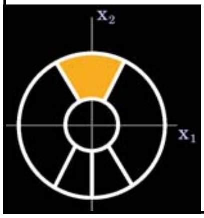
Case 2: We write g1 as a formula containing a rapidly converging one-dimensional integral

$$g_1(x) = \frac{e^{-ix_1}}{2\pi}\mathrm{Re}[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \\ + \frac{(N+1)!e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}}dt]$$

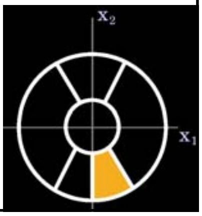


Cases 3 and 4: The integral in case 2 is modified using the residue theorem

$$-i\int_0^\infty \frac{e^{-x_2 s + ix_1 s}}{(-is-i)^{N+2}}ds$$



$$(1+i)\int_0^\infty \frac{e^{-is(x_2+x_1)+s(x_2-x_1)}}{(s+is-i)^{N+2}}ds$$



Cases 5 and 6 are reduced to cases 4 and 2 using the following symmetry:

$$g_1(-x_1, x_2) = \overline{g_1(x_1, x_2)}$$

Case 7: for |x|>25 we ignore the integral in case 2 and use the truncated sum

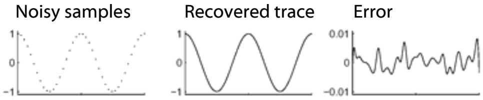
$$g_1(x) \approx \frac{e^{-ix_1}}{2\pi}\text{Re}\left[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}}\right]$$

Example 1: The harmonic case V=0

Choose harmonic function u:

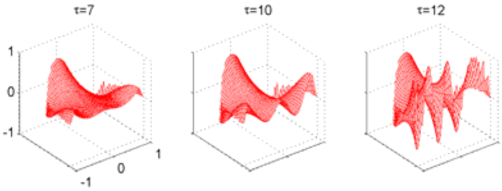
$$u(y_1, y_2) = \text{Re}(z^4), \qquad z = y_1 + iy_2$$

Produce computer simulated noisy Cauchy data:



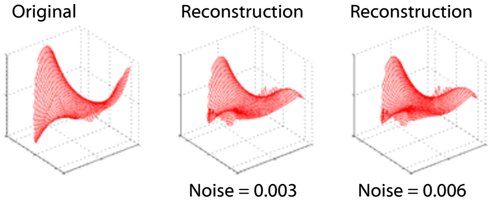
Standard deviation of Gaussian noise = 0.003

Example 1: Solution with noisy Cauchy data



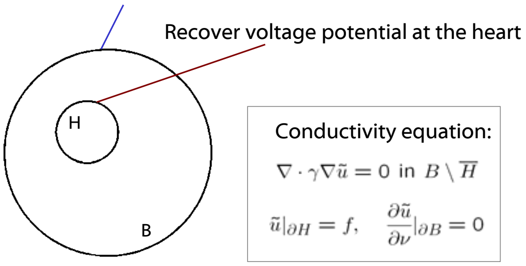
Small τ gives good reconstruction deep inside Ω, large τ gives good reconstruction near Γ

Example 1: Solution with two noise levels illustrates the stability of our method



Here τ is chosen as function of y

Example 2: Inverse potential problem of electrocardiography



Measure voltage potential at the skin

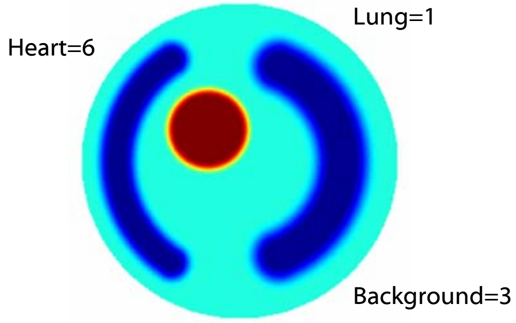
Recover voltage potential at the heart

Conductivity equation:

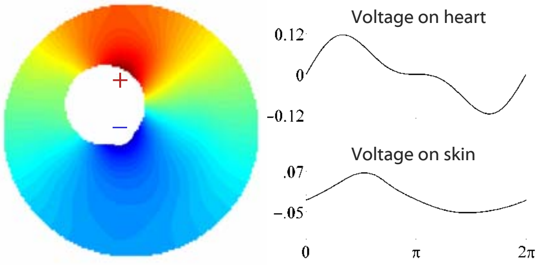
$$\nabla \cdot \gamma \nabla \tilde{u} = 0 \text{ in } B \setminus \overline{H}$$

$$\tilde{u}|_{\partial H} = f, \qquad \frac{\partial \tilde{u}}{\partial \nu}|_{\partial B} = 0$$

Example 2: We construct a conductivity modelling a cross section of human chest



Example 2: We compute voltage potential outside heart by Finite Element Method



Example 2: We transform the conductivity equation to the Schrödinger equation

Conductivity equation:

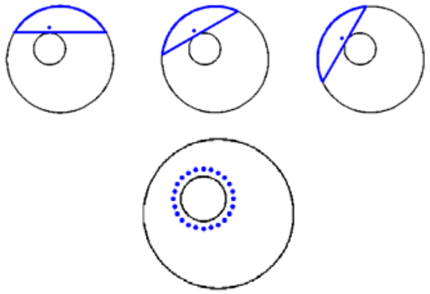
$$\nabla\cdot\gamma\nabla\tilde{u} = 0 \text{ in } B\setminus\overline{H}, \quad \tilde{u}|_{\partial H} = f, \quad \frac{\partial\tilde{u}}{\partial\nu}|_{\partial B} = 0$$

Define $u = \gamma^{1/2}\tilde{u}$, $\quad V(x) = \frac{\Delta\sqrt{\gamma(x)}}{\sqrt{\gamma(x)}}$

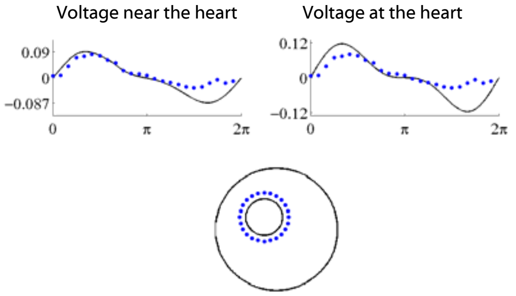
Then u satisfies the equation

$$(-\Delta+V)u = 0 \text{ in } B\setminus\overline{H}, \quad u|_{\partial B} = \sqrt{3}\tilde{u}|_{\partial B}, \quad \frac{\partial u}{\partial\nu}|_{\partial B} = 0$$

Example 2: We recover voltage near the heart by rotating the canonical geometry



Example 2: We reconstruct voltage near the heart from ECG data with 2% noise



Conclusion: we presented a new numerical solution method of the Cauchy problem

Our method is computationally fast:
no need to solve direct problems

In the inverse problem of ECG,
our method has 20% average relative error
on the anterior surface of the heart

Future work: 3D problems