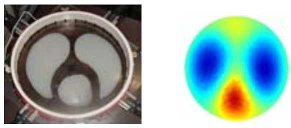
The d-bar reconstruction method for electrical impedance tomography



Samuli Siltanen (samuli.siltanen@iki.fi)
Instrumentarium Imaging (part of GE Healthcare), Finland

Istituto per le applicazioni del calcolo, Firenze, 27.9.2004

This is a joint work with

David Isaacson
Rensselaer Polytechnic Institute, USA

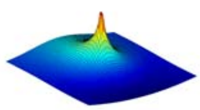
Kim Knudsen
Aalborg University, Denmark

Jennifer Mueller
Colorado State University, USA

Jon Newell
Rensselaer Polytechnic Institute, USA

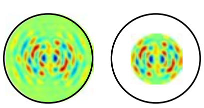
$$\Lambda_\gamma \rightarrow \mathbf{t} \rightarrow \gamma$$

1. Theory of d-bar method

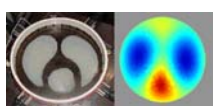


2. Numerical scattering transform

3. Regularized inversion



4. Reconstructions

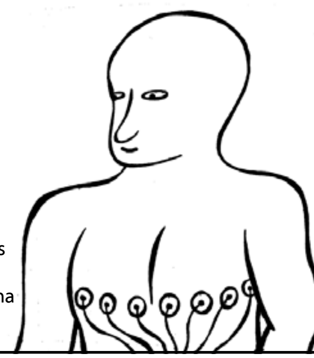


Electrical impedance tomography (EIT) is an emerging medical imaging method

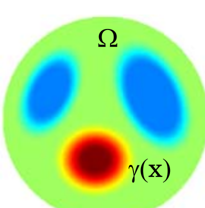
Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

Applications include:
monitoring heart and lungs of unconscious patients, detecting pulmonary edema (swollen lungs)



The inverse conductivity problem of Calderón is the mathematical model of EIT



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

Given the Dirichlet-to-Neumann map, how to reconstruct the conductivity?

The reconstruction problem is nonlinear and ill-posed.

EIT reconstruction algorithms can be divided roughly into the following classes:

Linearization

Iterative output least-squares methods

Statistical inversion

The inverse scattering approach, or **d-bar method**

Theoretical development of the d-bar method can be found in these studies

1980 Calderón
1987 Sylvester and Uhlmann
1987 R G Novikov
1988 Nachman
1996 Nachman
1997 Liu
1997 Brown and Uhlmann
2000 Francini
2001 Barceló, Barceló and Ruiz
2003 Knudsen and Tamasan
2003 Mueller and S
2003 Astala and Päivärinta

Throughout this talk we use these definitions:

Let $\Omega \subset \mathbb{R}^2$ be the unit disc.

Let $\gamma \in C^4(\Omega)$ be strictly positive.

Assume that $\gamma \equiv 1$ near $\partial\Omega$.

Define $q = \dfrac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \in C^2(\mathbb{R}^2)$ by zero extension.

Nachman's 1996 proof consists of two steps:

$$\Lambda_\gamma \longrightarrow \mathbf{t} \longrightarrow \gamma$$

The intermediate object t is a complex-valued function called *scattering transform* and defined as follows for complex k:

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

$$q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$$

$$(-\Delta + q)\psi(\cdot,k) = 0$$

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}$$

Step 1: from DN map to scattering transform

Solve traces of ψ from the boundary integral equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

where the single-layer operator has Faddeev Green's function as kernel.

Compute the scattering transform as

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(x,k)d\sigma(x).$$

Step 2: from scattering transform to γ

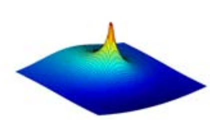
Define $\mu(x,k) = e^{-ikx}\psi(x,k)$

Then the following d-bar equation holds:

$$\frac{\partial}{\partial\bar{k}}\mu(x,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

The d-bar equation has a unique solution for all x.
The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k\to 0}\mu(x,k).$$



1. Theory of d-bar method	2. Numerical scattering transform
3. Regularized inversion	4. Reconstructions

In dimension two, the definition of Faddeev's Green function is as follows:

Let $k \in \mathbb{C} \setminus 0$. Define $g_k : \mathbb{R}^2 \to \mathbb{C}$ as

$$g_k(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} d\xi_1 d\xi_2.$$

Then g_k satisfies

$$(-\Delta - 4ik\overline{\partial})g_k = \delta,$$

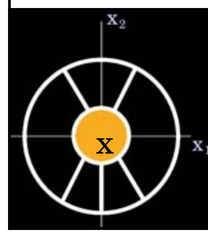
where

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Note the symmetry $g_k(x) = g_1(kx)$.

For x in the unit disc we compute $g_1(x)$ with a formula by [Boiti et al 1987]

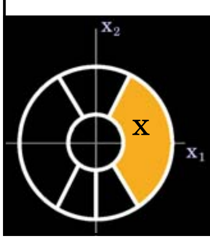
$$g_1(x) = -\frac{e^{-ix}}{4\pi}\left(2\gamma + \log|x|^2 + \sum_{n=1}^{\infty} \frac{(ix)^n + (-i\bar{x})^n}{nn!}\right)$$



Here γ is Euler's constant

We write $g_1(x)$ as a formula containing a rapidly converging one-dimensional integral

$$g_1(x) = \frac{e^{-ix_1}}{2\pi}\mathrm{Re}\Big[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \frac{(N+1)!e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}}dt\Big]$$



For other domains we use residue calculus and reflectional symmetry

Exponentially growing solutions in two dimensions

Let $q \in L^\infty(\mathbb{R}^2)$ be compactly supported and

$$(-\Delta + q)\psi(\cdot, k) = 0 \qquad \text{in } \mathbb{R}^2,$$

where $k \in \mathbb{C} \setminus 0$ and

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}$$

in the sense that

$$e^{-ikx}\psi(x,k) - 1 \in W^{1,p}(\mathbb{R}^2)$$

for some $p > 2$.

The solutions exist *for conductivity-type potentials* by [Nachman 1996].

Exponentially growing solutions are constructed using Faddeev's Green function

Let $k \in \mathbb{C} \setminus 0$ and denote

$$\mu(x,k) := e^{-ikx}\psi(x,k).$$

Then $\mu(\cdot, k)$ is the solution of

$$\mu(x,k) = 1 - \int_{\mathrm{supp}\, q} g_k(x-y)q(y)\mu(y,k)dy.$$

We take a square $S \supset \mathrm{supp}\, q$ and consider

$$w = 1 - \tilde{g}_k * (qw),$$

where $\tilde{g}_k$ is periodic extension of $g_k|_S$ and $*$ is convolution on the torus. It follows that

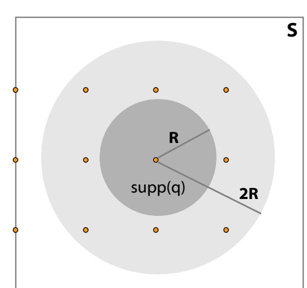
$$\mu|_{\mathrm{supp}\, q} = w|_{\mathrm{supp}\, q}.$$

We solve the periodic Lippmann-Schwinger equation following [Vainikko 2000]

We take a grid on S with (2^m x 2^m) points

Here m=2, in practice typically m=8.

This grid is suitable for the use of Fast Fourier Transform (FFT).



Vainikko's method is based on iterative solution of linear equations

We can solve the discretized periodic equation

$$[I + \tilde{g}_k * (q\cdot\ )]w = 1$$

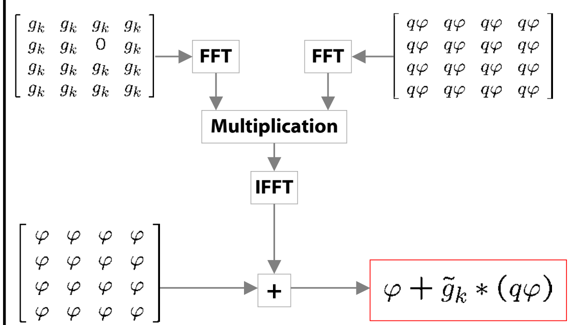
using the iterative GMRES method.

We just need to implement the linear operator

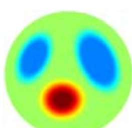
$$\varphi \mapsto \varphi + \tilde{g}_k * (q\varphi)$$

for a function φ given on the grid points.

The linear operator is implemented using Fast Fourier Transform (FFT)

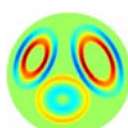


Given a conductivity, we compute the potential and scattering transform

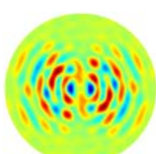


$\gamma(x)$

$|x| \leq 1$



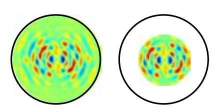
$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$$



$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

$|k| \leq 25$

1. Theory of d-bar method
2. Numerical scattering transform
3. Regularized inversion
4. Reconstructions



Practical step 1: compute stable approximation to scattering transform from noisy data

With noisy data, we cannot solve equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

so we introduce the approximate scattering transform:

$$\mathbf{t}^{\exp}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}d\sigma(x)$$

Further, we regularize the computation by truncation:

$$\mathbf{t}_R^{\exp}(k) := \begin{cases} \mathbf{t}^{\exp}(k), & |k| < R, \\ 0, & |k| \geq R. \end{cases}$$

Practical step 2: solve the d-bar equation with approximate kernel

Write the approximate dbar equation

$$\frac{\partial}{\partial\bar{k}}\mu_R(x,k) = \frac{\mathbf{t}_R^{\exp}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}$$

In integral form:

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathbf{t}_R^{\exp}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

This Lippmann-Schwinger -type equation can be solved numerically with modified Vainikko's algorithm. Then

$$\gamma_R^{1/2}(x) = \mu_R(x,0).$$

Truncation of scattering transform gives asymptotically correct reconstruction

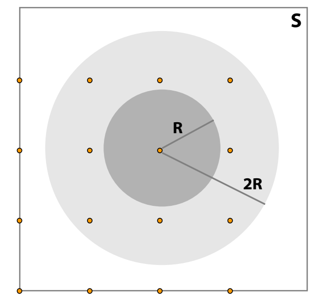
Theorem [Mueller & S 2003].
Let μ_R be the solution of the $\overline{\partial}$ equation

$$\frac{\partial}{\partial \bar{k}} \mu_R(x,k) = \frac{\mathrm{t}_R(k)}{4\pi \bar{k}} e^{-i(kx+\bar{k}\bar{x})} \overline{\mu_R(x,k)}.$$

Then the following estimate holds for large R:

$$\|\sqrt{\gamma} - \mu_R(\cdot,0)\|_{L^\infty(\Omega)} \le C R^{-1}.$$

We explain numerical solution of dbar equation; recall the grid for Vainikko's algorithm



The d-bar equation can be solved in a bounded domain using periodization

Instead of the d-bar equation

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathrm{t}_R^{\exp}(k)}{4\pi \bar{k}} e^{-i(kx+\bar{k}\bar{x})} \overline{\mu_R(x,k)} \right)$$

valid in the k-plane, we solve the S-periodic equation

$$[I + \frac{1}{\pi k} * (T_R \cdot \overline{})] w = 1$$

$$T_R(k) = -\frac{\mathrm{t}_R^{\exp}(k)}{4\pi \bar{k}} e^{-i(kx+\bar{k}\bar{x})}$$

The d-bar equation is also solved since it can be shown that

$$\mu_R(x,\cdot)|_{B(0,R)} = w|_{B(0,R)}$$

Vainikko's method is based on iterative solution of linear equations

We can solve the discretized equation

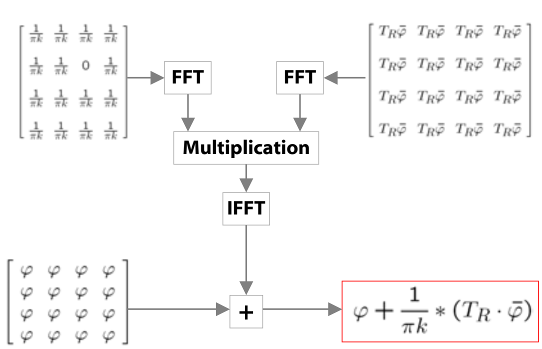
$$[I + \frac{1}{\pi k} * (T_R \cdot \overline{})] w = 1$$

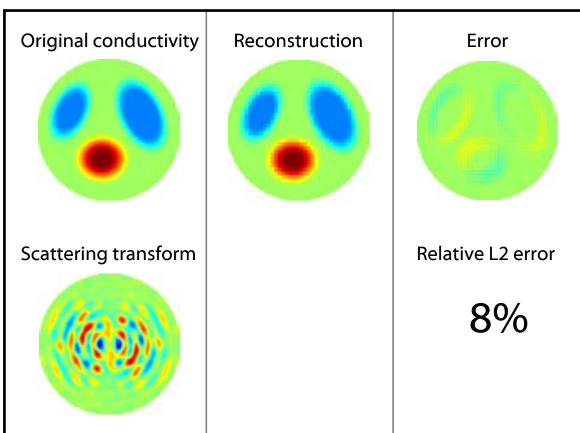
using the iterative GMRES method.
Note: real and imaginary parts must be kept separate!
We just need to implement the formula

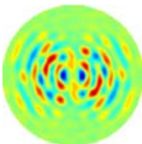
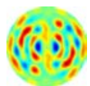
$$\varphi + \frac{1}{\pi k} * (T_R \cdot \bar{\varphi})$$

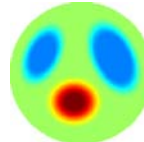
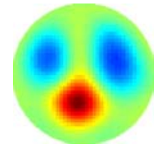
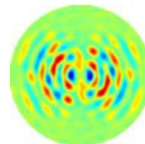
for a function ϕ given on the grid points.

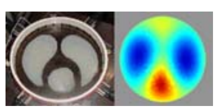
The convolution is effectively implemented using Fast Fourier Transform (FFT)



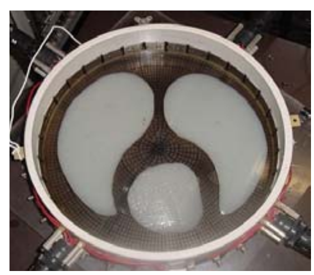


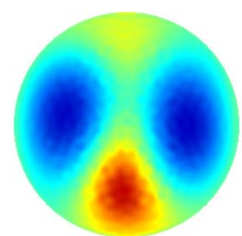
Original conductivity	Reconstruction	Error
		
Scattering transform	Truncation |k|<15	Relative L2 error of reconstruction
		8%

Original conductivity	Reconstruction	Error
		
Scattering transform	Truncation |k|<5	Relative L2 error of reconstruction
		14%

1. Theory of d-bar method	2. Numerical scattering transform
3. Regularized inversion	4. Reconstructions 

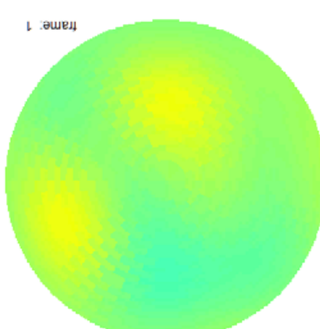
Reconstruction from data measured from a chest phantom consisting of saline and agar





Relative error 23% (lung) and 12% (heart).
Dynamical range of is 94% of the true range.

Reconstruction from data measured from a person



References

Isaacson D, Mueller J L, Newell J and Siltanen S 2004
Reconstructions of chest phantoms by the d-bar method for electrical impedance tomography
IEEE Transactions on Medical Imaging **23**(7), pp. 821-828

Knudsen K, Mueller J L and Siltanen S 2004
Numerical solution method for the dbar-equation in the plane
Journal of Computational Physics **198**(2), pp. 500-517

Mueller J L and Siltanen S 2003
Direct reconstructions of conductivities from boundary measurements
SIAM Journal of Scientific Computation **24**(4), pp. 1232-1266

Siltanen S, Mueller J L and Isaacson D 2000
An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem
Inverse Problems **16**, pp. 681-699;
Erratum Inverse problems **17** , pp. 1561-1563