The d-bar reconstruction method for electrical impedance tomography

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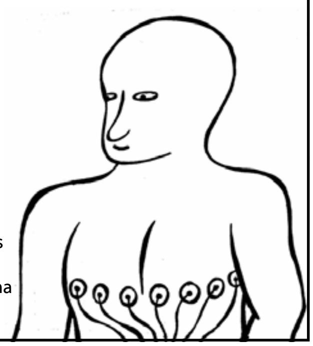
Electrical impedance tomography (EIT) is an emerging medical imaging method

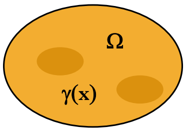
Feed electric currents
through electrodes,
measure voltages

Reconstruct the image of
electric conductivity in
a two-dimensional slice

Applications include:
monitoring heart and lungs
of SARS patients,
detecting pulmonary edema
(swollen lungs)



The inverse conductivity problem of Calderón is the mathematical model of EIT



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

$$\begin{aligned}\nabla\cdot\gamma\nabla u &= 0 \quad \text{in } \Omega,\\ u &= f \quad \text{on } \partial\Omega.\end{aligned}$$

Given the Dirichlet-to-Neumann map,
how to reconstruct the conductivity?

The reconstruction problem is nonlinear and ill-posed.

Theoretical development of the d-bar method can be found in these studies

- **1980 Calderón**
- **1987 Sylvester and Uhlmann**
- **1987 R G Novikov**
- **1988 Nachman**
- **1996 Nachman**
  - uniqueness proof for the 2D problem
- **1997 Brown and Uhlmann**
  - generalization of Nachman 1996
- **2003 Knudsen and Tamasan**
  - reconstruction for Brown-Uhlmann
- **2003 Astala and Päivärinta**
  - uniqueness for essentially bounded conductivities

Nachman's 1996 proof consists of two steps:

$$\Lambda_\gamma \longrightarrow \mathbf{t} \longrightarrow \gamma$$

The intermediate object t is a complex-valued function called *scattering transform* and defined as follows for complex k:

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)\,dx$$

$$q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$$

$$(-\Delta + q)\psi(\cdot,k) = 0$$

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}$$

Step 1: from DN map to scattering transform

Solve traces of ψ from the boundary integral equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

where the single-layer operator has Faddeev Green's function as kernel.

Compute the scattering transform as

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(x,k)\,d\sigma(x).$$

Step 2: from scattering transform to γ

Define $\mu(x,k) = e^{-ikx}\psi(x,k)$

Then the following d-bar equation holds:

$$\frac{\partial}{\partial\bar{k}}\mu(x,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

The d-bar equation has a unique solution for all x.
The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k\to 0}\mu(x,k).$$

Practical step 1: compute stable approximation to scattering transform from noisy data

With noisy data, we cannot solve equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

so we introduce the approximate scattering transform:

$$\mathbf{t}^{\mathrm{exp}}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}\,d\sigma(x)$$

Further, we regularize the computation by truncation:

$$\mathbf{t}_R^{\mathrm{exp}}(k) := \begin{cases} \mathbf{t}^{\mathrm{exp}}(k), & |k| < R, \\ 0, & |k| \geq R. \end{cases}$$

Practical step 2: solve the d-bar equation with approximate kernel

Write the approximate dbar equation

$$\frac{\partial}{\partial\bar{k}}\mu_R(x,k) = \frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}$$

In integral form:

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

This Lippmann-Schwinger -type equation can be solved numerically with modified Vainikko's algorithm. Then

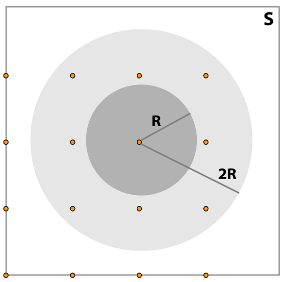
$$\gamma_R^{1/2}(x) = \mu_R(x,0).$$

We define a grid for Vainikko's algorithm

We define a grid with
(2^m x 2^m) points
in the square S
in the k-plane.

Here m=2, in practice
typically m=8.

This grid is suitable for
the use of Fast Fourier
Transform (FFT).



The d-bar equation can be solved in a bounded domain using periodization

Instead of the d-bar equation

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

valid in the k-plane, we solve the S-periodic equation

$$[I + \frac{1}{\pi k} * (T_R \cdot {}^{-})]w = 1$$

$$T_R(k) = -\frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}$$

The d-bar equation is also solved since it can be shown that

$$\mu_R(x,\cdot)|_{B(0,R)} = w|_{B(0,R)}$$

Vainikko's method is based on iterative solution of linear equations

We can solve the discretized equation

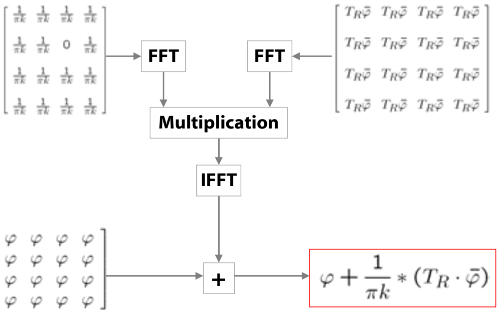
$$[I+\frac{1}{\pi k}*(T_R\cdot{}^{-})]w=1$$

using the iterative GMRES method.
We just need to implement the formula

$$\varphi+\frac{1}{\pi k}*(T_R\cdot\bar{\varphi})$$

for a function ϕ given on the grid points.

The convolution is effectively implemented using Fast Fourier Transform (FFT)



$$\begin{bmatrix} \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\ \frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\ \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\ \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \end{bmatrix} \qquad \begin{bmatrix} T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} \\ T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} \\ T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} \\ T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} \end{bmatrix}$$

FFT

FFT

Multiplication

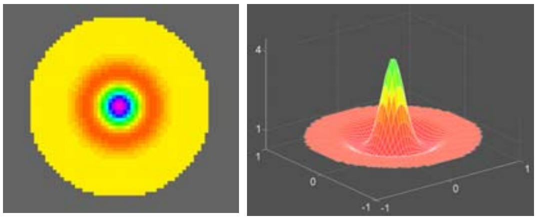
IFFT

$$\begin{bmatrix} \varphi & \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi & \varphi \end{bmatrix}$$

+

$$\varphi+\frac{1}{\pi k}*(T_R\cdot\bar{\varphi})$$

We construct a symmetric example conductivity

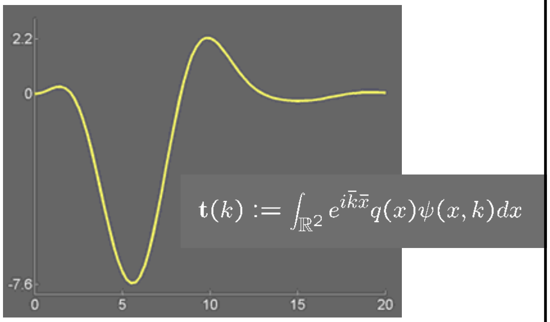


Minimum and maximum values of γ are 0.6 and 4

Near the boundary γ equals 1

Here γ is 4 times continuously differentiable

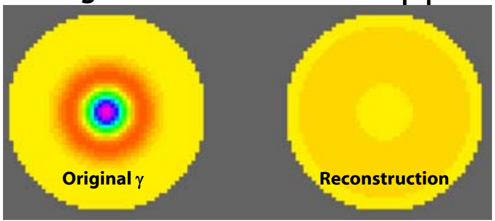
We compute the scattering transform of γ numerically from the definition



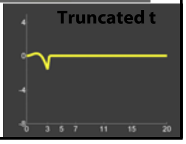
$$\mathrm{t}(k):=\int_{\mathbb{R}^2}e^{i\bar{k}\bar{x}}q(x)\psi(x,k)dx$$

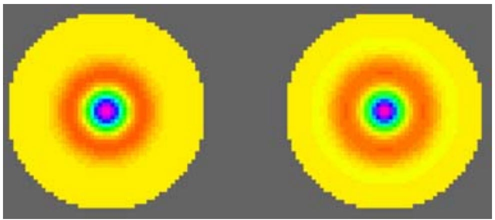
Profile of t (radial symmetry of q implies that of t)

We compute the reconstruction with scattering transform truncated at |k|=R

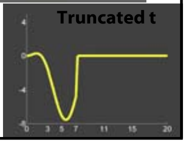


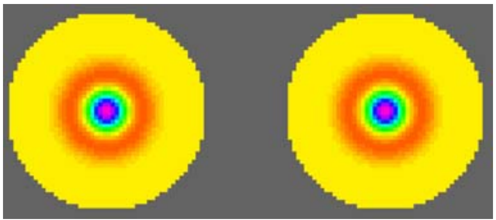
R=3



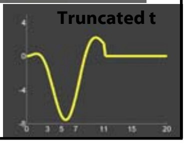


R=7

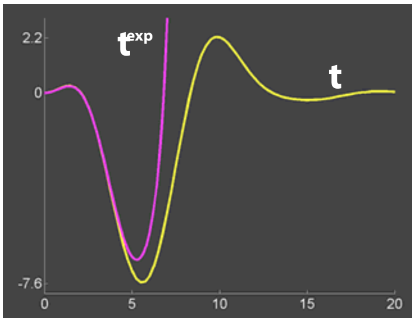




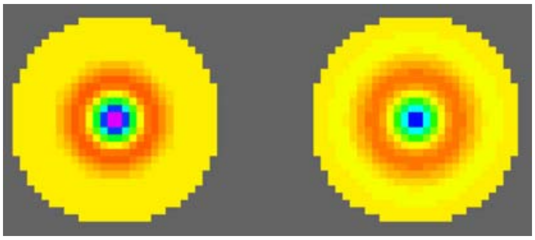
R=11



We compute the approximate scattering transform from noisy data

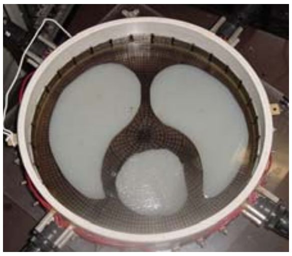


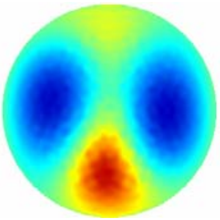
Reconstruction from noisy data



Original **Reconstruction**

Reconstruction from data measured from a chest phantom consisting of saline and agar





Relative error 23% (lung) and 12% (heart).
Dynamical range of is 94% of the true range.

In the future we face the following challenges:

Reconstruction method for discontinuous conductivities

Removing the requirement $\gamma=1$ near boundary

Modelling electrode measurements properly

Analysing the approximation and truncation of t(k)

Developing a 3D algorithm

Details are available in these articles:

Siltanen, Mueller and Isaacson: *An implementation of the reconstruction algorithm of A Nachman for the 2D inverse conductivity problem,*
Inverse Problems 16 (2000), pp. 681-699.
Erratum, Inverse Problems 17 (2001), pp. 1561-1563.

Siltanen, Mueller and Isaacson: *A direct reconstruction algorithm for electrical impedance tomography,*
IEEE Transactions on Medical Imaging 21 (2002), pp. 555-559.

Mueller and Siltanen: *Direct reconstructions of conductivities from boundary measurements,*
SIAM Journal of Scientific Computation 24 (2003), pp. 1232–1266

Isaacson, Mueller, Newell and Siltanen: *Reconstructions of chest phantoms by the d-bar method for electrical impedance tomography,* submitted.

Knudsen, Mueller and Siltanen: *Numerical solution method for the d-bar equation in the plane,* submitted.