New computational methods for electrical impedance tomography

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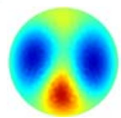

This is a joint work with

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Kim Knudsen	Aalborg University, Denmark
Matti Lassas	Helsinki University of Technology, Finland
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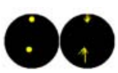
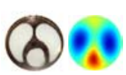
$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ Introduction to EIT

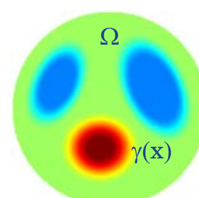


EIT meets Mittag-Leffler



D-bar method for EIT

The inverse conductivity problem of Calderón is mathematical model of EIT



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

Given the Dirichlet-to-Neumann map,
how to reconstruct the conductivity?
The reconstruction problem is nonlinear and ill-posed.

The uniqueness question in 2-D has been studied by these authors

1980 Calderón
1985 Kohn and Vogelius
1987 Sylvester and Uhlmann (n>2)
1987 R G Novikov (n>2)
1988 Nachman (n>2)
1996 Nachman
1997 Brown and Uhlmann
2003 Astala and Päivärinta

Applications of EIT to medical imaging

Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

Clinical tasks: monitoring heart and lungs of SARS patients, detecting pulmonary edema (swollen lungs)

Nondestructive testing and EIT

Finding cracks or defects in known background.
Photo from: Shigo, A.L., 1983. Tree Defects: A Photo Guide. USDA Forest Service, No. Cent. For. Exp. Sta., GTRNE-82.

imagination at work

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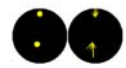
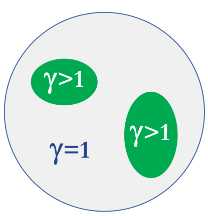


EIT meets Mittag-Leffler

We consider recovering an inclusion in homogeneous background



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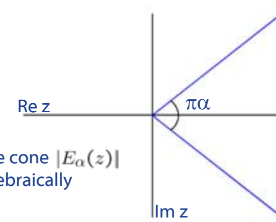
Mittag-Leffler's function: definition and asymptotic behavior

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}$$



In the cone $|E_\alpha(z)|$ grows exponentially as $|z| \to \infty$

Outside the cone $|E_\alpha(z)|$ decays algebraically as $|z| \to \infty$

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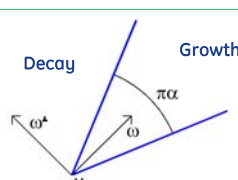
Indicator function is the key object for recovering inclusions from ND map

$$I^\alpha_{(y,\omega)}(\tau) = \int_{\partial\Omega} \overline{\frac{\partial e^\alpha_\tau}{\partial\nu}}(R_1 - R_\gamma)\frac{\partial e^\alpha_\tau}{\partial\nu}\, d\sigma,$$

$$e^\alpha_\tau(x; y, \omega) = E_\alpha(\tau\{(x-y)\cdot\omega + i(x-y)\cdot\omega^\perp\}).$$

Asymptotic behavior of e^α_τ as $|z| \to \infty$:



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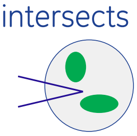
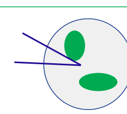
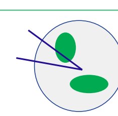
Theorem: we can find out if a given cone intersects the inclusion



$$\lim_{\tau\to\infty} |I^\alpha_{(y,\omega)}(\tau)| = 0$$



$$\liminf_{\tau\to\infty} |I^\alpha_{(y,\omega)}(\tau)| > 0$$



$$\lim_{\tau\to\infty} |I^\alpha_{(y,\omega)}(\tau)| = \infty$$

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Indicator function can be written in terms of the measured ND map

Write $I^{\alpha}_{(y,\omega)}(\tau) = \int_{\partial\Omega} \overline{\frac{\partial e^{\alpha}_{\tau}}{\partial \nu}}(R_1 - R_\gamma)\frac{\partial e^{\alpha}_{\tau}}{\partial \nu} d\sigma,$

and expand the function $\frac{\partial e^{\alpha}_{\tau}}{\partial \nu}$ in Fourier basis to get

$$I^{1/n,nN}_{(y,\omega)}(\tau) = 2\pi \sum_{1\le m,\ell\le nN} \frac{\tau^{m+\ell}\overline{\omega^m}\omega^\ell}{\Gamma(\frac{m}{n}+1)\Gamma(\frac{\ell}{n}+1)}$$

$$\times \sum_{r_1=1}^{m}\sum_{r_2=1}^{\ell}\binom{m}{r_1}\binom{\ell}{r_2} r_1 r_2 (-1)^{m+\ell-r_1-r_2} y^{m-r_1}\overline{y^{\ell-r_2}}$$

$$\times(\mathcal{R}_1[r_2,r_1] - \mathcal{R}_\gamma[r_2,r_1]).$$

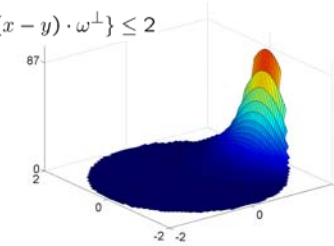
Due to finite number of measurements, we cannot take τ to infinity

The truncated power series of Mittag-Leffler's function gives relative accuracy of 1% for |z|<2.
We evaluate Mittag-Leffler's function at

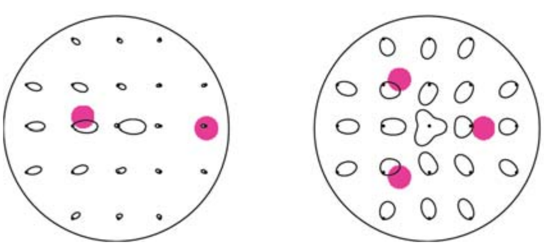
$$\tau\{(x-y)\cdot\omega + i(x-y)\cdot\omega^\perp\} \le 2$$

Thus we choose

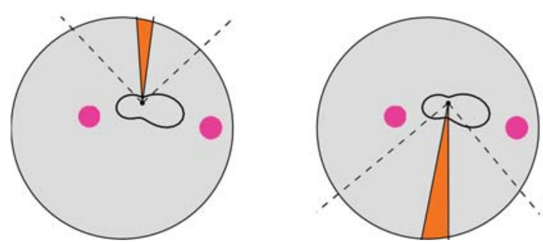
$$\tau(y) = \frac{1}{1+|y|}$$



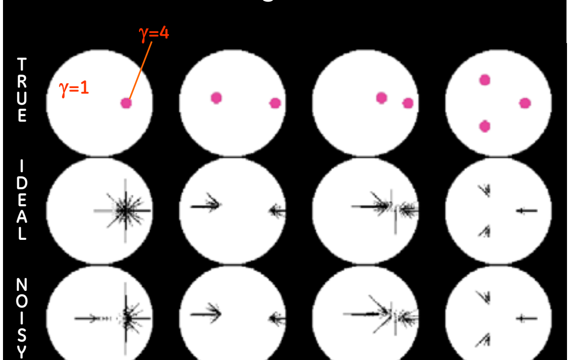
Indicator functions with fixed y and plotting |I(y,ω)| in polar coordinates



We exclude cones corresponding to local minima of the indicator function



Numerical recovery of disc inclusions



D-bar method for EIT

Nachman's reconstruction method [Ann of Math 1996] consists of two steps

$$\Lambda_\gamma \rightarrow \mathbf{t} \rightarrow \gamma$$

The intermediate object t is the *scattering transform*

Step 1: given noisy data, compute approximate scattering transform

Approximate scattering transform:

$$\mathbf{t}^{\exp}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}d\sigma(x)$$

Regularize computation by truncation:

$$\mathbf{t}^{\exp}_R(k) := \begin{cases} \mathbf{t}^{\exp}(k), & |k| < R, \\ 0, & |k| \geq R. \end{cases}$$

Step 2: solve d-bar equation with approximate kernel

Write the approximate dbar equation

$$\frac{\partial}{\partial \bar{k}}\mu_R(x,k) = \frac{\mathbf{t}^{\exp}_R(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}$$

In integral form:

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathbf{t}^{\exp}_R(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

This Lippmann-Schwinger -type equation can be solved numerically with modified Vainikko's algorithm. Then

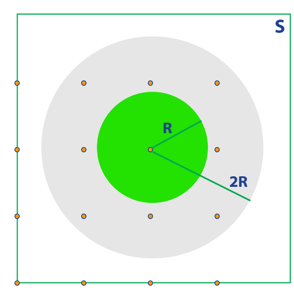
$$\gamma_R^{1/2}(x) = \mu_R(x,0).$$

Define a grid for Vainikko's algorithm

(2^m x 2^m) points in the square S in the k-plane.

Here m=2, in practice typically m=8.

This grid is suitable for the use of Fast Fourier Transform (FFT).



The d-bar equation can be solved using periodization

Instead of the d-bar equation

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathbf{t}^{\exp}_R(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

valid in the k-plane, we solve the S-periodic equation

$$[I + \frac{1}{\pi k} * (T_R \cdot {}^{-})]w = 1$$

$$T_R(k) = -\frac{\mathbf{t}^{\exp}_R(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}$$

The d-bar equation is also solved since

$$\mu_R(x,\cdot)|_{B(0,R)} = w|_{B(0,R)}$$

Vainikko's method is based on iterative solution of linear equations

We can solve the discretized equation

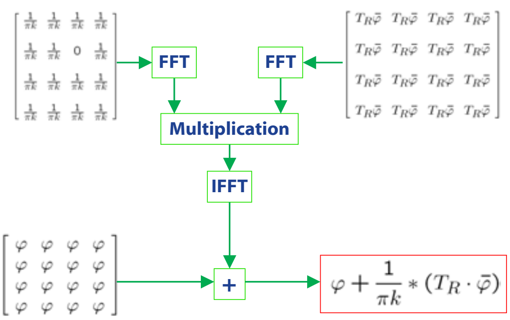
$$[I + \frac{1}{\pi k} * (T_R \cdot {}^{-})]w = 1$$

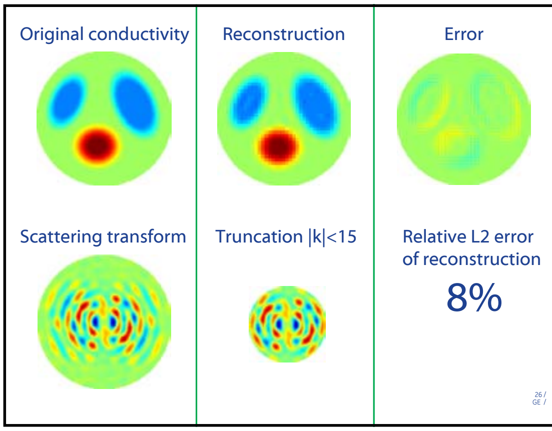
using the iterative GMRES method.
We just need to implement the formula

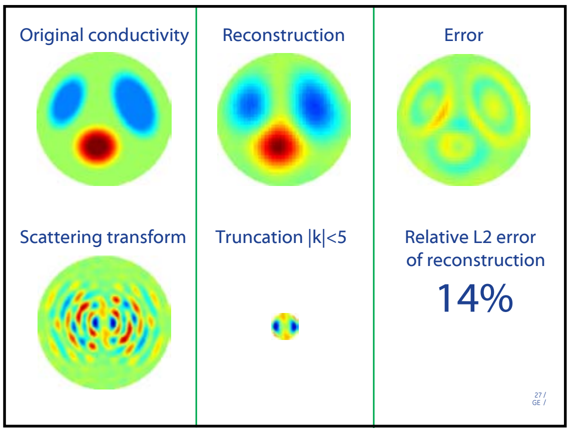
$$\varphi + \frac{1}{\pi k} * (T_R \cdot \bar{\varphi})$$

for a function ϕ given on the grid points.

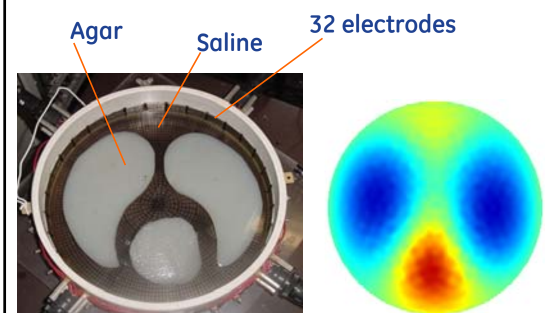
Convolution is implemented using FFT







Reconstruction from data measured from a chest phantom



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