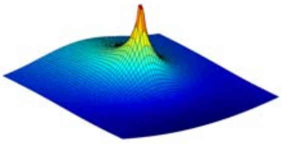
Numerical inversion methods based on Faddeev's Green function



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Perspectives on Inverse Problems
Helsinki, May 31, 2004

This is a joint work with

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Matti Lassas	Helsinki University of Technology, Finland
Jennifer Mueller	Colorado State University, USA
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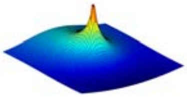
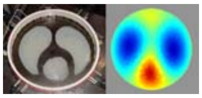


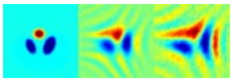
1. Faddeev's Green function

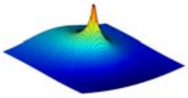


2. D-bar method for EIT

3. Inverse problem of ECG

4. KdV in dimension (2+1)





1. Faddeev's Green function

Faddeev's Green function and exponentially growing solutions

Numerical computation

In dimension two, the definition of Faddeev's Green function is as follows:

Let $k \in \mathbb{C} \setminus 0$. Define $g_k : \mathbb{R}^2 \to \mathbb{C}$ as

$$g_k(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} d\xi_1 d\xi_2.$$

Then g_k satisfies

$$(-\Delta - 4ik\bar{\partial})g_k = \delta,$$

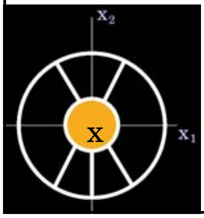
where

$$\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}).$$

Note the symmetry $g_k(x) = g_1(kx)$.

For x in the unit disc we compute $g_1(x)$ with a formula by [Boiti et al 1987]

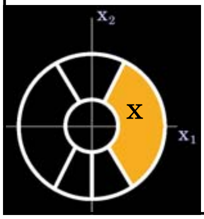
$$g_1(x) = -\frac{e^{-ix}}{4\pi}(2\gamma + \log|x|^2 + \sum_{n=1}^{\infty} \frac{(ix)^n + (-i\bar{x})^n}{nn!})$$



Here γ is Euler's constant

We write g₁(x) as a formula containing a rapidly converging one-dimensional integral

$$g_1(x) = \frac{e^{-ix_1}}{2\pi}\mathrm{Re}[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \\ + \frac{(N+1)!e^{ix_1}}{(-x)^{N+1}}\boxed{\int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}}dt}]$$



For other domains we use residue calculus and reflectional symmetry

Exponentially growing solutions in two dimensions

Let $q \in L^\infty(\mathbb{R}^2)$ be compactly supported and

$$(-\Delta + q)\psi(\cdot, k) = 0 \qquad \text{in } \mathbb{R}^2,$$

where $k \in \mathbb{C} \setminus 0$ and

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}$$

in the sense that

$$e^{-ikx}\psi(x,k) - 1 \in W^{1,p}(\mathbb{R}^2)$$

for some $p > 2$.

The solutions exist *for conductivity-type potentials* by [Nachman 1996].

Exponentially growing solutions are constructed using Faddeev's Green function

Let $k \in \mathbb{C} \setminus 0$ and denote

$$\mu(x,k) := e^{-ikx}\psi(x,k).$$

Then $\mu(\cdot,k)$ is the solution of

$$\mu(x,k) = 1 - \int_{\text{supp}\, q} g_k(x-y)q(y)\mu(y,k)dy.$$

We take a square $S \supset \text{supp}\, q$ and consider

$$w = 1 - \tilde{g}_k * (qw),$$

where $\tilde{g}_k$ is periodic extension of $g_k|_S$ and $*$ is convolution on the torus. It follows that

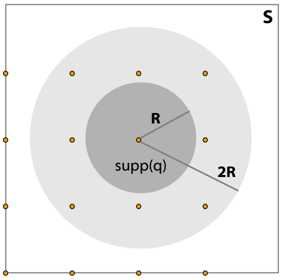
$$\mu|_{\text{supp}\, q} = w|_{\text{supp}\, q}.$$

We solve the periodic Lippmann-Schwinger equation following [Vainikko 2000]

We take a grid on S with (2^m x 2^m) points

Here m=2, in practice typically m=8.

This grid is suitable for the use of Fast Fourier Transform (FFT).



Vainikko's method is based on iterative solution of linear equations

We can solve the discretized periodic equation

$$[I + \tilde{g}_k * (q\cdot\ )]w = 1$$

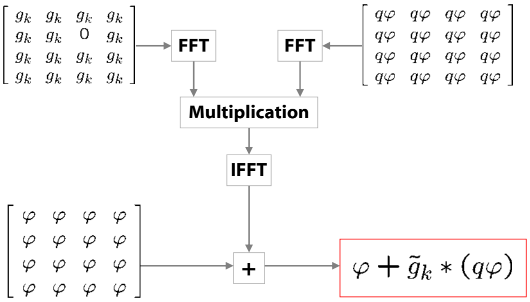
using the iterative GMRES method.

We just need to implement the linear operator

$$\varphi \mapsto \varphi + \tilde{g}_k * (q\varphi)$$

for a function φ given on the grid points.

The linear operator is implemented using Fast Fourier Transform (FFT)



$$\begin{bmatrix} g_k & g_k & g_k & g_k \\ g_k & g_k & 0 & g_k \\ g_k & g_k & g_k & g_k \\ g_k & g_k & g_k & g_k \end{bmatrix}$$

FFT

FFT

$$\begin{bmatrix} q\varphi & q\varphi & q\varphi & q\varphi \\ q\varphi & q\varphi & q\varphi & q\varphi \\ q\varphi & q\varphi & q\varphi & q\varphi \\ q\varphi & q\varphi & q\varphi & q\varphi \end{bmatrix}$$

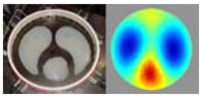
Multiplication

IFFT

$$\begin{bmatrix} \varphi & \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi & \varphi \\ \varphi & \varphi & \varphi & \varphi \end{bmatrix}$$

+

$$\boxed{\varphi + \tilde{g}_k * (q\varphi)}$$



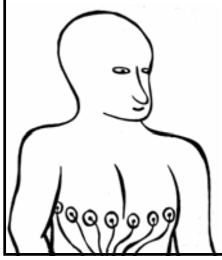
2. D-bar method for EIT

Electrical impedance tomography (EIT) and Calderón’s problem

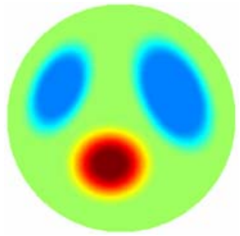
Numerical analysis of the d-bar method

Electrical impedance tomography (EIT) is an emerging medical imaging method

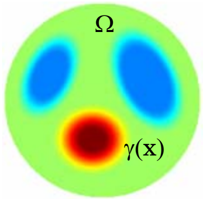
Feed electric currents through electrodes, measure voltages



Reconstruct the image of electric conductivity in a two-dimensional slice



The inverse conductivity problem of Calderón is the mathematical model of EIT



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

Given the Dirichlet-to-Neumann map,
how to reconstruct the conductivity?

The reconstruction problem is nonlinear and ill-posed.

Throughout this talk we use these definitions:

Let $\Omega \subset \mathbb{R}^2$ be the unit disc.

Let $\gamma \in C^4(\Omega)$ be strictly positive.

Assume that $\gamma \equiv 1$ near $\partial\Omega$.

Define $q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \in C^2(\mathbb{R}^2)$ by zero extension.

Theoretical development of the d-bar method for EIT can be found in these studies

1980 Calderón
1987 Sylvester and Uhlmann
1987 R G Novikov
1988 Nachman
1996 Nachman (uniqueness proof for the 2D problem)
1997 Brown and Uhlmann
2003 Knudsen and Tamasan
2003 Astala and Päivärinta

Nachman’s 1996 proof consists of two steps:

$$\Lambda_\gamma \to \mathbf{t} \to \gamma$$

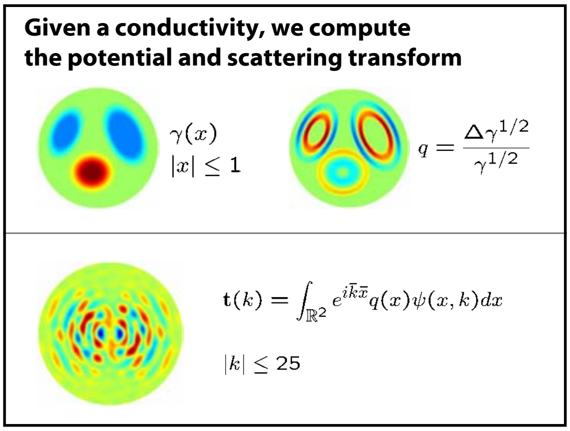
The intermediate object t is a complex-valued function called *scattering transform* and defined as follows for complex k:

$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x) \psi(x,k)\, dx$$

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$$

$$(-\Delta + q)\psi(\cdot, k) = 0$$

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1 + ik_2)(x_1 + ix_2)}$$



Given a conductivity, we compute the potential and scattering transform

$\gamma(x)$

$|x| \leq 1$

$q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$

$t(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$

$|k| \leq 25$

Step two: given scattering transform t, find γ. Theorem [Nachman 1996]

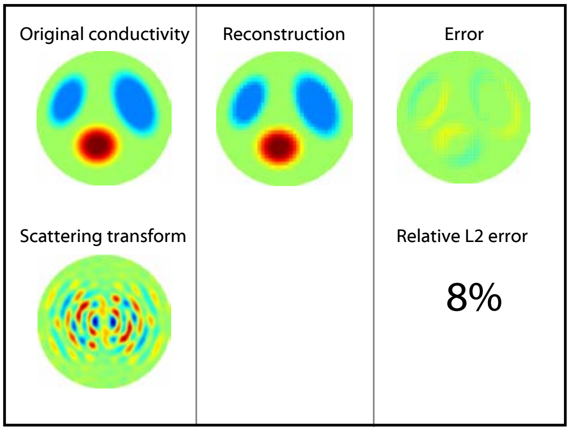
Define $\mu(x,k) = e^{-ikx}\psi(x,k)$.

Then the following d-bar equation holds:

$$\frac{\partial}{\partial \bar{k}}\mu(x,k) = \frac{t(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

The d-bar equation has a unique solution for all x.
The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k\to 0} \mu(x,k).$$



Original conductivity

Reconstruction

Error

Scattering transform

Relative L2 error

8%

Step two in practice: t must be truncated.

Theorem [Mueller & S 2003].
Let μ_R be the solution of the $\bar{\partial}$ equation

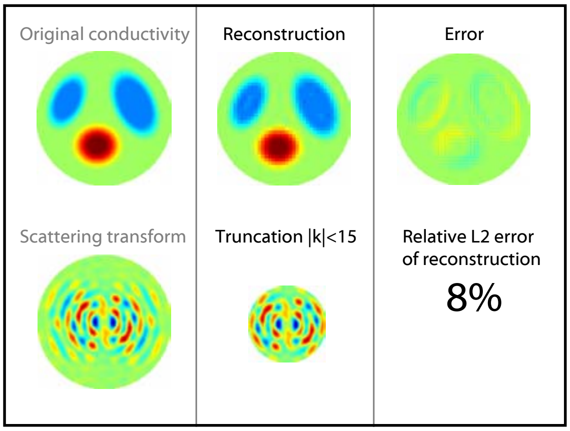
$$\frac{\partial}{\partial \bar{k}}\mu_R(x,k) = \frac{t_R(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)},$$

where t_R is the truncated scattering transform:

$$t_R(k) := \begin{cases} t(k), & |k| < R, \\ 0, & |k| \geq R. \end{cases}$$

Then the following estimate holds for large R:

$$\|\sqrt{\gamma} - \mu_R(\cdot,0)\|_{L^\infty(\Omega)} \leq CR^{-1}.$$



Original conductivity

Reconstruction

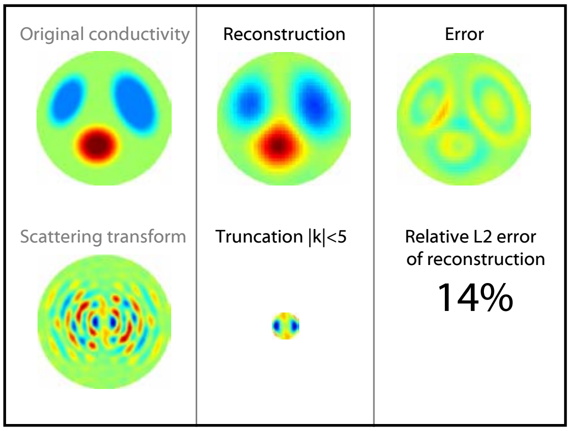
Error

Scattering transform

Truncation |k|<15

Relative L2 error of reconstruction

8%



Original conductivity

Reconstruction

Error

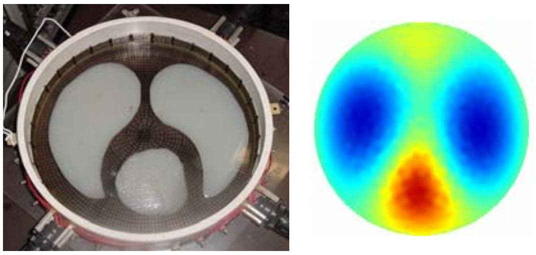
Scattering transform

Truncation |k|<5

Relative L2 error of reconstruction

14%

Reconstruction from data measured from a chest phantom consisting of saline and agar



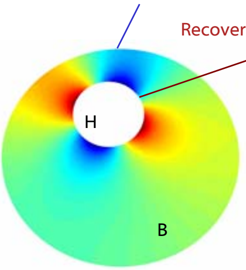
More details are given in the talk of Jennifer Mueller later in this conference.

We assume that electric conductivity inside patient is known

We solve the inverse problem of electrocardiography (ECG) using exponentially growing solutions

3. Inverse problem of ECG

We study the inverse potential problem of electrocardiography (ECG)



Measure voltage potential at the skin

Recover voltage potential at the heart

Conductivity equation:

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } B \setminus \overline{H}$$

$$u|_{\partial H} = f, \qquad \frac{\partial u}{\partial \nu}|_{\partial B} = 0$$

The conductivity equation is transformed into the Schrödinger equation

Conductivity equation:

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } B \setminus \overline{H}, \qquad u|_{\partial H} = f, \qquad \frac{\partial u}{\partial \nu}|_{\partial B} = 0$$

Define $\tilde{u} = \gamma^{1/2} u, \qquad q(x) = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$

Then ũ satisfies the Schrödinger equation

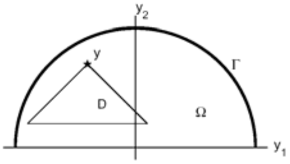
$$(-\Delta + q)\tilde{u} = 0 \text{ in } B \setminus \overline{H}, \qquad \tilde{u}|_{\partial B} = u|_{\partial B}, \qquad \frac{\partial \tilde{u}}{\partial \nu}|_{\partial B} = 0$$

Solve the Cauchy problem for the Schrödinger equation in canonical geometry as follows

Let ψ be the exponentially growing solution of

$$(-\Delta + q(x))\psi(x,k) = \chi_D \, e^{-k(x_2 - y_2)} e^{-ikx_1}$$

for real, negative k.



We get a regularized reconstruction method by using finite k in the following theorem

Theorem [Ikehata 2001]: we have

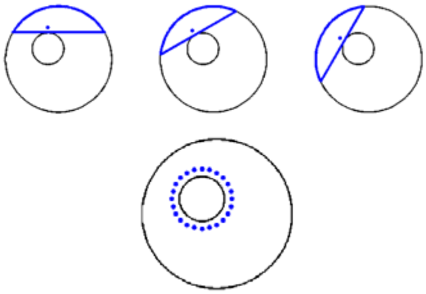
$$\tilde{u}(y) = \lim_{k \to -\infty} \tilde{u}_k(y),$$

where

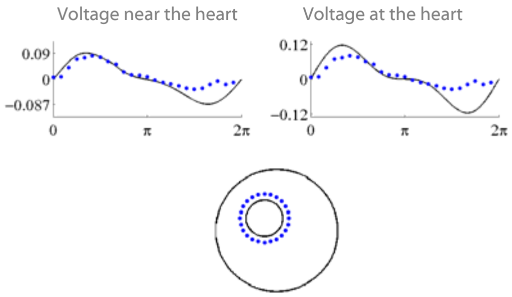
$$\tilde{u}_k(y) := \frac{2k^2 e^{-iky_1}}{C_D} \int_\Gamma \left(\frac{\partial \tilde{u}}{\partial \nu} \overline{\psi(\cdot,k)} - \frac{\partial \overline{\psi(\cdot,k)}}{\partial \nu} \tilde{u}\right) d\sigma$$

and C_D is a known constant.

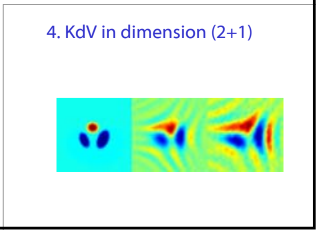
We recover voltage near the heart by rotating the canonical geometry



We reconstruct voltage near the heart from simulated ECG data with 2% noise



We introduce a generalization of the Korteveg-de Vries (KdV) equation to (2+1) dimensions



4. KdV in dimension (2+1)

The inverse scattering scheme of Novikov and Veselov

$$\begin{tikzcd}
\mathbf{t} \arrow[r, "e^{i\tau(k^3+\bar{k}^3)}\cdot"] & \mathbf{t}_\tau \arrow[d, "Q"] \\
q \arrow[u, "T"] \arrow[r, "\text{Nonlinear evolution}"'] & q_\tau
\end{tikzcd}$$

References for (2+1) dimensional KdV equations and the IST scheme

1967	Gardner, Greene, Kruskal and Miura (1+1)
1968	Lax (1+1)
1984	Novikov and Veselov (periodic setting)
1987	Boiti, Leon, Manna and Pempinelli
1993	Tsai
1996	Nachman

The inverse scattering scheme is well-defined for conductivity-type initial data

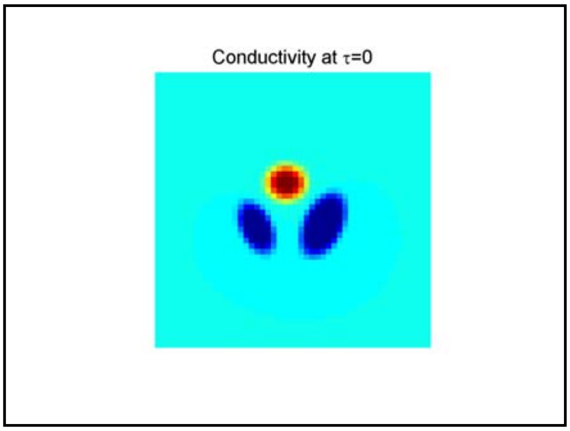
Theorem [Lassas, Mueller & S 2004].
The function

$$q_\tau = Q(e^{i\tau(k^3+\bar{k}^3)}(Tq)(k))$$

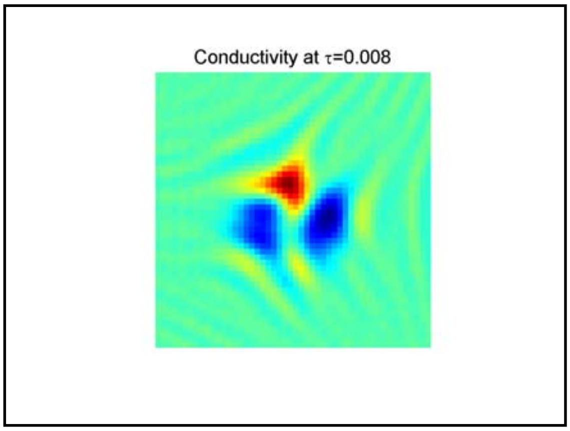
is a real-valued, smooth element of $L^p(\mathbb{R}^2)$ with some $1 < p < 2$, has no exceptional points, and $Tq_\tau = e^{i\tau(k^3+\bar{k}^3)}Tq$. Moreover,

$$q_\tau = \frac{\Delta \gamma_\tau^{1/2}}{\gamma_\tau^{1/2}}$$

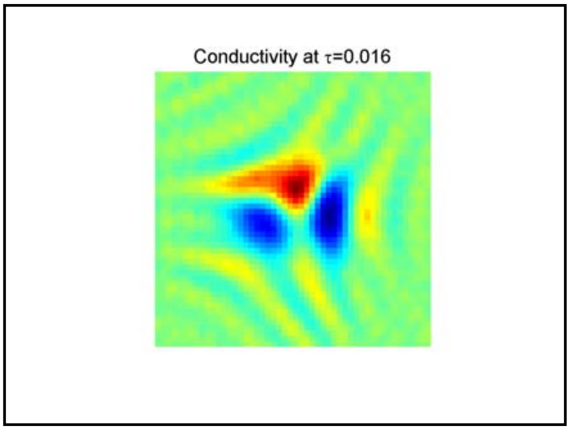
for some strictly positive $\gamma_\tau \in L^\infty(\mathbb{R}^2)$.



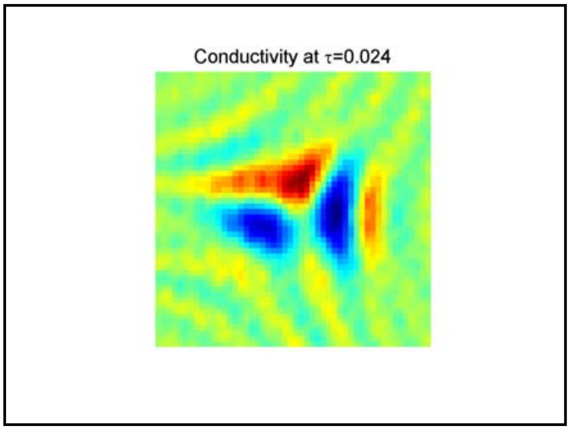
Conductivity at $\tau=0$



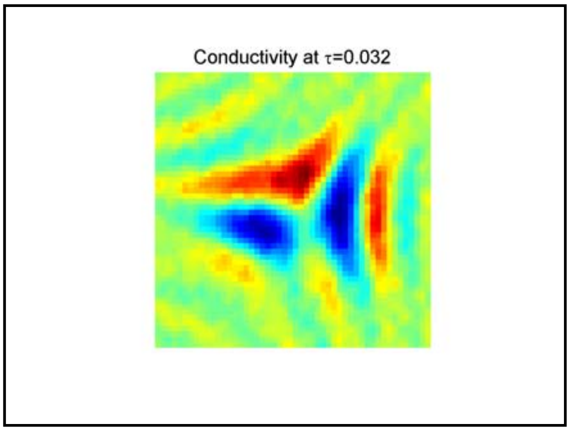
Conductivity at $\tau=0.008$



Conductivity at $\tau=0.016$



Conductivity at $\tau=0.024$



Conductivity at $\tau=0.032$

Conclusion

Numerical inversion methods based on exponentially growing solutions can be used to solve practical inverse problems

Future challenge: three-dimensional inverse problems

This work was supported in part by
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