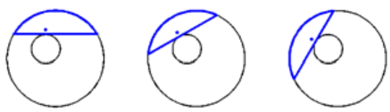
Numerical solution of the inverse problem of ECG using Faddeev's Green function



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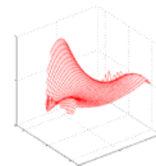
Outline of the talk

$$u(y) = \lim_{\tau\to\infty} u_\tau(y)$$

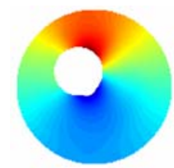
1. Background

2. Computation

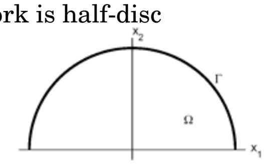
3. Harmonic example



4. Inverse problem of ECG



Geometry of the Cauchy problem considered in this work is half-disc



$$(-\Delta + V)u = 0 \qquad \text{in } \Omega,$$

$$u \in H^2(\Omega), \quad V(x) \in L^\infty(\Omega)$$

Problem: Recover u from its Cauchy data

$$(u|_\Gamma, \frac{\partial u}{\partial \nu}|_\Gamma)$$

Cauchy problem has many applications

Recovering temperature distribution inside a known physical body from surface temperature and heat flux

Recovering voltage potential on the heart from voltage measurements on the skin

Several numerical solution methods for the Cauchy problem have been presented

Klibanov and Santosa 1991
(based on Lattés and Lions 1969, Lavrentyev 1956)
Kabanikhin and Karchevsky 1995
Leitão 2000 (based on Maz'ya 1991)
Háo and Lesnic 2000 (V=0)
Berntsson and Eldén 2001 (V=0)
Cheng, Hon, Wei and Yamamoto 2001 (V=0)

We present a new non-iterative solution method for V≠0 that does not involve solution of boundary value problems.

We solve the Cauchy problem using Faddeev's Green function

Ikehata [2001] proved that

$$u(y) = \lim_{\tau\to\infty} u_\tau(y)$$

where

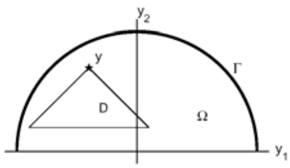
$$u_\tau(y) := \frac{2\tau^2 e^{-i\tau y_1}}{C_D} \int_\Gamma \left(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u\right) d\sigma(x)$$

Computation with finite τ provides a regularised reconstruction method.

The reconstruction method uses Faddeev's exponentially growing solutions

$$-\Delta v''_\tau + \tilde{V} v''_\tau = \chi_D e^{\tau(x_2-y_2)} e^{i\tau x_1} \text{ in } \mathbb{R}^2$$

$$v_\tau = v''_\tau|_\Omega$$



The proof is based on Green's formula

$$\begin{aligned}\int_D e^{\tau(x_2-y_2)} e^{i\tau x_1} u(x)dx &= -\int_\Omega (\Delta v_\tau)u + \int_\Omega V v_\tau u \\ &= \int_{\partial\Omega} (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u) \\ &= (\int_\Gamma + \int_{\partial\Omega\setminus\Gamma})(\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u)\end{aligned}$$

$$\int_D e^{\tau(x_2-y_2)} e^{i\tau x_1} u(x)dx \sim \frac{C_D}{2\tau^2} e^{i\tau y_1} u(y) \quad \text{as } \tau \to \infty$$

$$\int_{\partial\Omega\setminus\Gamma} (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u) \to 0 \quad \text{exponentially as } \tau \to \infty$$

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Numerical implementation of the method is divided into 5 problems

$$u_\tau(y) = \frac{2\tau^2 e^{-i\tau y_1}}{C_D} \int_\Gamma (\frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u) d\sigma(x)$$

1. Integration on Γ
2. Choosing the triangle D=D(y)
3. How to choose τ?
4. Computing exponentially growing solutions
5. Computing normal derivatives of exponentially growing solutions

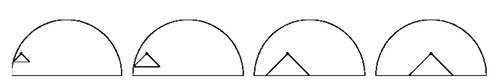
Implementation Step 1: Numerical integration on Γ

We choose a set of integration quadrature points and weights on Γ. Then integral of *f(x)* over Γ is approximated by the following sum:

$$\int_\Gamma f d\sigma \approx \sum_{k=1}^{K} w^{(k)} f(x^{(k)})$$

Implementation Step 2: Choosing the triangle D

We take D to be the largest possible triangular patch such that D is a subset of Ω and the base of D is twice its height.



The choice is based on theory: Maximizing |CD| minimizes error. We always have |CD|<=2, and with this choice we have $C_D = 2$

Implementation Step 3: Choosing the regularization parameter τ

If τ is too small, the recovered solution is not close to the true solution

If τ is too large, noise will be amplified

Choice of τ depends on the Cauchy data, a priori bound on u in Ω and noise level; we do not have a general practical choice

Implementation Step 4: Computing exponentially growing solutions

Piecewise smooth V: solve

$$w'_\tau(x) + g_\tau * (\tilde{V} w'_\tau) = g_\tau * \chi_D$$

with an adaptation of Vainikko's Lippmann-Schwinger equation solver [Mueller-S 2003] based on FFT and GMRES

Case V=0: evaluate convolution

$$w'_\tau(x) = g_\tau * \chi_D$$

Exponentially growing solutions are defined using Faddeev's fundamental solution

The function

$$g_\tau(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\tau(\xi_1 - i\xi_2)} d\xi$$

satisfies

$$(-\Delta - 2i\tau(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}))g_\tau(x) = \delta(x)$$

Implementation Step 5: normal derivatives of exponentially growing solutions

Piecewise smooth V:

$$\frac{\partial v_\tau}{\partial \nu} = \frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi} \times \left[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * (\tilde{V} w'_\tau - \chi_D)\right]$$

Case V=0:

$$\frac{\partial v_\tau}{\partial \nu} = -\frac{e^{\tau(x_2-y_2)}e^{i\tau x_1}}{4\pi} \times \left[(\nu_1(\frac{1}{\bar{x}} + \frac{e^{-i2\tau x_1}}{x}) + \nu_2(\frac{1}{i\bar{x}} - \frac{e^{-i2\tau x_1}}{ix})) * \chi_D\right]$$

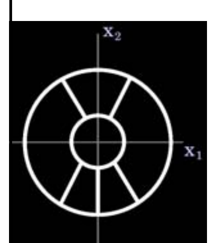
Computing exponentially growing solutions and their derivatives is reduced to evaluating

$$g_1(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2(\xi_1 - i\xi_2)} d\xi$$

This is due to the symmetry

$$g_\tau(x) = g_1(\tau x)$$

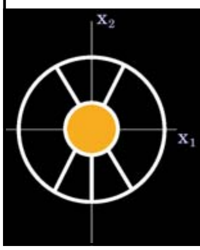
Computation of Faddeev's fundamental solution is divided into 7 cases



The x-plane is divided into 7 disjoint regions, each leading to a different algorithm

Case 1: For x in the disc of radius 5.5 we use a formula by [Boiti et al 1987]

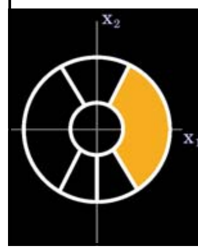
$$g_1(x) = -\frac{e^{-ix}}{4\pi}\left(2\gamma+\log|x|^2+\sum_{n=1}^{\infty}\frac{(ix)^n+(-i\bar{x})^n}{n n!}\right)$$



Here γ is Euler's constant

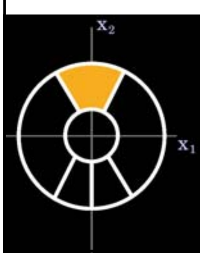
Case 2: We write g1 as a formula containing a rapidly converging 1-D integral

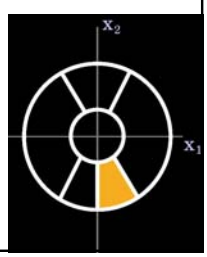
$$g_1(x) = \frac{e^{-ix_1}}{2\pi}\mathrm{Re}\Big[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \frac{(N+1)!e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}}dt\Big]$$



Cases 3 and 4: The integral in case 2 is modified using the residue theorem

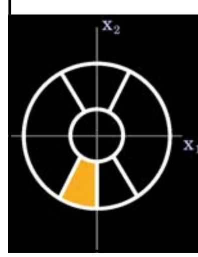
$$-i\int_0^\infty \frac{e^{-x_2 s+ix_1 s}}{(-is-i)^{N+2}}ds \qquad (1+i)\int_0^\infty \frac{e^{-is(x_2+x_1)+s(x_2-x_1)}}{(s+is-i)^{N+2}}ds$$

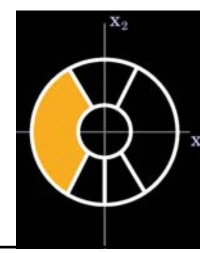




Cases 5 and 6 are reduced to cases 4 and 2 using the following symmetry:

$$g_1(-x_1,x_2)=\overline{g_1(x_1,x_2)}$$





Case 7: for |x|>25 we ignore the integral in case 2 and use the truncated sum

$$g_1(x) \approx \frac{e^{-ix_1}}{2\pi}\mathrm{Re}\left[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}}\right]$$

We get 8 correct digits

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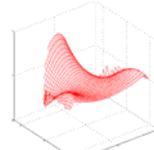


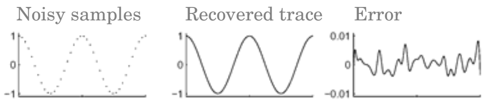
4. Inverse problem of ECG

Example 1: The harmonic case V=0

Choose harmonic function u:

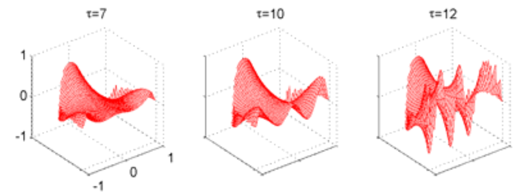
$$u(y_1, y_2) = \mathrm{Re}(z^4), \qquad z = y_1 + iy_2$$

Produce computer simulated noisy Cauchy data:



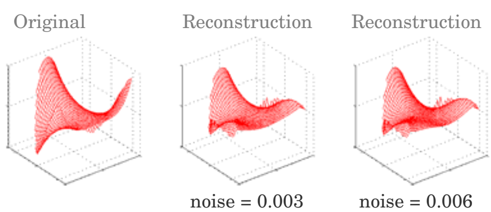
Standard deviation of Gaussian noise = 0.003

Example 1: Solution with noisy Cauchy data



Small τ gives good reconstruction deep inside Ω,
large τ gives good reconstruction near Γ

Example 1: Solution with two noise levels illustrates the stability of our method



Here τ is chosen as function of y

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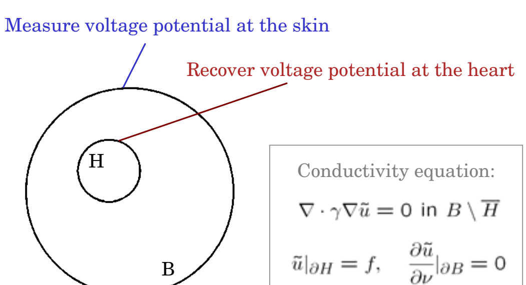
Example 2: Inverse potential problem of electrocardiography

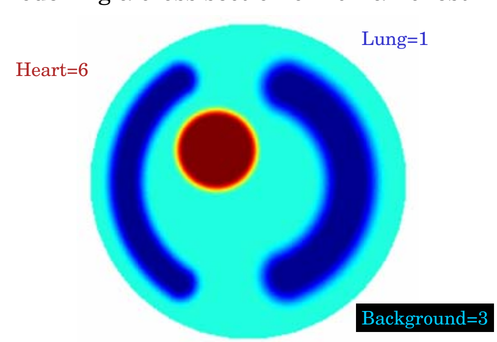


Conductivity equation:

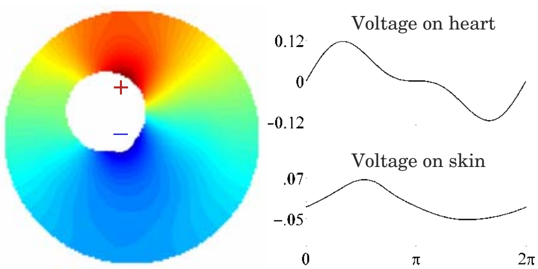
$$\nabla \cdot \gamma \nabla \tilde{u} = 0 \text{ in } B \setminus \overline{H}$$

$$\tilde{u}|_{\partial H} = f, \qquad \frac{\partial \tilde{u}}{\partial \nu}|_{\partial B} = 0$$

Example 2: We construct a conductivity modelling a cross section of human chest



Example 2: We compute voltage potential outside heart by Finite Element Method



Example 2: We transform the conductivity equation to the Schrödinger equation

Conductivity equation:

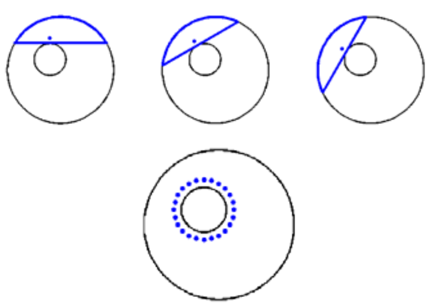
$$\nabla\cdot\gamma\nabla\tilde{u} = 0 \text{ in } B\backslash\overline{H}, \quad \tilde{u}|_{\partial H} = f, \quad \frac{\partial\tilde{u}}{\partial\nu}|_{\partial B} = 0$$

Define $u = \gamma^{1/2}\tilde{u}, \qquad V(x) = \frac{\Delta\sqrt{\gamma(x)}}{\sqrt{\gamma(x)}}$

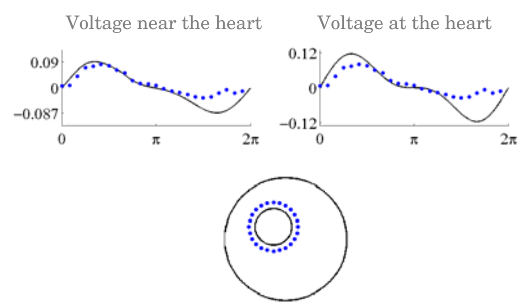
Then u satisfies the equation

$$(-\Delta + V)u = 0 \text{ in } B\backslash\overline{H}, \quad u|_{\partial B} = \sqrt{3}\tilde{u}|_{\partial B}, \quad \frac{\partial u}{\partial\nu}|_{\partial B} = 0$$

Example 2: We recover voltage near the heart by rotating the canonical geometry



Example 2: We reconstructed voltage near the heart from ECG data with 2% noise



Conclusion: we presented a new numerical solution method of the Cauchy problem

Our method is computationally fast:
no need to solve direct problems

In the inverse problem of ECG,
our method has 20% average relative error
on the anterior surface of the heart

Future work: 3D problems