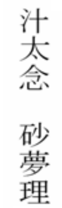
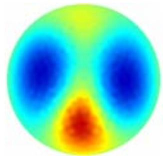
The d-bar reconstruction method for electrical impedance tomography





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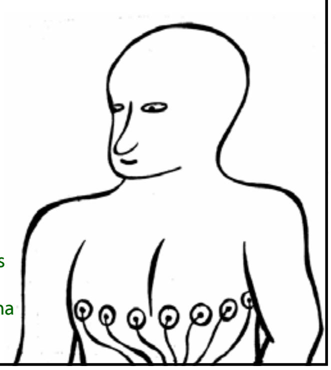
Electrical impedance tomography (EIT) is an emerging medical imaging method

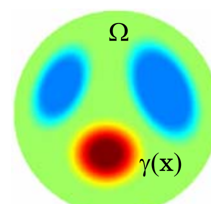
Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

Applications include:
monitoring heart and lungs of unconscious patients, detecting pulmonary edema (swollen lungs)



The inverse conductivity problem of Calderón is the mathematical model of EIT



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial\Omega},$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

Given the Dirichlet-to-Neumann map, how to reconstruct the conductivity?

The reconstruction problem is nonlinear and ill-posed.

Nonlinearity of Calderón's problem

The weak formulation of the DN map as an operator

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

is given by

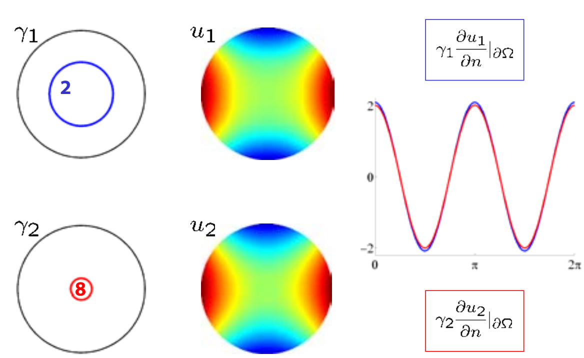
$$\langle \Lambda_\gamma f, g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v,$$

where v is any H^1 function with trace g, and u satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Now the map $\gamma \mapsto \Lambda_\gamma$ is nonlinear because u depends on γ.

Íll-posedness of Calderón's problem



EIT reconstruction algorithms can be divided roughly into the following classes:

Linearization

Iterative output least-squares methods

Statistical inversion

The inverse scattering approach, or **d-bar method**

Theoretical development of the d-bar method can be found in these studies

1980 Calderón
1987 Sylvester and Uhlmann
1987 R G Novikov
1988 Nachman
1996 Nachman
1997 Liu
1997 Brown and Uhlmann
2000 Francini
2001 Barceló, Barceló and Ruiz
2003 Knudsen and Tamasan
2003 Mueller and S
2003 Astala and Päivärinta

Throughout this talk we use these definitions:

Let $\Omega \subset \mathbb{R}^2$ be the unit disc.

Let $\gamma \in C^4(\Omega)$ be strictly positive.

Assume that $\gamma \equiv 1$ near $\partial\Omega$.

Define $q = \dfrac{\Delta\gamma^{1/2}}{\gamma^{1/2}} \in C^2(\mathbb{R}^2)$ by zero extension.

Nachman's 1996 proof consists of two steps:

$$\Lambda_\gamma \rightarrow \mathbf{t} \rightarrow \gamma$$

The intermediate object t is a complex-valued function called *scattering transform* and defined as follows for complex k:

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

$$q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$$

$$(-\Delta + q)\psi(\cdot,k) = 0$$

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}$$

Step 1: from DN map to scattering transform

Solve traces of ψ from the boundary integral equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

where the single-layer operator has Faddeev Green's function as kernel.

Compute the scattering transform as

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(x,k)d\sigma(x).$$

Step 2: from scattering transform to γ

Define $\mu(x,k) = e^{-ikx}\psi(x,k)$

Then the following d-bar equation holds:

$$\frac{\partial}{\partial\bar{k}}\mu(x,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

The d-bar equation has a unique solution for all x.
The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k\to 0}\mu(x,k).$$

To evaluate the scattering transform we need exponentially growing solutions

Recall the definition of the scattering transform:

$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

Given a conductivity-type potential, we wish to evaluate numerically the solutions

$$(-\Delta + q)\psi(\cdot, k) = 0, \qquad \psi(x,k) \sim e^{ikx}.$$

This can be done via the Lippmann-Schwinger type equation

$$\mu(x,k) = 1 - \int_{\text{supp}\, q} g_k(x-y)q(y)\mu(y,k)dy,$$
$$\mu(x,k) = e^{-ikx}\psi(x,k).$$

In dimension two, the definition of Faddeev's Green function is as follows:

Let $k \in \mathbb{C} \setminus 0$. Define $g_k : \mathbb{R}^2 \to \mathbb{C}$ as

$$g_k(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} d\xi_1 d\xi_2.$$

Then g_k satisfies

$$(-\Delta - 4ik\bar{\partial})g_k = \delta,$$

where

$$\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}).$$

Note the symmetry $g_k(x) = g_1(kx)$.

Equation for exponentially growing solutions can be rewritten in periodic setting

We wish to solve the equation

$$\mu(x,k) = 1 - \int_{\text{supp}\, q} g_k(x-y)q(y)\mu(y,k)dy.$$

We take $R > 0$ such that $\text{supp}\, q \subset B(0,R)$ and choose a square $S \supset B(0,2R)$.
Then we consider the S-periodic equation

$$w = 1 - \tilde{g}_k * (qw),$$

where $\tilde{g}_k$ is periodic extension of $g_k|_S$ and $*$ is convolution on the torus. It follows that

$$\mu|_{\text{supp}\, q} = w|_{\text{supp}\, q}.$$

We need three steps for numerical computation of the scattering transform

$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

1. Evaluation of Faddeev's Green function

$$g_1(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2(\xi_1 + i\xi_2)} d\xi_1 d\xi_2.$$

2. Solution of the periodic equation

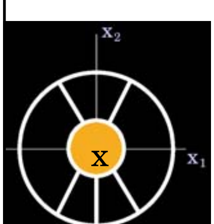
$$w = 1 - \tilde{g}_k * (qw)$$

3. Substituting the result to definition of t(k)

$$\psi(x,k)|_{\text{supp}\, q} = e^{ikx} w(x,k)|_{\text{supp}\, q}$$

For x in the unit disc we compute $g_1(x)$ with a formula by [Boiti et al 1987]

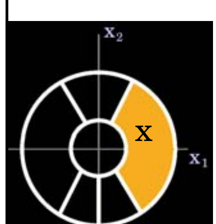
$$g_1(x) = -\frac{e^{-ix}}{4\pi}(2\gamma + \log|x|^2 + \sum_{n=1}^{\infty} \frac{(ix)^n + (-i\bar{x})^n}{nn!})$$



Here γ is Euler's constant

We write $g_1(x)$ as a formula containing a rapidly converging one-dimensional integral

$$g_1(x) = \frac{e^{-ix_1}}{2\pi}\text{Re}[-e^{ix_1}\sum_{j=0}^{N} \frac{j!}{(ix)^{j+1}} + \\ + \frac{(N+1)!e^{ix_1}}{(-x)^{N+1}} \int_0^\infty \frac{e^{-t(x_1+ix_2)}}{(t-i)^{N+2}} dt]$$

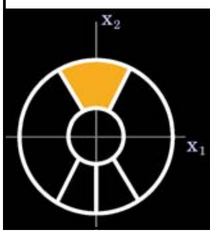


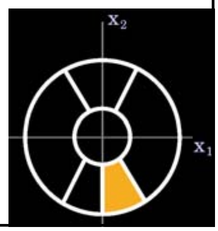
For other domains we use residue calculus and reflectional symmetry

Cases 3 and 4: The integral in case 2 is modified using the residue theorem

$$-i\int_0^\infty \frac{e^{-x_2 s + i x_1 s}}{(-is-i)^{N+2}} ds$$

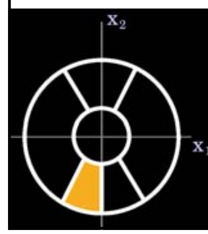
$$(1+i)\int_0^\infty \frac{e^{-is(x_2+x_1)+s(x_2-x_1)}}{(s+is-i)^{N+2}} ds$$

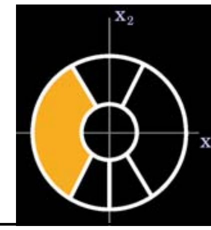




Cases 5 and 6 are reduced to cases 4 and 2 using the following symmetry:

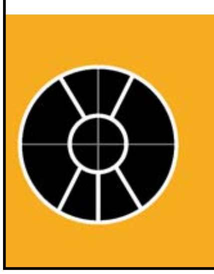
$$g_1(-x_1, x_2) = \overline{g_1(x_1, x_2)}$$



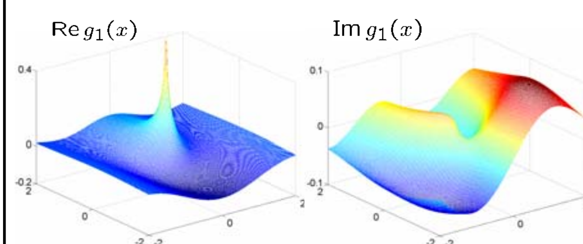


Case 7: for |x|>25 we ignore the integral in case 2 and use the truncated sum

$$g_1(x) \approx \frac{e^{-ix_1}}{2\pi} \mathrm{Re}\left[-e^{ix_1}\sum_{j=0}^{N} \frac{j!}{(ix)^{j+1}}\right]$$



We now have a complete algorithm for the evaluation of Faddeev's Green function

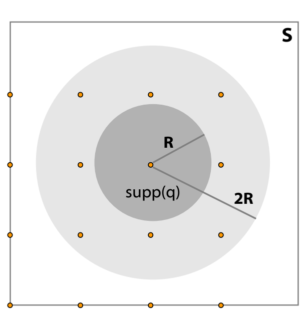


We solve the equation $w = 1 - \tilde{g}_k * (qw)$ following [Vainikko 2000]

We take a grid on S with (2^m x 2^m) points

Here m=2, in practice typically m=8.

This grid is suitable for the use of Fast Fourier Transform (FFT).



Vainikko's method is based on iterative solution of linear equations

We can solve the discretized periodic equation

$$[I + \tilde{g}_k * (q\cdot\ )]w = 1$$

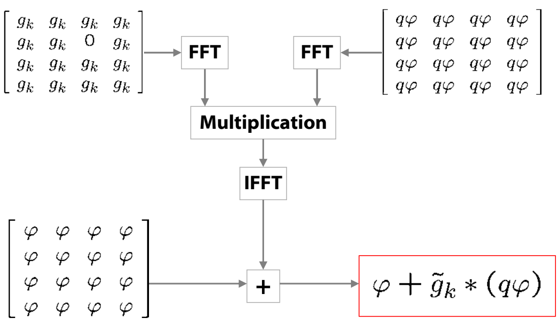
using the iterative GMRES method.

We just need to implement the linear operator

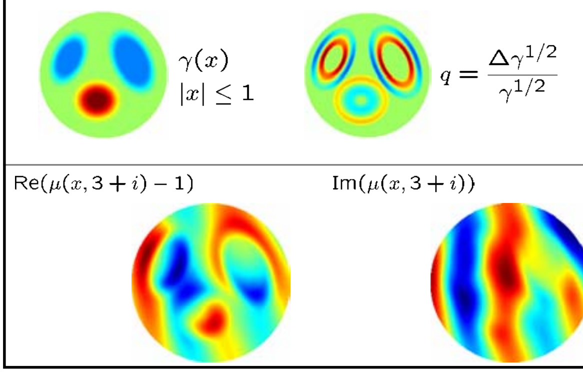
$$\varphi \ \mapsto\ \varphi + \tilde{g}_k * (q\varphi)$$

for a function φ given on the grid points.

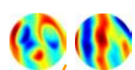
The linear operator is implemented using Fast Fourier Transform (FFT)



Given a conductivity, we compute the potential and Faddeev solutions μ



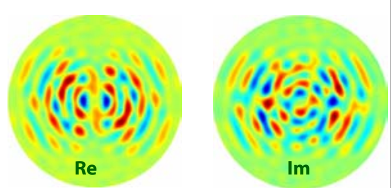
We can now use μ to compute the scattering transform



$$\psi(x,k) = e^{ikx}\mu(x,k)$$

$$\mathrm{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)\,dx$$

$|k| \leq 25$



Practical step 1: compute stable approximation to scattering transform from noisy data

With noisy data, we cannot solve equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

so we introduce the approximate scattering transform:

$$\mathrm{t}^{\mathrm{exp}}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)e^{ikx}\,d\sigma(x)$$

Further, we regularize the computation by truncation:

$$\mathrm{t}_R^{\mathrm{exp}}(k) := \begin{cases} \mathrm{t}^{\mathrm{exp}}(k), & |k| < R, \\ 0, & |k| \geq R. \end{cases}$$

Practical step 2: solve the d-bar equation with approximate kernel

Write the approximate dbar equation

$$\frac{\partial}{\partial \bar{k}}\mu_R(x,k) = \frac{\mathrm{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}$$

In integral form:

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathrm{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

This Lippmann-Schwinger -type equation can be solved numerically with modified Vainikko's algorithm. Then

$$\gamma_R^{1/2}(x) = \mu_R(x,0).$$

The d-bar equation can be solved in a bounded domain using periodization

Instead of the d-bar equation

$$\mu_R(x,k) = 1 + \frac{1}{\pi k} * \left(\frac{\mathrm{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}\right)$$

valid in the k-plane, we solve the S-periodic equation

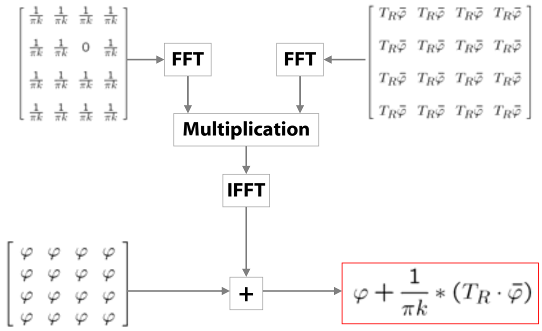
$$[I + \frac{1}{\pi k} * (T_R \cdot^{-})]w = 1$$

$$T_R(k) = -\frac{\mathrm{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}$$

The d-bar equation is also solved since it can be shown that

$$\mu_R(x,\cdot)|_{B(0,R)} = w|_{B(0,R)}$$

The convolution is effectively implemented using Fast Fourier Transform (FFT)



Note: real and imaginary parts must be kept separate!

Truncation of scattering transform gives asymptotically correct reconstruction

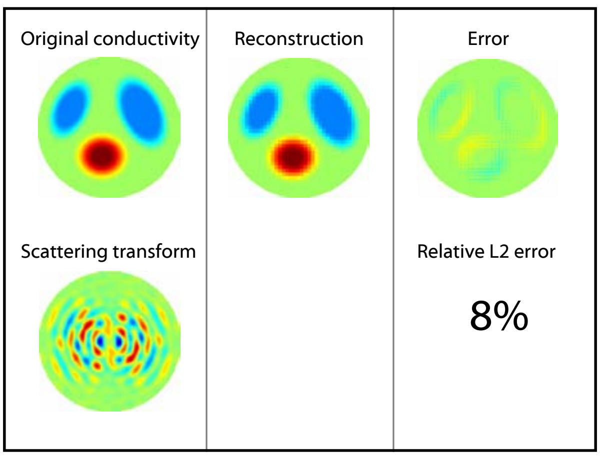
Theorem [Mueller & S 2003].
Let μ_R be the solution of the $\bar{\partial}$ equation

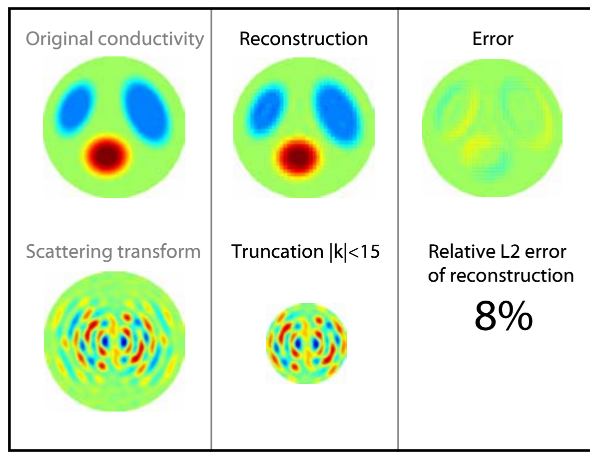
$$\frac{\partial}{\partial \bar{k}} \mu_R(x,k) = \frac{\mathbf{t}_R(k)}{4\pi \bar{k}} e^{-i(kx+\bar{k}\bar{x})} \overline{\mu_R(x,k)}.$$

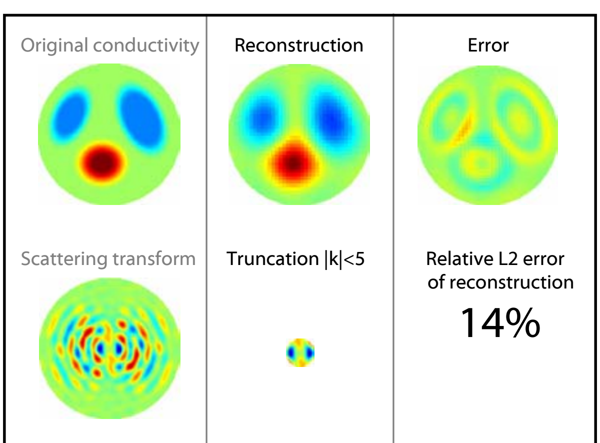
Then the following estimate holds for large R:

$$\|\sqrt{\gamma} - \mu_R(\cdot,0)\|_{L^\infty(\Omega)} \leq C R^{-1}.$$

Next we demonstrate this theorem numerically.







Current-to-voltage measurements are described with the ND map

The Neumann-to-Dirichlet (ND) map is defined by

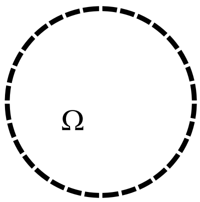
$$R_\gamma f = u|_{\partial\Omega} - \frac{1}{|\partial\Omega|}\int_{\partial\Omega} u,$$

where u satisfies the Neumann problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega, \\ \gamma \frac{\partial u}{\partial \nu} = f & \text{on } \partial\Omega. \end{cases}$$

In practice, currents are applied and voltages measured.
This is because
the ND operator is smoothing and suppresses noise, while
the DN operator is roughing and amplifies noise.

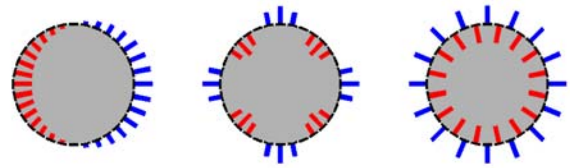
This is a typical configuration for electrode measurements in EIT



Here we have N=32 electrodes.
The machine is in Rensselaer Polytechnic Institute, USA.

There is a finite number of linearly independent current patterns

Here are three examples with N=32:



Altogether, there are N-1 linearly independent current patterns due to conservation of charge.

We represent a finite set of EIT measurements with a complex 31x31 matrix

Define trigonometric current patterns as

$$T_\ell^k = \begin{cases} M\cos(k\theta_\ell), & k = 1,\dots,\frac{N}{2}-1, \\ M\cos(\pi\ell), & k = N/2, \\ M\sin((k-N/2)\theta_\ell), & k = \frac{N}{2}+1,\dots,N-1, \end{cases}$$

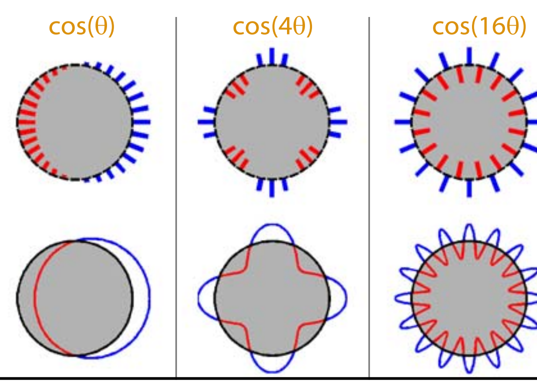
where M is current amplitude. Let V_ℓ^k denote the voltage measured on the ℓth electrode corresponding to the kth current pattern with

$$\sum_{\ell=1}^{N} V_\ell^k = 0.$$

Up to normalization, the ND matrix is given by

$$R_\gamma[m,n] := \sum_\ell T_\ell^m V_\ell^n.$$

We use trigonometric current patterns in both discrete and continuous form



The approximate scattering transform can now be written in terms of measured data

Expand the exponential function as series:

$$e^{ikz} = \sum_{n=-\infty}^{\infty} a_n(k)e^{in\theta} \quad \text{with} \quad a_n(k) = \begin{cases} \frac{(ik)^n}{n!}, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

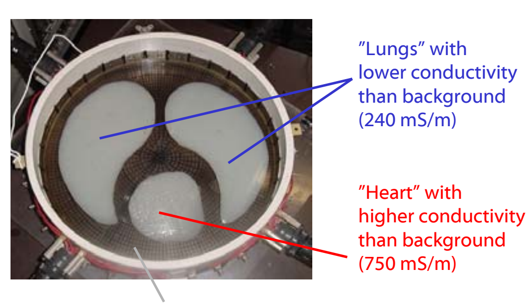
Then we can write

$$\mathrm{t}^{\exp}(k) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m(\bar{k})a_n(k)\int_{\partial\Omega} e^{im\theta}(\Lambda_\gamma - \Lambda_1)e^{in\theta}d\sigma(\theta).$$

The relation between DN map the measurements is, roughly,

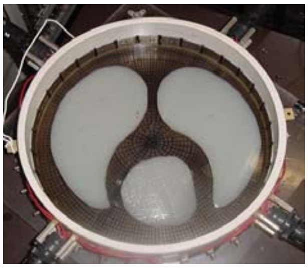
$$\Lambda_\gamma = R_\gamma^{-1}.$$

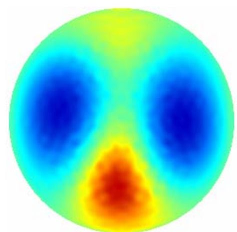
At the RPI lab, we construct a chest phantom consisting of saline and agar



Background of salt water, conductivity 424 mS/m.
Diameter of the tank is 30cm.

Reconstruction from measured data





Relative error 23% (lung) and 12% (heart).
Dynamical range is 94% of the true range.

Conclusion

We have developed a new kind of EIT algorithm
Large contrast details recovered remarkably well
The algorithm has rigorous mathematical background

Future challenges

Understanding reconstruction of discontinuities
Removing the requirement $\gamma=1$ near boundary
Taking the shape of domain into account
Developing a 3D algorithm

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