The d-bar reconstruction method for electrical impedance tomography

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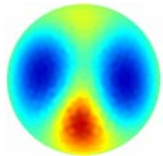

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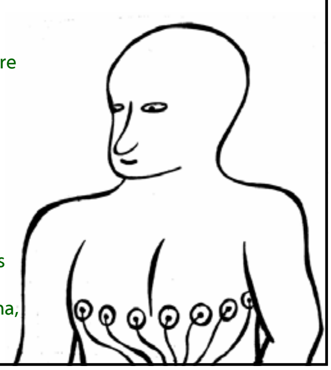
Electrical impedance tomography (EIT) is an emerging medical imaging method

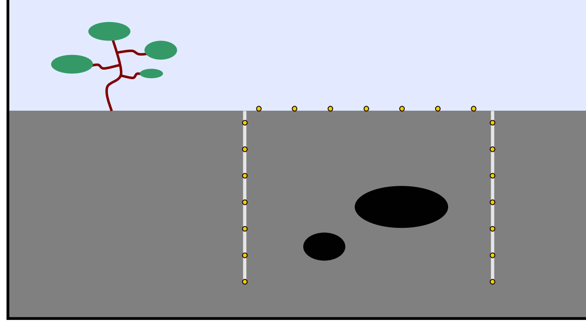
Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

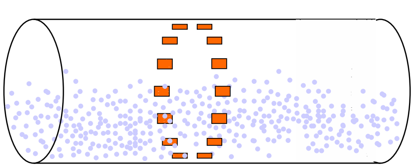
Applications include:
monitoring heart and lungs of unconscious patients,
detecting pulmonary edema,
enhancing ECG and EEG



Geological sensing of oil or metals is another application of EIT

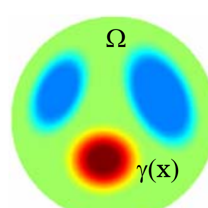


EIT can be used as well in industrial process monitoring



Monitoring fluid level
Detection of particles or bubbles
Imaging concentration profile

The inverse conductivity problem of Calderón is the mathematical model of EIT



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

Given the Dirichlet-to-Neumann map, how to reconstruct the conductivity?

The reconstruction problem is nonlinear and ill-posed.

Nonlinearity of Calderón's problem

The weak formulation of the DN map as an operator

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

is given by

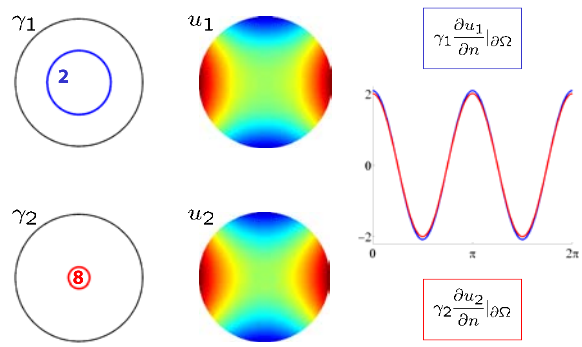
$$\langle \Lambda_\gamma f, g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v,$$

where v is any H^1 function with trace g, and u satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Now the map $\gamma \mapsto \Lambda_\gamma$ is nonlinear because u depends on γ.

Current measurements corresponding to the two conductivities are almost the same



EIT reconstruction algorithms can be divided roughly into the following classes:

Linearization

Iterative output least-squares methods

Statistical inversion

The inverse scattering approach, or **d-bar method**

Theoretical development of the d-bar method can be found in these studies

1980 Calderón
1987 Sylvester and Uhlmann
1987 R G Novikov
1988 Nachman
1996 Nachman
1997 Liu
1997 Brown and Uhlmann
2000 Francini
2001 Barceló, Barceló and Ruiz
2003 Knudsen and Tamasan
2003 Mueller and S
2003 Astala and Päivärinta

Throughout this talk we use these definitions:

Let $\Omega \subset \mathbb{R}^2$ be the unit disc.

Let $\gamma \in C^4(\Omega)$ be strictly positive.

Assume that $\gamma \equiv 1$ near $\partial\Omega$.

Define $q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \in C^2(\mathbb{R}^2)$ by zero extension.

Nachman's 1996 proof consists of two steps:

$$\Lambda_\gamma \to \mathbf{t} \to \gamma$$

The intermediate object t is a complex-valued function called *scattering transform* and defined as follows for complex k:

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x) \psi(x,k) dx$$

$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$	$(-\Delta + q)\psi(\cdot, k) = 0$ $\psi(x,k) \sim e^{ikx} = e^{i(k_1 + ik_2)(x_1 + ix_2)}$

Step 1: from DN map to scattering transform

Solve traces of ψ from the boundary integral equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

where the single-layer operator has Faddeev Green's function as kernel.

Compute the scattering transform as

$$\mathrm{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(x,k)d\sigma(x).$$

Step 2: from scattering transform to γ

Define $\mu(x,k) = e^{-ikx}\psi(x,k)$

Then the following d-bar equation holds:

$$\frac{\partial}{\partial\bar{k}}\mu(x,k) = \frac{\mathrm{t}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

The d-bar equation has a unique solution for all x.
The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k\to 0}\mu(x,k).$$

To evaluate the scattering transform we need exponentially growing solutions

Recall the definition of the scattering transform:

$$\mathrm{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}}q(x)\psi(x,k)dx$$

Given a conductivity-type potential, we wish to evaluate numerically the solutions

$$(-\Delta + q)\psi(\cdot,k) = 0, \qquad \psi(x,k) \sim e^{ikx}.$$

This can be done via the Lippmann-Schwinger type equation

$$\mu(x,k) = 1 - \int_{\mathrm{supp}\, q} g_k(x-y)q(y)\mu(y,k)dy,$$
$$\mu(x,k) = e^{-ikx}\psi(x,k).$$

Equation for exponentially growing solutions can be rewritten in periodic setting

We wish to solve the equation

$$\mu(x,k) = 1 - \int_{\mathrm{supp}\, q} g_k(x-y)q(y)\mu(y,k)dy.$$

We take $R > 0$ such that $\mathrm{supp}\, q \subset B(0,R)$ and choose a square $S \supset B(0,2R)$.
Then we consider the S-periodic equation

$$w = 1 - \tilde{g}_k * (qw),$$

where $\tilde{g}_k$ is periodic extension of $g_k|_S$ and $*$ is convolution on the torus. It follows that

$$\mu|_{\mathrm{supp}\, q} = w|_{\mathrm{supp}\, q}.$$

We need three steps for numerical computation of the scattering transform

$$\mathrm{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}}q(x)\psi(x,k)dx$$

1. Evaluation of Faddeev's Green function
$$g_1(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{e^{ix\cdot\xi}}{|\xi|^2 + 2(\xi_1 + i\xi_2)}d\xi_1 d\xi_2.$$
2. Solution of the periodic equation
$$w = 1 - \tilde{g}_k * (qw)$$
3. Substituting the result to definition of t(k)
$$\psi(x,k)|_{\mathrm{supp}\, q} = e^{ikx}w(x,k)|_{\mathrm{supp}\, q}$$

In dimension two, the definition of Faddeev's Green function is as follows:

Let $k \in \mathbb{C}\setminus 0$. Define $g_k : \mathbb{R}^2 \to \mathbb{C}$ as

$$g_k(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{e^{ix\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)}d\xi_1 d\xi_2.$$

Then g_k satisfies

$$(-\Delta - 4ik\bar{\partial})g_k = \delta,$$

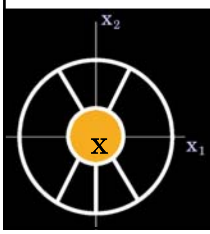
where

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Note the symmetry $g_k(x) = g_1(kx)$.

For x in the unit disc we compute $g_1(x)$ with a formula by [Boiti et al 1987]

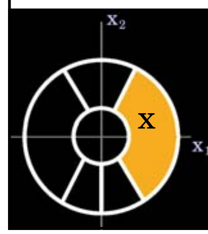
$$g_1(x) = -\frac{e^{-ix}}{4\pi}\left(2\gamma + \log|x|^2 + \sum_{n=1}^{\infty}\frac{(ix)^n + (-i\bar{x})^n}{n\,n!}\right)$$



Here γ is Euler's constant

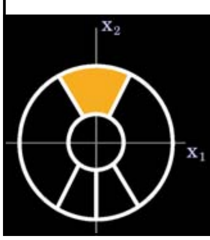
We write $g_1(x)$ as a formula containing a rapidly converging one-dimensional integral

$$g_1(x) = \frac{e^{-ix_1}}{2\pi}\mathrm{Re}\Big[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}} + \frac{(N+1)!\,e^{ix_1}}{(-x)^{N+1}}\int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}}dt\Big]$$

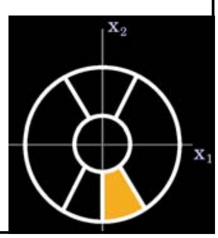


Cases 3 and 4: The integral in case 2 is modified using the residue theorem

$$-i\int_0^\infty \frac{e^{-x_2 s + i x_1 s}}{(-is-i)^{N+2}}ds$$

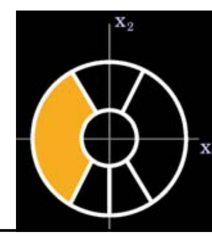


$$(1+i)\int_0^\infty \frac{e^{-is(x_2+x_1)+s(x_2-x_1)}}{(s+is-i)^{N+2}}ds$$



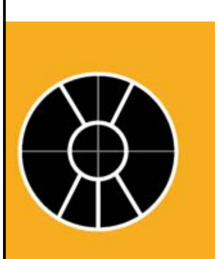
Cases 5 and 6 are reduced to cases 4 and 2 using the following symmetry:

$$g_1(-x_1, x_2) = \overline{g_1(x_1, x_2)}$$



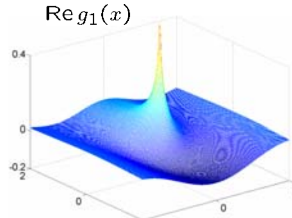
Case 7: for |x|>25 we ignore the integral in case 2 and use the truncated sum

$$g_1(x) \approx \frac{e^{-ix_1}}{2\pi}\mathrm{Re}\left[-e^{ix_1}\sum_{j=0}^{N}\frac{j!}{(ix)^{j+1}}\right]$$

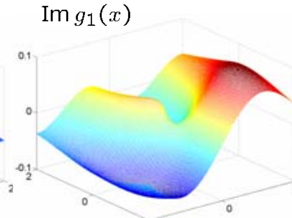


We now have a complete algorithm for the evaluation of Faddeev's Green function

Re $g_1(x)$



Im $g_1(x)$

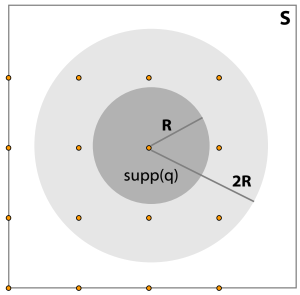


We solve the equation $w = 1 - \tilde{g}_k * (qw)$ following [Vainikko 2000]

We take a grid on S with (2^m x 2^m) points

Here m=2, in practice typically m=8.

This grid is suitable for the use of Fast Fourier Transform (FFT).



Vainikko's method is based on iterative solution of linear equations

We can solve the discretized periodic equation

$$[I + \tilde{g}_k * (q \cdot\ )]w = 1$$

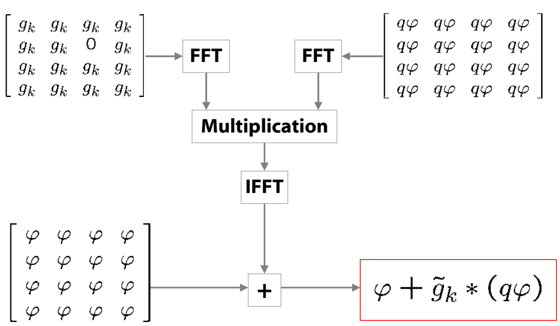
using the iterative GMRES method.

We just need to implement the linear operator

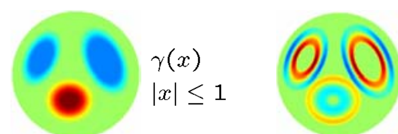
$$\varphi \ \mapsto \ \varphi + \tilde{g}_k * (q\varphi)$$

for a function φ given on the grid points.

The linear operator is implemented using Fast Fourier Transform (FFT)



Given a conductivity, we compute the potential and Faddeev solutions μ

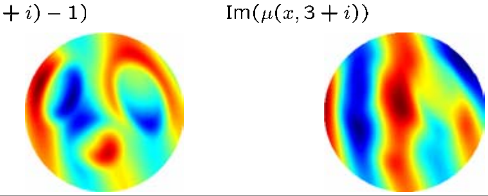


$\gamma(x)$

$|x| \le 1$

$q = \dfrac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$

Re$(\mu(x, 3+i) - 1)$ Im$(\mu(x, 3+i))$

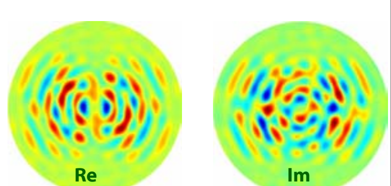


We can now use μ to compute the scattering transform

$$\psi(x,k) = e^{ikx}\mu(x,k)$$

$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

$|k| \le 25$



Practical step 1: compute stable approximation to scattering transform from noisy data

With noisy data, we cannot solve equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

so we introduce the approximate scattering transform:

$$\mathbf{t}^{\exp}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}d\sigma(x)$$

Further, we regularize the computation by truncation:

$$\mathbf{t}_R^{\exp}(k) := \begin{cases} \mathbf{t}^{\exp}(k), & |k| < R, \\ 0, & |k| \ge R. \end{cases}$$

Practical step 2: solve the d-bar equation with approximate kernel

Write the approximate dbar equation

$$\frac{\partial}{\partial \bar{k}}\mu_R(x,k) = \frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}$$

In integral form:

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

This Lippmann-Schwinger -type equation can be solved numerically with modified Vainikko's algorithm. Then

$$\gamma_R^{1/2}(x) = \mu_R(x,0).$$

The d-bar equation can be solved in a bounded domain using periodization

Instead of the d-bar equation

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

valid in the k-plane, we solve the S-periodic equation

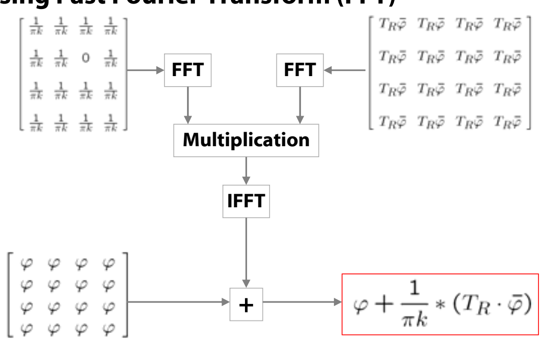
$$[I + \frac{1}{\pi k} * (T_R \cdot \bar{\ })]w = 1$$

$$T_R(k) = -\frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}$$

The d-bar equation is also solved since it can be shown that

$$\mu_R(x,\cdot)|_{B(0,R)} = w|_{B(0,R)}$$

The convolution is effectively implemented using Fast Fourier Transform (FFT)



Note: real and imaginary parts must be kept separate!

Truncation of scattering transform gives asymptotically correct reconstruction

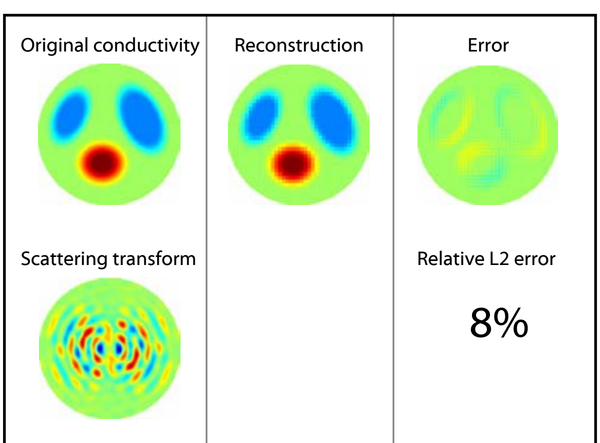
Theorem [Mueller & S 2003].
Let μ_R be the solution of the $\bar{\partial}$ equation

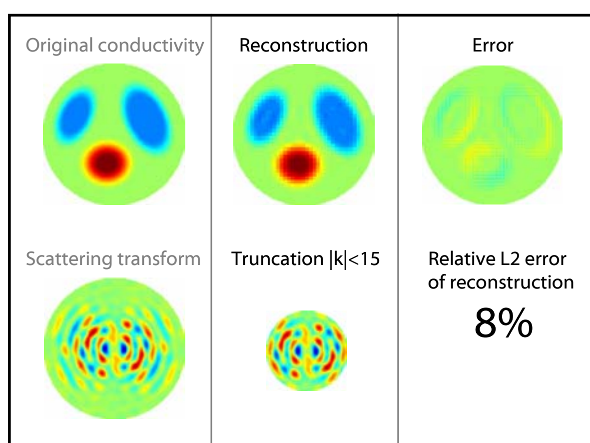
$$\frac{\partial}{\partial \bar{k}}\mu_R(x,k) = \frac{\mathbf{t}_R(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}.$$

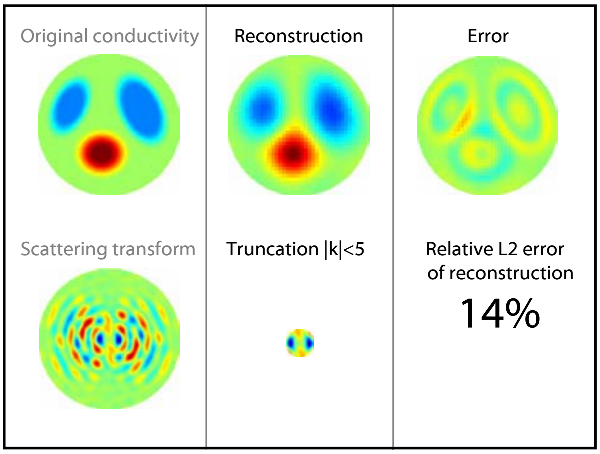
Then the following estimate holds for large R:

$$\|\sqrt{\gamma} - \mu_R(\cdot,0)\|_{L^\infty(\Omega)} \leq CR^{-1}.$$

Next we demonstrate this theorem numerically.







Current-to-voltage measurements are described with the ND map

The Neumann-to-Dirichlet (ND) map is defined by

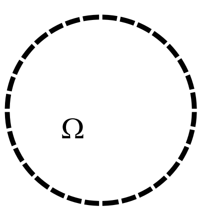
$$R_\gamma f = u|_{\partial\Omega} - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u,$$

where u satisfies the Neumann problem

$$\begin{cases} \nabla\cdot\gamma\nabla u = 0 & \text{in } \Omega, \\ \gamma\frac{\partial u}{\partial\nu} = f & \text{on } \partial\Omega. \end{cases}$$

In practice, currents are applied and voltages measured.
This is because
the ND operator is smoothing and suppresses noise, while
the DN operator is roughing and amplifies noise.

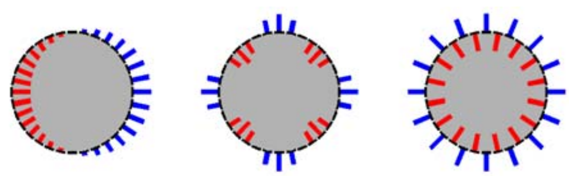
This is a typical configuration for electrode measurements in EIT



Here we have N=32 electrodes.
The machine is in Rensselaer Polytechnic Institute, USA.

There is a finite number of linearly independent current patterns

Here are three examples with N=32:



Altogether, there are N-1 linearly independent current patterns due to conservation of charge.

We represent a finite set of EIT measurements with a complex 31x31 matrix

Define trigonometric current patterns as

$$T_\ell^k = \begin{cases} M\cos(k\theta_\ell), & k = 1,\dots,\frac{N}{2}-1, \\ M\cos(\pi\ell), & k = N/2, \\ M\sin((k-N/2)\theta_\ell), & k = \frac{N}{2}+1,\dots,N-1, \end{cases}$$

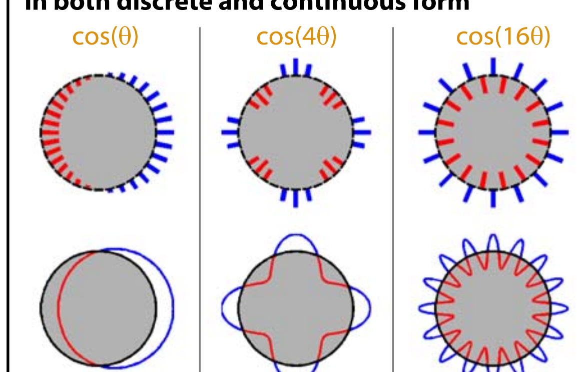
where M is current amplitude. Let V_ℓ^k denote the voltage measured on the ℓth electrode corresponding to the kth current pattern with

$$\sum_{\ell=1}^{N} V_\ell^k = 0.$$

Up to normalization, the ND matrix is given by

$$R_\gamma[m,n] := \sum_\ell T_\ell^m V_\ell^n.$$

We use trigonometric current patterns in both discrete and continuous form



The approximate scattering transform can now be written in terms of measured data

Expand the exponential function as series:

$$e^{ikz} = \sum_{n=-\infty}^{\infty} a_n(k) e^{in\theta} \quad \text{with} \quad a_n(k) = \begin{cases} \frac{(ik)^n}{n!}, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

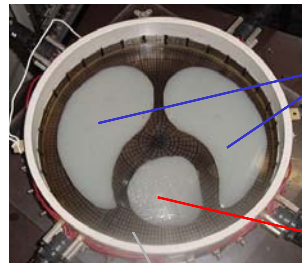
Then we can write

$$t^{\exp}(k) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m(\bar{k}) a_n(k) \int_{\partial\Omega} e^{im\theta} (\Lambda_\gamma - \Lambda_1) e^{in\theta} d\sigma(\theta).$$

The relation between DN map the measurements is, roughly,

$$\Lambda_\gamma = R_\gamma^{-1}.$$

At the RPI lab, we construct a chest phantom consisting of saline and agar

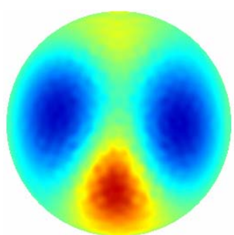


"Lungs" with lower conductivity than background (240 mS/m)

"Heart" with higher conductivity than background (750 mS/m)

Background of salt water, conductivity 424 mS/m.
Diameter of the tank is 30cm.

Reconstruction from measured data



Relative error 23% (lung) and 12% (heart).
Dynamical range is 94% of the true range.

Conclusion

We have developed a new kind of EIT algorithm
Large contrast details recovered remarkably well
The algorithm has rigorous mathematical background

Future challenges

Understanding reconstruction of discontinuities
Removing the requirement γ=1 near boundary
Taking the shape of domain into account
Developing a 3D algorithm

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