Modeling and simulation for electric impedance tomography

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Joint work with
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Electric impedance imaging

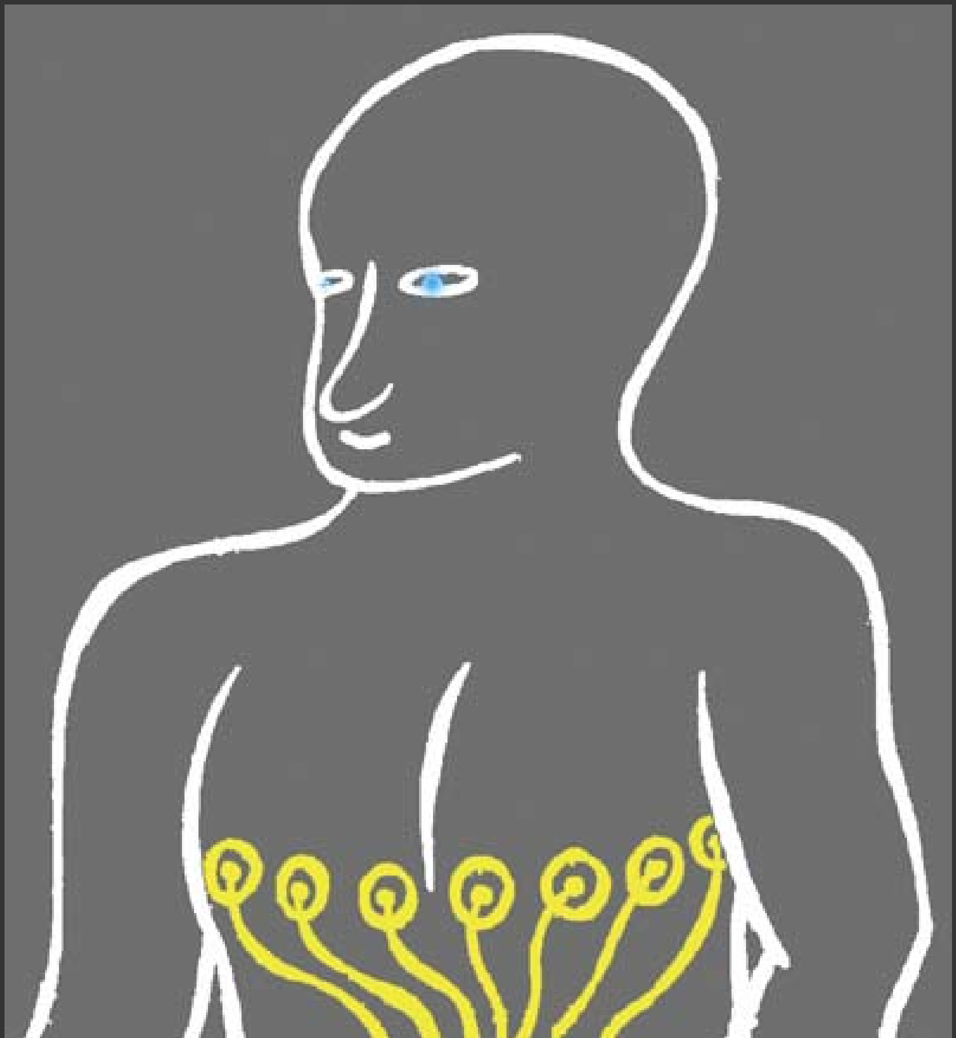
Feed electric currents through electrodes on the skin

Measure voltages at the electrodes

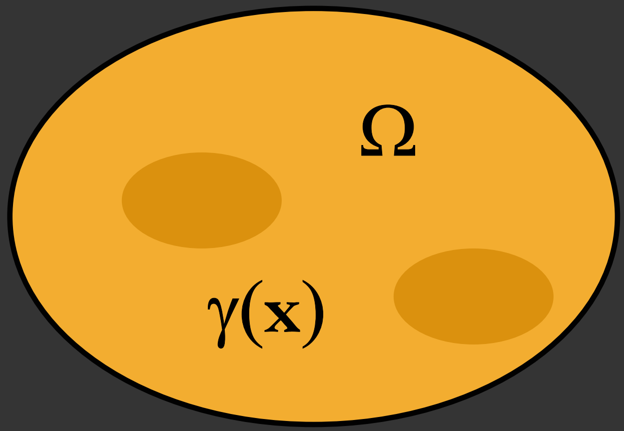
Form the image of electric conductivity in a two-dimensional slice

Applications:

monitoring heart and lungs, detecting pulmonary edema



Inverse conductivity problem



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

Given measurements, or the Dirichlet-to-Neumann map, how to reconstruct the conductivity?

- The map $\Lambda_\gamma \mapsto \gamma$ is nonlinear.
- The reconstruction problem is ill-posed.
- In practical imaging, only partial and noisy knowledge of the D-to-N map is available.

Numerical approaches to EIT

- Linearization
- Iterative LS methods for the nonlinear problem
- Layer stripping
- Statistical inversion
- Finding partial information
- The inverse scattering approach (dbar method)

Theory for the dbar method

- 1980 Calderón
- 1987 Sylvester and Uhlmann
- 1987 Novikov
- 1988 Nachman
- 1996 Nachman: uniqueness proof for the 2-D inverse conductivity problem. Outlines a reconstruction method for twice differentiable conductivities. Proof does not use non-constructive techniques such as analytic continuation.

Structure of Nachman's proof

In Nachman's proof of the 2-D inverse conductivity problem the conductivity is recovered from boundary data in two steps:

$$\Lambda_\gamma \longrightarrow \mathbf{t} \longrightarrow \gamma.$$

The intermediate object t is called *scattering transform*.

The scattering transform

Define a potential

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}.$$

For any nonzero complex number k, consider exponentially growing solutions of the equation

$$(-\Delta + q)\psi(\cdot, k) = 0$$

satisfying the following asymptotic condition:

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}.$$

Define the scattering transform of q by the formula

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

Step 1: from D-to-N map to t

Solve traces of the exponentially growing solutions from the boundary integral equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

where the single-layer operator has Faddeev Green's function as kernel.

Compute t from the formula

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(x,k)\,d\sigma(x).$$

Step 2: from t to γ

Define $$\mu(x,k) = e^{-ikx}\psi(x,k).$$

Then the following dbar equation holds:

$$\frac{\partial}{\partial \bar{k}}\mu(x,k) = \frac{\mathrm{t}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

The dbar equation can be solved numerically with a fast FFT-based algorithm. The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k\to 0} \mu(x,k).$$

Regularized reconstruction

When data is noisy, equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

cannot be solved numerically.
Instead, we introduce the following approximation to t:

$$\mathbf{t}^{\mathrm{exp}}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}d\sigma(x).$$

This function can be computed from noisy data for small |k|.

We replace t by truncated approximation in the dbar equation of Step 2. This is a nonlinear low-pass filter.

Two results

(1) We have the following asymptotic result:

Theorem: Let $\gamma \in C^4(\Omega)$. Assume that $\gamma \equiv 1$ near $\partial\Omega$. Define for any $R > 0$

$$\mathbf{t}_R(k) := \begin{cases} \mathbf{t}(k), & |k| < R, \\ 0, & |k| \geq R. \end{cases}$$

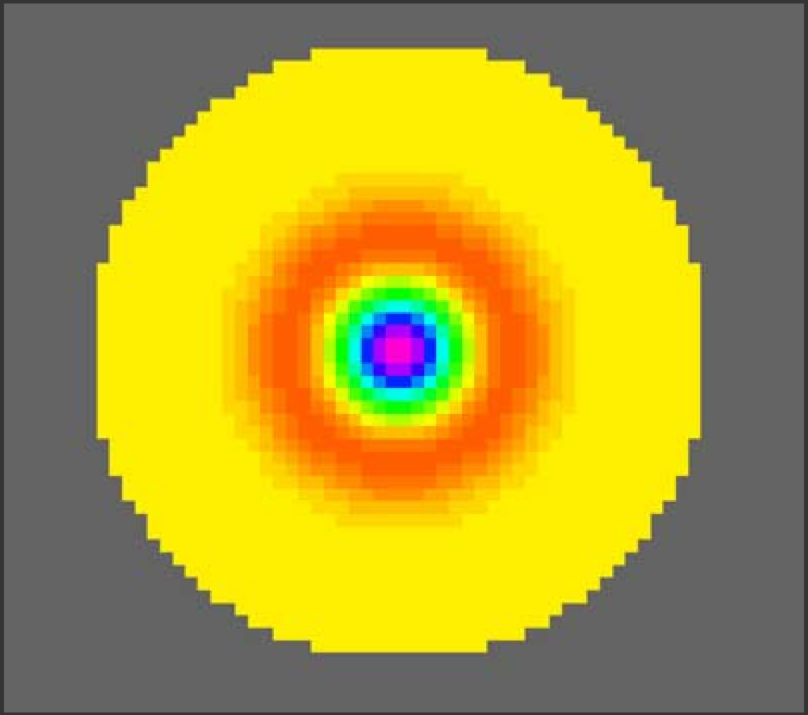
The following estimate holds for large R:

$$\|\sqrt{\gamma} - \sqrt{\gamma_R}\|_{L^\infty(\Omega)} \leq \frac{C}{R}.$$

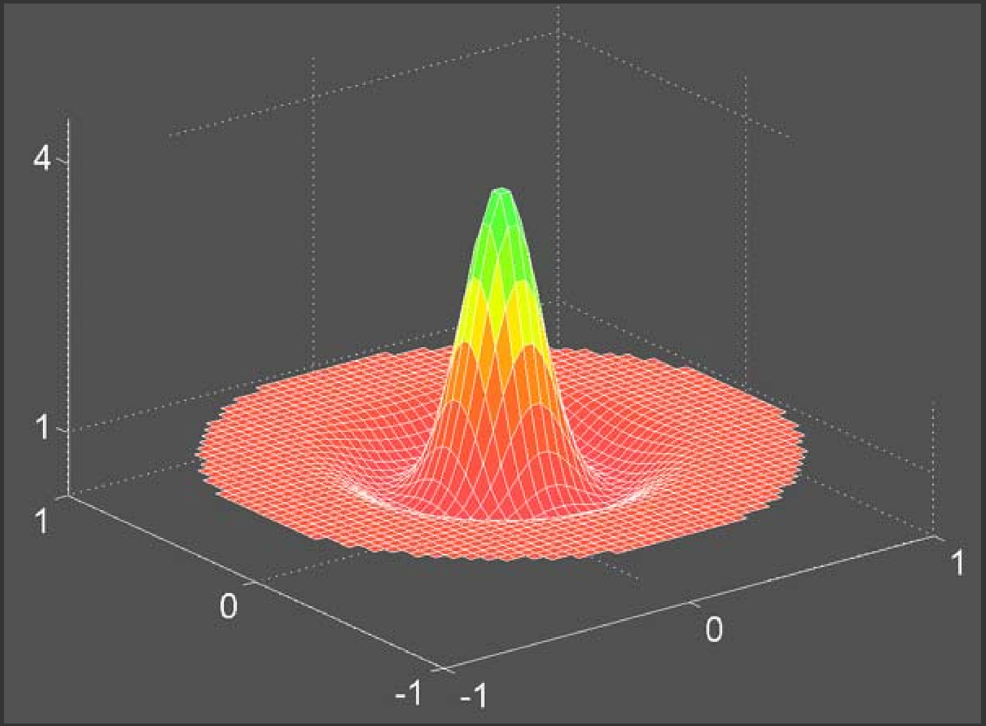
(2) For k very near to zero, the approximate scattering transform is close to t.

Example conductivity

Image of γ



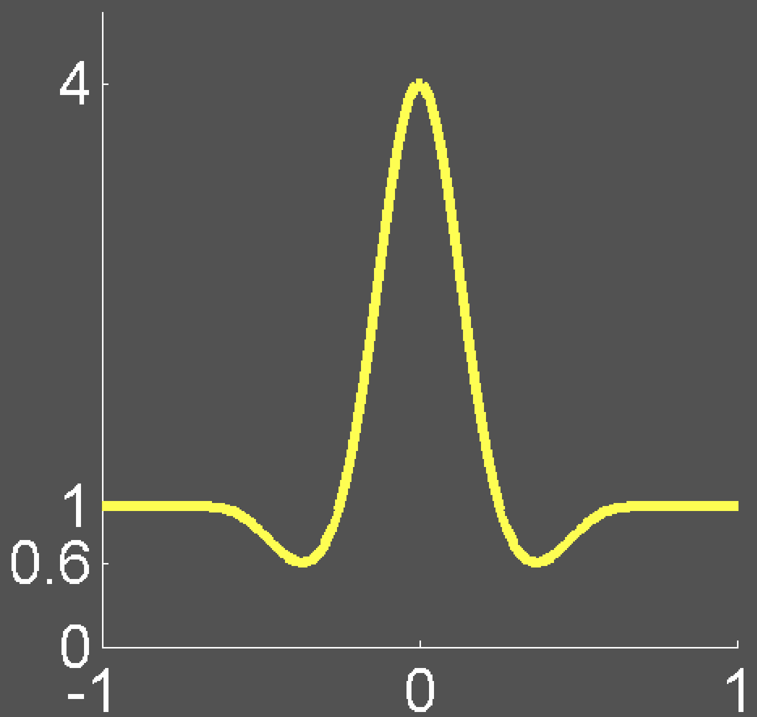
3D plot of γ



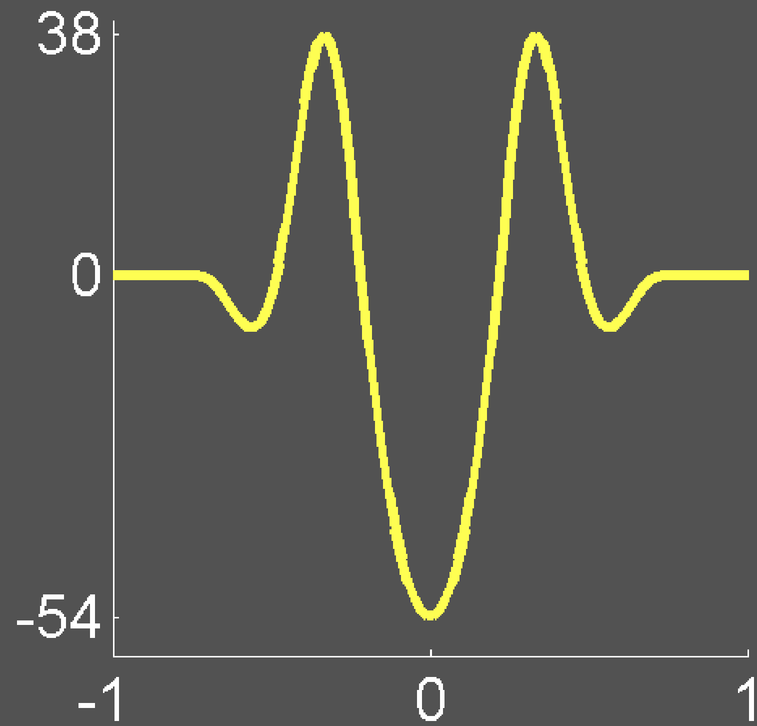
- Maximum value of γ is 4
- Minimum value of γ is 0.6
- Near the boundary γ equals 1
- Here γ is 4 times continuously differentiable

The potential q associated to γ

Profile of γ



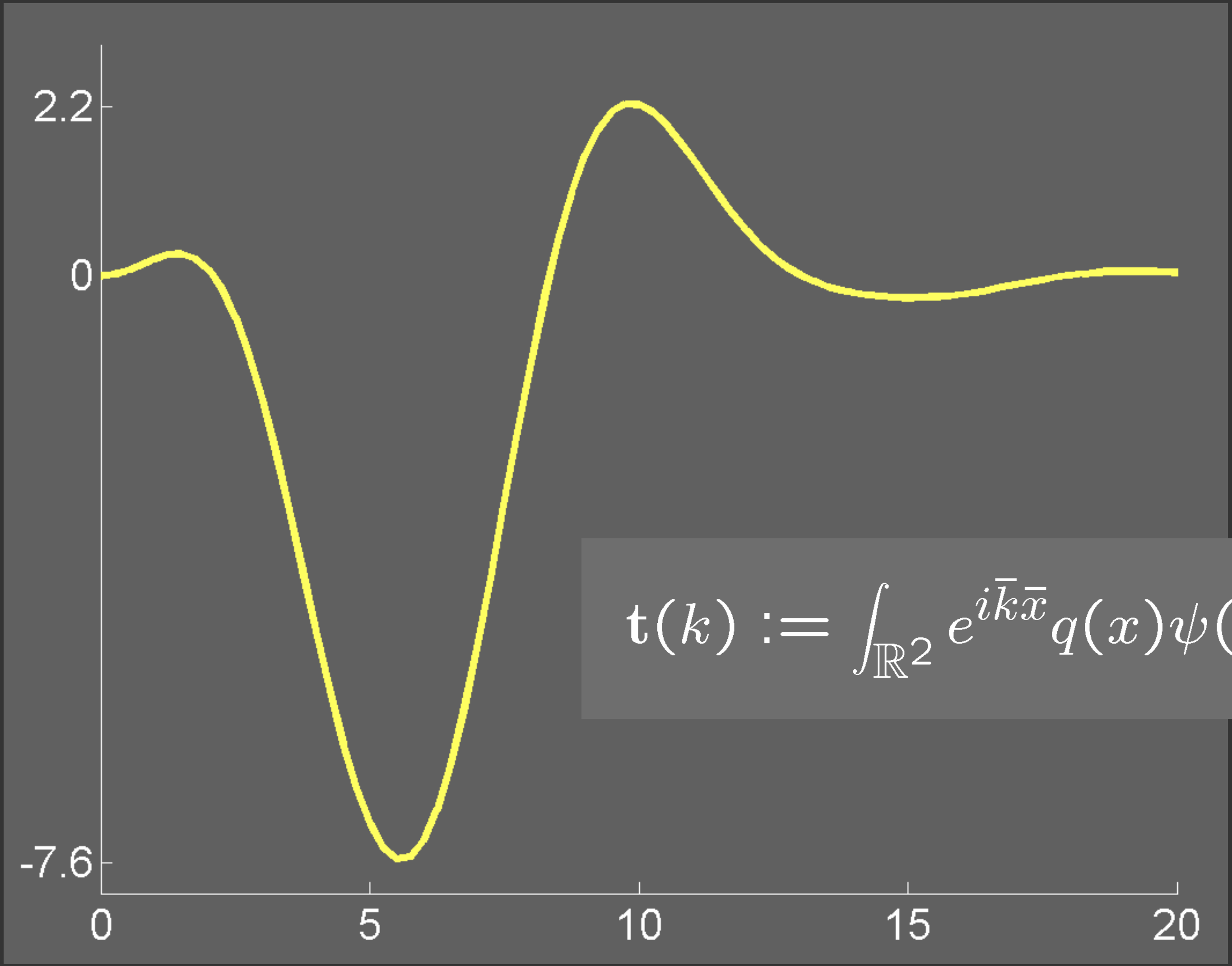
Profile of q



$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}.$$

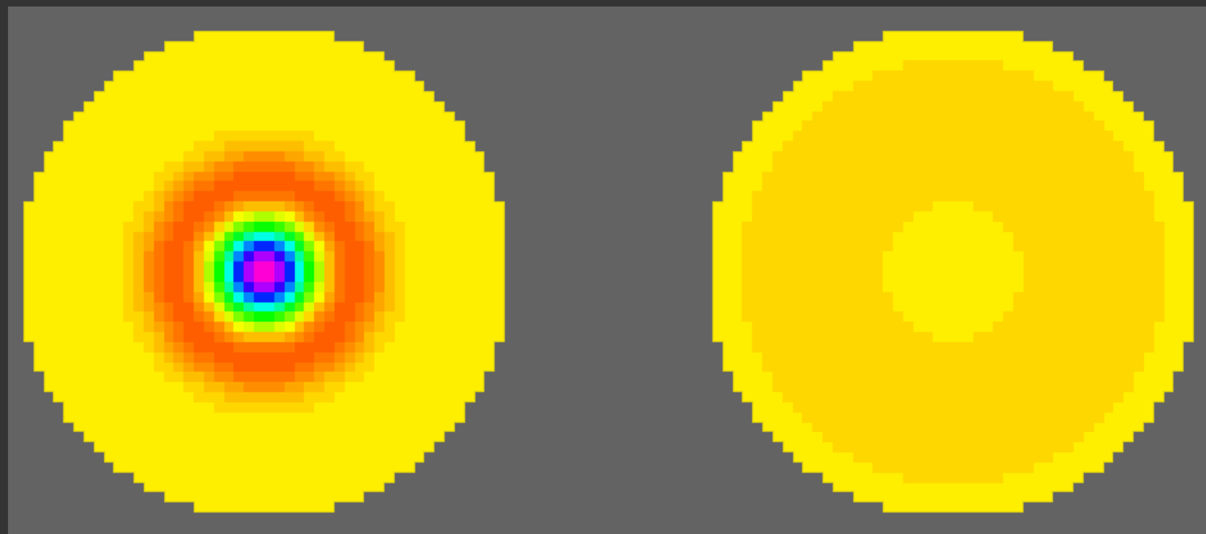
The scattering transform of γ

Profile of t (radial symmetry of q implies that of t)



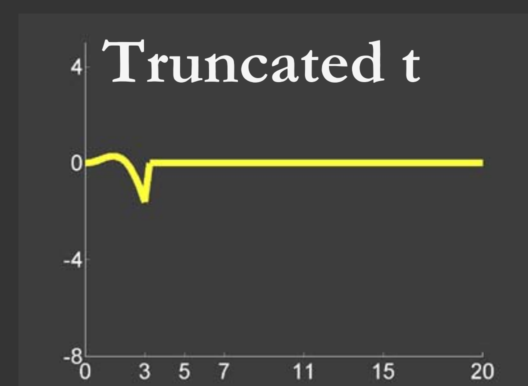
$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x) \psi(x,k)\, dx$$

Step 2 with truncated t

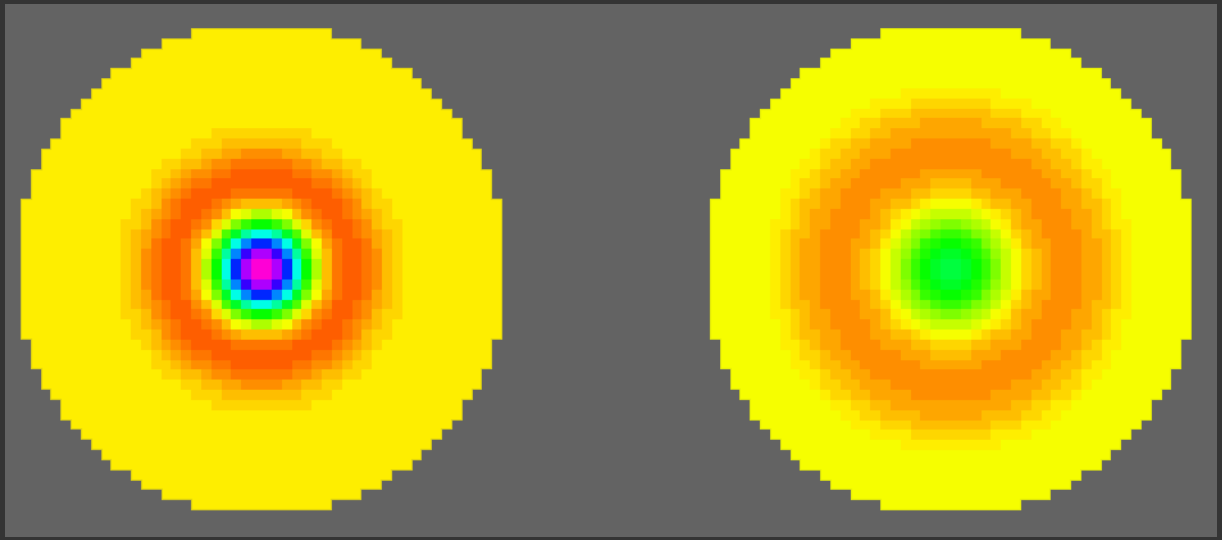


Original　　　　Reconstruction

R=3

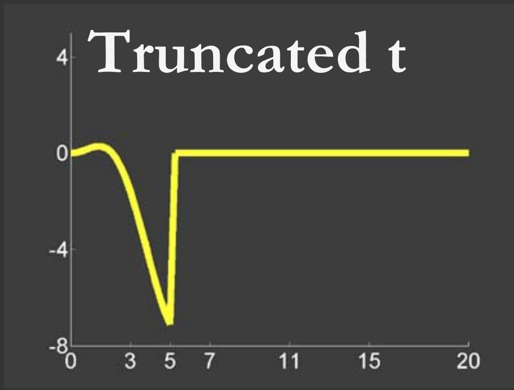


Step 2 with truncated t

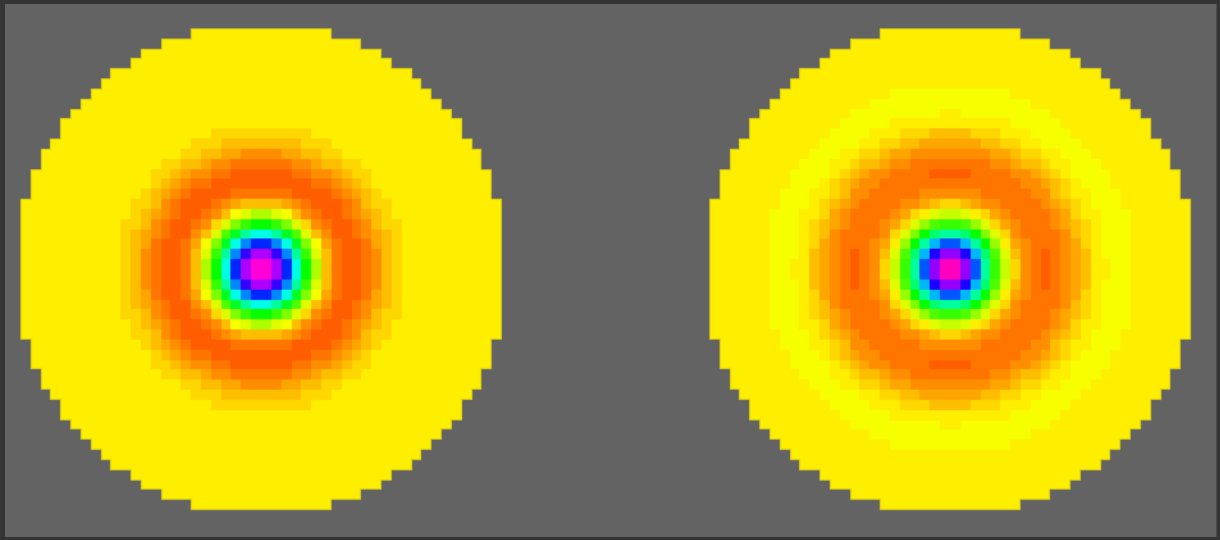


Original Reconstruction

R=5



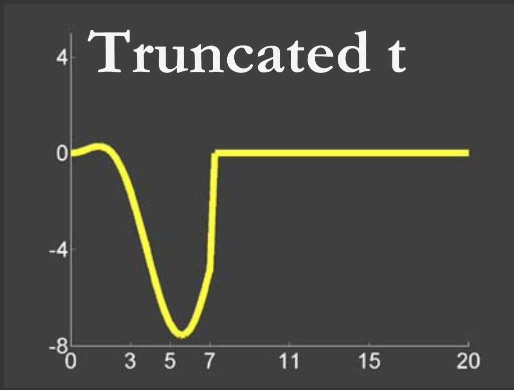
Step 2 with truncated t



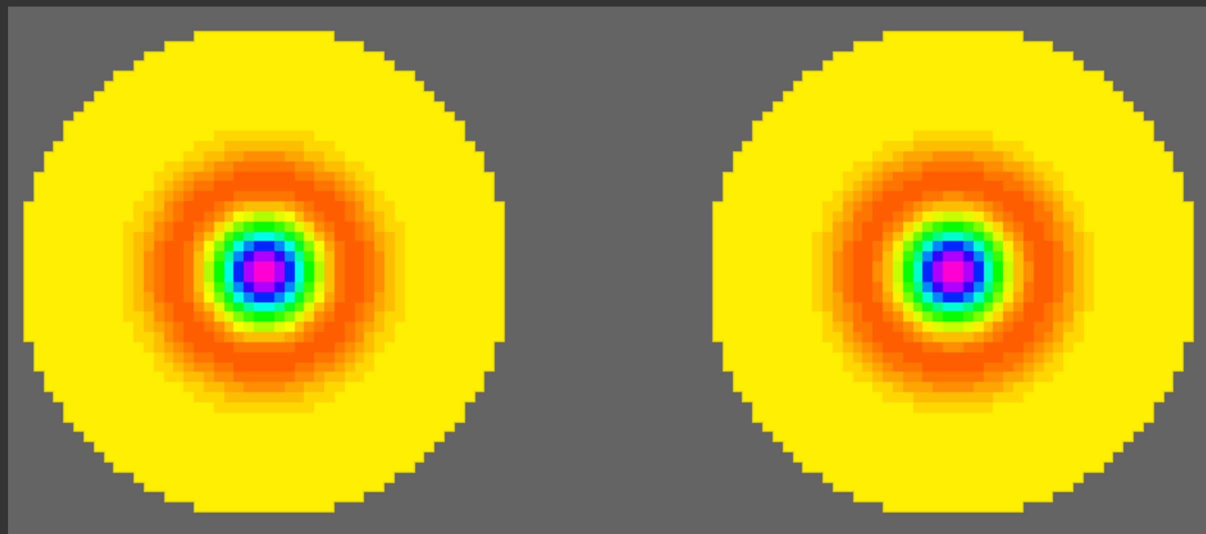
Original

Reconstruction

R=7

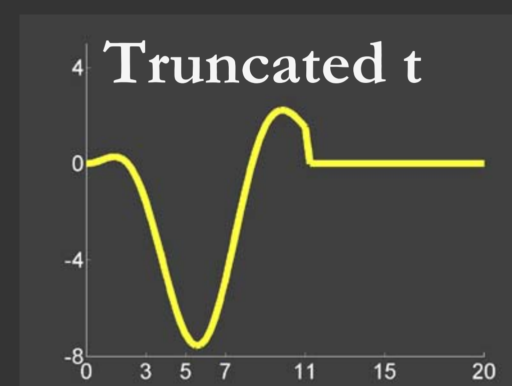


Step 2 with truncated t

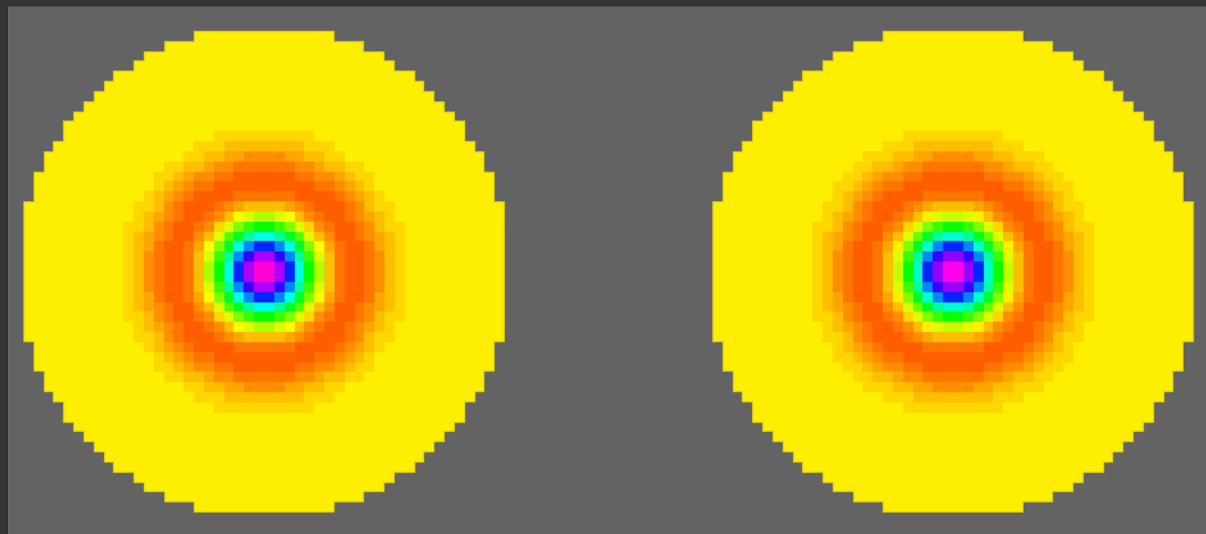


Original Reconstruction

R=11

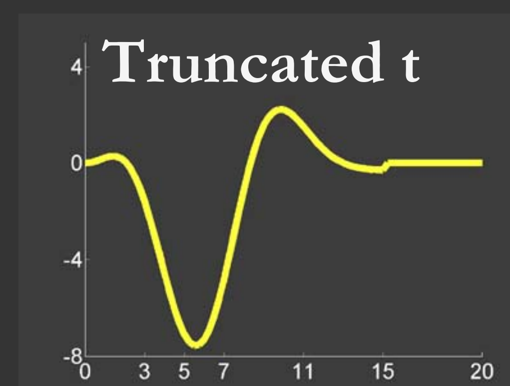


Step 2 with truncated t



Original Reconstruction

R=15



Simulating noisy data

1	-0.045
2	-0.052
3	-0.025
4	-0.010
5	-0.0040
6	-0.0016
7	-0.00064
8	-0.00026
9	-0.00011
10	-0.000046
11	-0.000020
12	-0.0000088
13	-0.0000039
14	-0.0000018
15	-0.00000080
16	-0.00000037
17	-0.00000017
18	-0.000000080
19	-0.000000038
20	-0.000000018
21	-0.0000000085
22	-0.0000000041
23	-0.0000000020
24	-0.00000000097
25	-0.00000000047
26	-0.00000000023
27	-0.00000000011
28	-0.000000000057
29	-0.000000000028
30	-0.000000000014
31	-0.0000000000071
32	-0.0000000000036

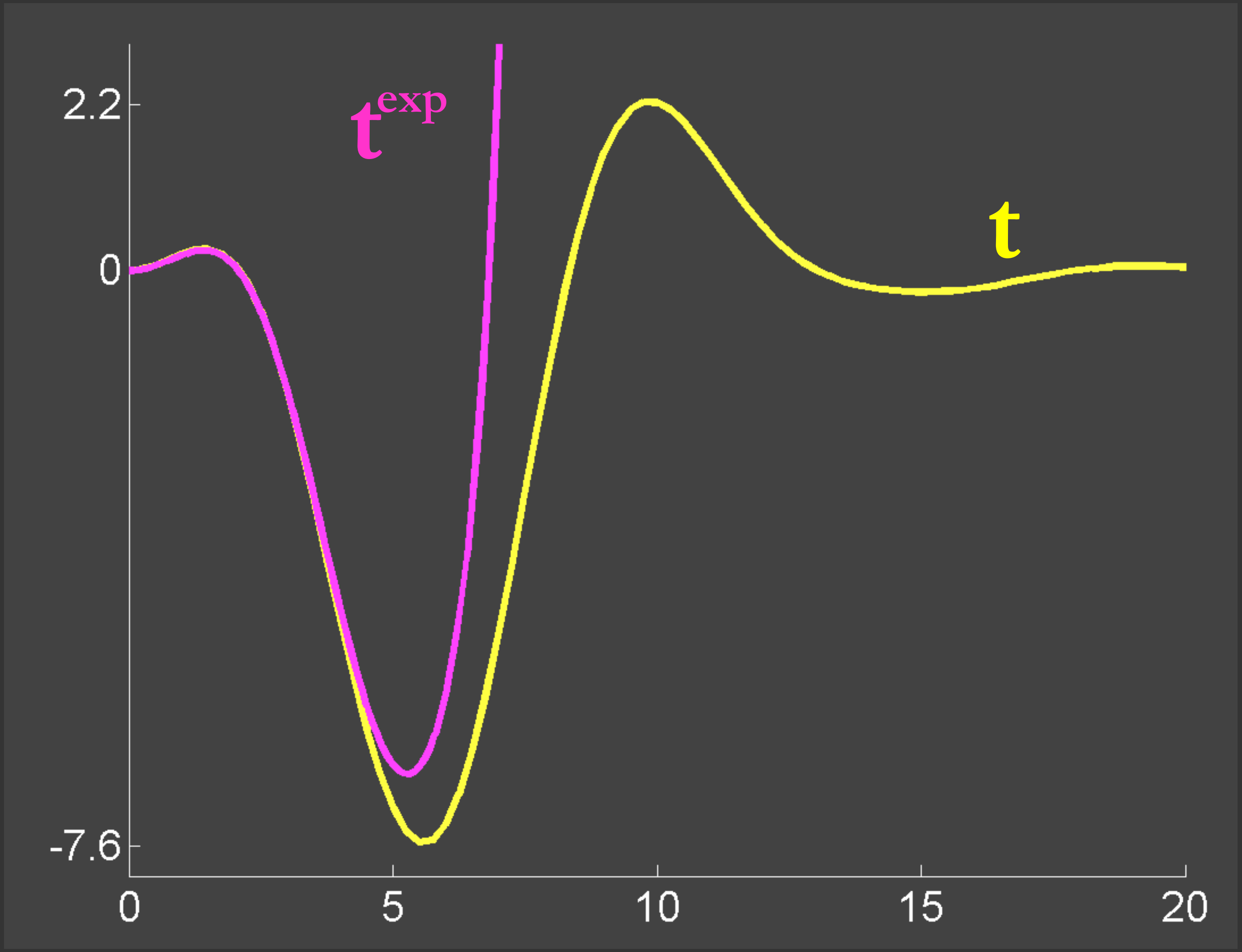
Since our example conductivity is radially symmetric, the D-to-N map is diagonal in the trigonometric basis on the unit circle.

On the left we have λ(n)-n for n=1,…,32, where λ(n) is an eigenvalue of the D-to-N map.

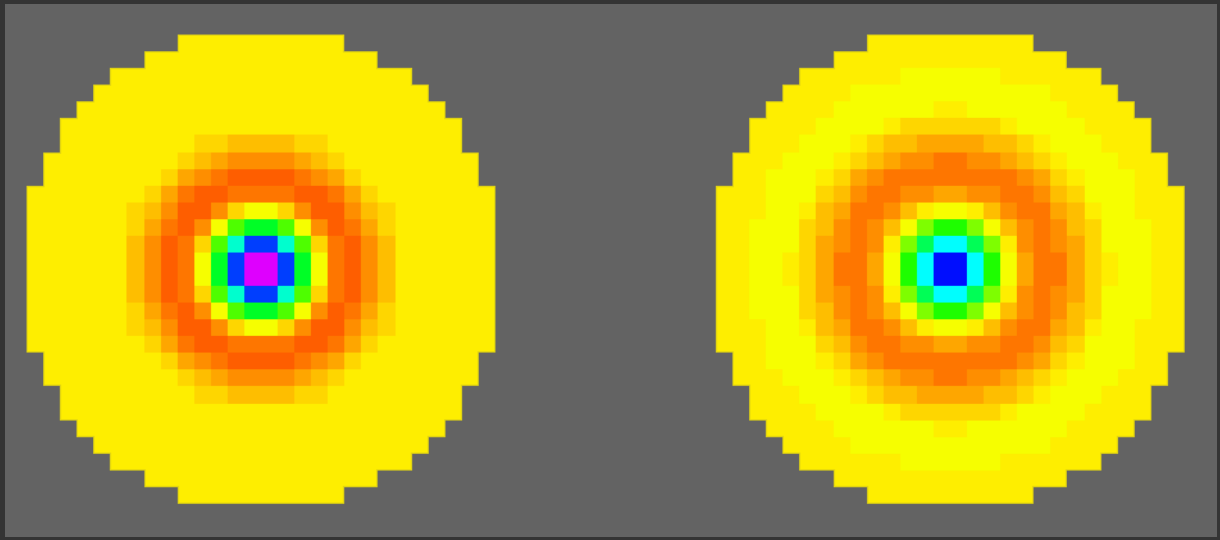
To simulate noise, we add random errors to all elements of the diagonal matrix representing the D-to-N map and compute

$$\mathbf{t}^{\mathrm{exp}}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}\,d\sigma(x).$$

Approximate scattering transform

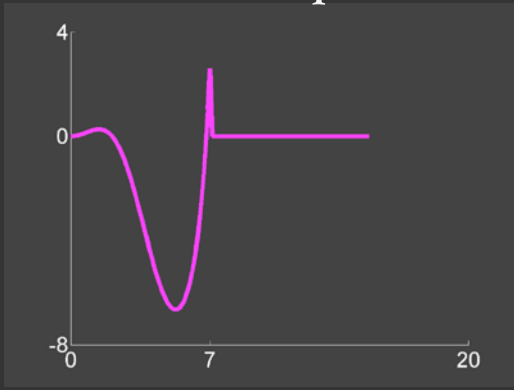


Reconstruction from noisy data

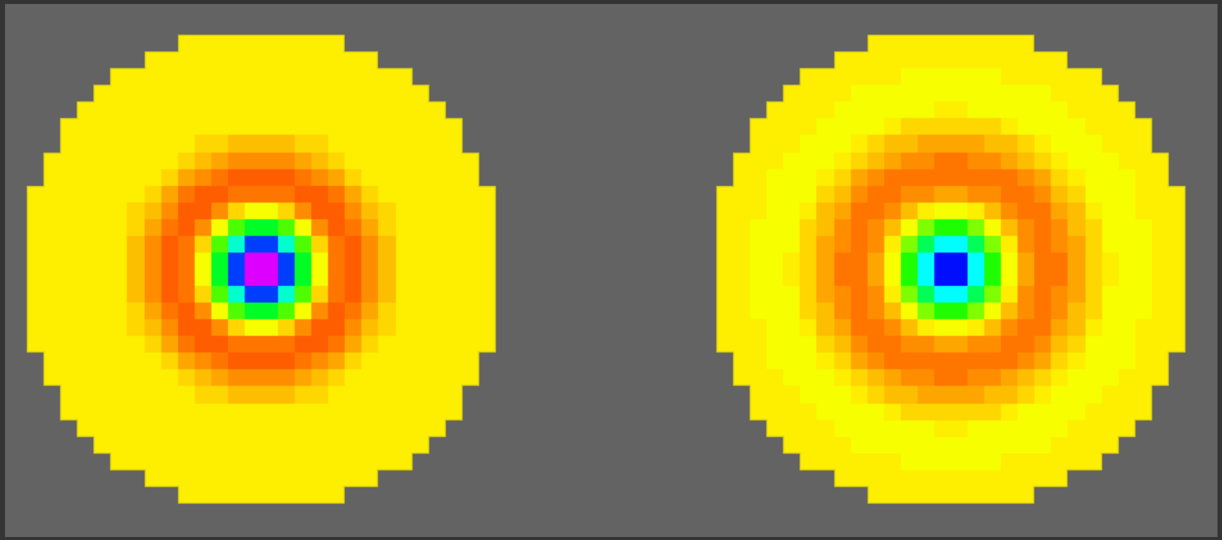


Original　　　Reconstruction

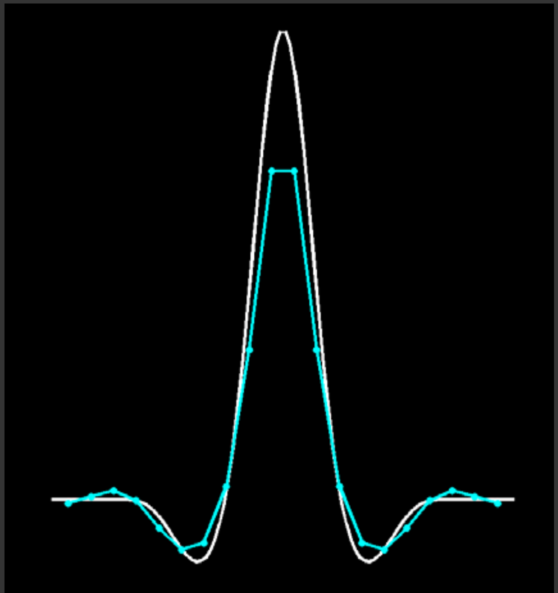
Truncated Texp



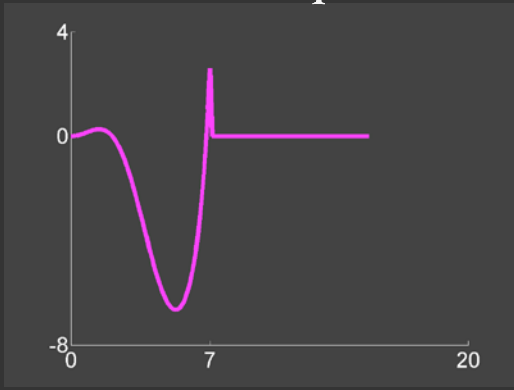
Reconstruction from noisy data



Original Reconstruction

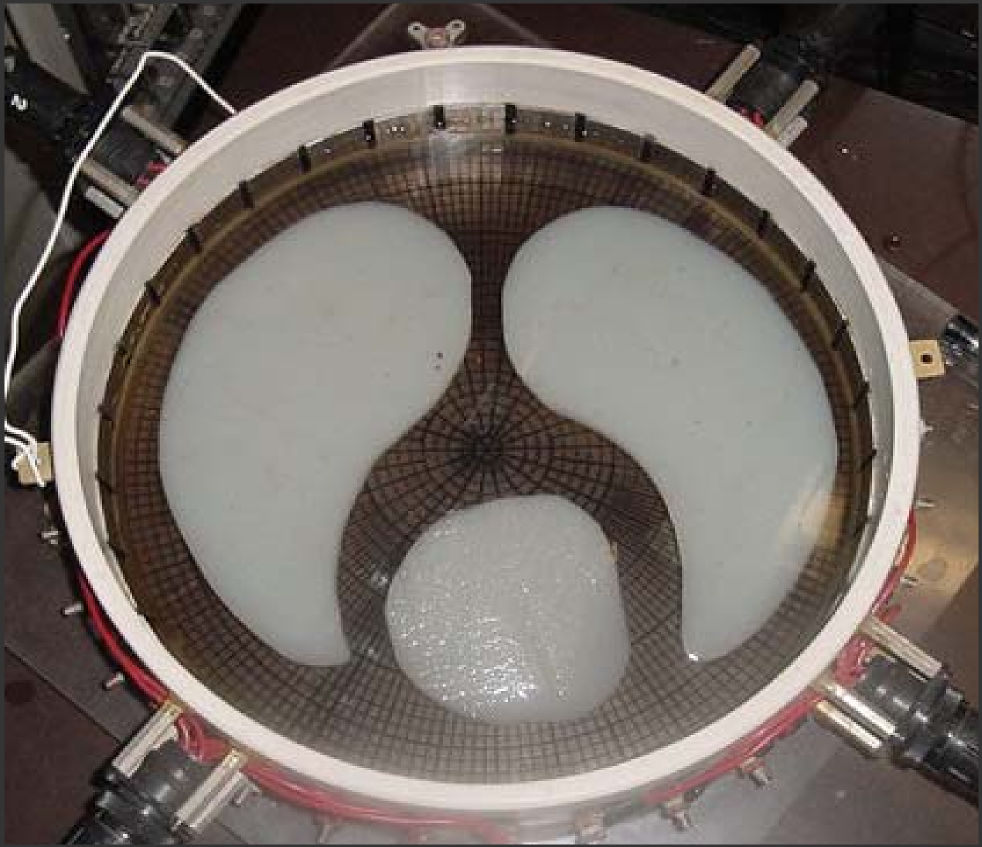


Truncated Texp

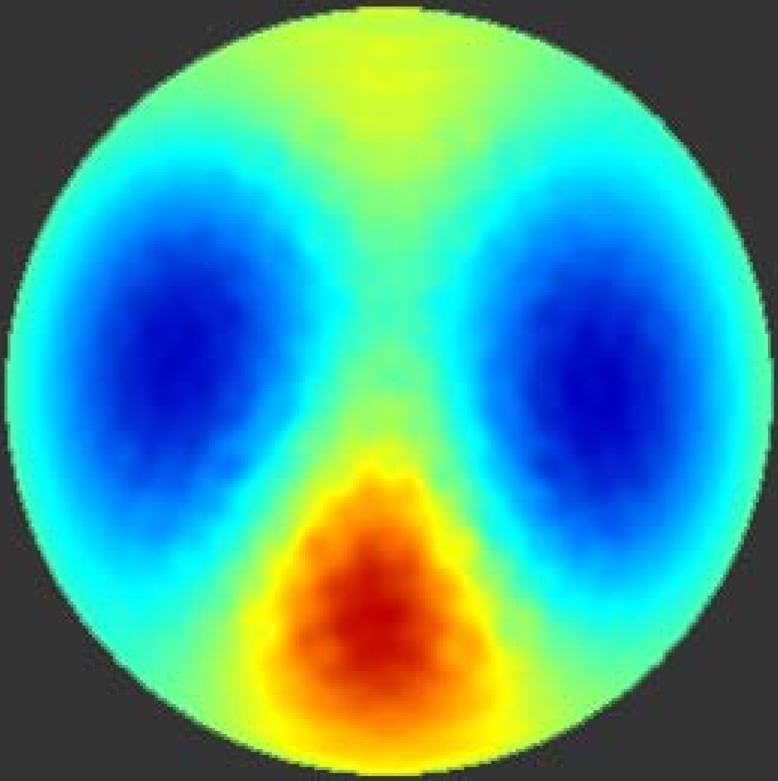


Reconstruction from real data

Phantom tank with saline and agar



Reconstruction



Relative error in conductivity values 23% (lung) and 12% (heart).

Dynamical range of the reconstruction is 94% of the true range.

Future work

- Theory for realistic (discontinuous) conductivities
- Removing the requirement γ=1 near boundary
- Proper modeling of electrode measurements
- Justification of approximations to t(k)
- Analysis of effect of truncation
- Real-time computation
- 3D algorithm

Publications

- Siltanen, Mueller and Isaacson: *An implementation of the reconstruction algorithm of A Nachman for the 2D inverse conductivity problem*,
  Inverse Problems 16 (2000), pp. 681-699.
  Erratum, Inverse Problems 17 (2001), pp. 1561-1563.

- Siltanen, Mueller and Isaacson: *Reconstruction of high contrast 2-D conductivities by the algorithm of A. Nachman*,
  Contemporary Mathematics 278 (2001), pp. 241-254.

- Siltanen, Mueller and Isaacson: *A direct reconstruction algorithm for electrical impedance tomography,*
  IEEE Transactions on Medical Imaging 21 (2002), pp. 555-559.

- Mueller and Siltanen: *Direct reconstructions of conductivities from boundary measurements,*
  to appear in SIAM Journal of Scientific Computation.

- Isaacson, Mueller, Newell and Siltanen: *Reconstructions of chest phantoms by the D-bar method for Electrical Impedance Tomography,*
  in preparation.

- Knudsen, Mueller and Siltanen: *FIST: a fast inverse scattering transform solver,*
  in preparation.