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ITERATIVE TIME-REVERSAL CONTROL FOR INVERSE PROBLEMS

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Abstract. A novel method to solve inverse problems for the wave equation is introduced. The method is a combination of the boundary control method and an iterative time reversal scheme, leading to adaptive imaging of coefficient functions of the wave equation using focusing waves in unknown medium. The approach is computationally effective since the iteration lets the medium do most of the processing of the data.

The iterative time reversal scheme also gives an algorithm for approximating a given wave in a subset of the domain without knowing the coefficients of the wave equation.

1. **Introduction.**

We present a novel inversion method for the wave equation. Suppose that we can send waves from the boundary into an unknown body with spatially varying wave speed $c(x)$. Using a combination of the boundary control (BC) method and an iterative time reversal scheme, we show how to focus waves near a point x_0 inside the medium and simultaneously recover $c(x_0)$ if the wave speed is isotropic. In the anisotropic case we can reconstruct the wave speed up to a change of coordinates.

Let us describe the simple isotropic case more precisely. Take a closed and bounded set $M \subset \mathbb{R}^3$ with smooth boundary ∂M, and let $c(x)$ be a scalar-valued

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wave speed in M. Consider the wave equation

$$\begin{aligned} u_{tt} - c(x)^2 \Delta u = 0 \qquad & \text{in } M \times \mathbb{R}_+, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0, & \\ -c(x)^{-2} \partial_n u = f(x,t) \qquad & \text{in } \partial M \times \mathbb{R}_+, \end{aligned} \tag{1}$$

where ∂_n denotes the Euclidean normal derivative. We denote by $u^f = u^f(x,t)$ the solution of (1) corresponding to the boundary source term f.

The inverse problem is to reconstruct the wave speed $c(x)$ from the set

$$\{f|_{\partial M \times (0,2T)},\, u^f|_{\partial M \times (0,2T)} :\ f \in C_0^\infty(\partial M \times (0,2T))\},$$

that is, the Cauchy data of solutions corresponding to all possible boundary sources $f \in C_0^\infty(\partial M \times (0,2T))$ with large enough observation time $2T$. These data constitute the *response operator*

$$\Lambda_{2T} : f \mapsto u^f|_{\partial M \times (0,2T)}, \tag{2}$$

also called the *non-stationary Neumann-to-Dirichlet map*. Physically, $\Lambda_{2T} f$ describes the measurement of the medium response to any applied boundary source f, see [26].

In the classical BC method material parameters are reconstructed from Λ_{2T} using hyperbolic techniques, see [1, 6, 7, 9, 23, 24, 25]. Straightforward numerical implementation of the BC method is difficult: some procedures involved (such as the Gram-Schmidt orthogonalization) are unstable, as seen in [8]. Inspired by Isaacson's iterative measurement scheme [20] for electrical impedance tomography (EIT), we overcome these problems by combining the BC method with an adaptive time reversal iteration that lets the medium do most processing of the data.

Traditional time reversal methods record waves, invert them in time, and send them back into the medium. If the recorded signal originates from point sources, or is the reflection of an input wave from small scatterers in the medium, the time reversed waves focus at the source or scatterer points. This is useful for time reversal mirrors in communication technologies and medical therapies. The efficient use of feedback is the main advantage of time reversal: outputs of previous measurements dictate the input of the next one. For early references on time reversal (using ultrasound in air), see the seminal works of Fink et al. [17, 16, 15]. For a microlocal discussion of time reversal, see [3, 4], and for another mathematical treatment see [27].

Time reversal in known background medium with random fluctuations has also been extensively studied, see e.g. [5, 12], and applied to medical imaging, nondestructive testing and underwater acoustics. These methods are outside the scope of this paper.

Iterative time reversal has been used to find best measurements for inverse problems. By "best" we here mean "optimal for detecting the presence of an object". This *distinguishability problem* has been studied for fixed-frequency problems in EIT [20] and acoustic scattering [36]. The connection between optimal measurements and iterative time-reversal experiments was pointed out in [36, 37, 38]. For the wave equation, the best measurement problem has been studied in [14], where the optimal incident field for probing a half space was found by an iteration involving time reversal.

Take $T > 0$ larger than the radius of M in travel time metric, so that any point inside M can be reached by waves sent from the boundary before time T. The new method introduced in this paper is based on a family of boundary sources $\widetilde{h}(\alpha)$ obtained from an arbitrary initial source f. For $\alpha > 0$, the sources $\widetilde{h}(\alpha)$ produce waves $u^{\widetilde{h}(\alpha)}(T)$ that converge to $C_0(x_0)u^f(x_0,T)\delta(x-x_0)$ as $\alpha \to 0$, where $x_0 \in M$ has prescribed travel time coordinates. We call $u^{\widetilde{h}(\alpha)}(x,t)$ the focusing waves, as at time $t = T$ the function $u^{\widetilde{h}(\alpha)}(x,T)$ is concentrated near x_0 when α is small.

The sources $\widetilde{h}(\alpha)$ are constructed iteratively for any $\alpha > 0$. Each step in the iteration uses time reversal, two very simple linear operators, and the response $\Lambda_{2T}g_j$ for one specific source g_j. We call this process *the iterative time reversal control* (ITRC).

The ITRC method proposed here has the advantage of the BC-method to be applicable to fully non-linear inverse problems with an unknown background. Simultaneously, ITRC exploits feedback properties of the time reversal schemes with the output of the previous measurement producing the input of the next measurement. The iterative nature of ITRC avoids non-physical sources which occur in the BC-method and provides robustness against measurement noise.

The ITRC method is valid not only for the equation (1) but for a more general equation

$$u_{tt}(x,t) + Au(x,t) = 0 \quad \text{in} \quad M \times \mathbb{R}, \tag{3}$$

where A is a formally self-adjoint elliptic partial differential operator. In particular, ITRC applies to anisotropic cases with lower order terms specified on Riemannian manifolds.

For the acoustic equation, we can use the focusing waves to determine the geodesics starting from the boundary points. For more general hyperbolic equations, we use those geodesics to determine the travel time distance between any interior point and any boundary point, leading to the recovery of the wave speed in the medium.

The paper is organized as follows. In Section 2 we formulate our main results separately for acoustic equation and for more general hyperbolic equations. In Section 3 we prove the convergence of iterative time-reversal control. Section 4 is devoted to the analysis of the focusing properties of the waves, and in Section 5 we show how to reconstruct the metric using the boundary distance function. In the last section we discuss our method in the case of noisy measurements.

2. Main results.

In this section we will describe the iteration procedure for the time reversal and apply it to solve the inverse problem for the acoustic wave equation (1). We will also describe the results on the inverse problems for the general wave equation (3). Proof of results are postponed to later sections.

2.1. Notations.

Let us consider the closure $M \subset \mathbb{R}^m$, $m \geq 1$, of an open smooth set, or a (non-compact or compact) complete Riemannian manifold (M, g) of dimension m with a non-empty boundary. For simplicity, we assume that the boundary ∂M is compact. Let u solve the wave equation

$$\begin{aligned} u_{tt}(x,t) + Au(x,t) = 0 \quad &\text{in} \quad M \times \mathbb{R}_+, \\ u|_{t=0} = 0, \quad & u_t|_{t=0} = 0, \\ B_{\nu,\eta}u|_{\partial M \times \mathbb{R}_+} = f. & \end{aligned} \tag{4}$$

Here, $f \in L^2(\partial M \times \mathbb{R}_+)$ is a real valued function, A is a formally self-adjoint elliptic partial differential operator of the form (in local coordinates in the case when M is a manifold)

(5)
$$A v=-\sum_{j,k=1}^{m}\mu(x)^{-1}|g(x)|^{-\frac{1}{2}}\frac{\partial}{\partial x^{j}}\left(\mu(x)|g(x)|^{\frac{1}{2}}g^{jk}(x)\frac{\partial v}{\partial x^{k}}(x)\right)+q(x)v(x),$$

where $g^{jk}(x)$ is a smooth real positive definite matrix, $|g| = \det(g^{jk}(x))^{-1}$, and $\mu(x) > 0$ and $q(x)$ are smooth real valued functions. Note that the form (5) is slightly different from the form used in [26]. Furthermore,

$$B_{\nu,\eta}v = -\partial_\nu v + \eta v,$$

where $\eta : \partial M \to \mathbb{R}$ is a smooth function and

$$\partial_\nu v = \sum_{j,k=1}^{m}\mu(x)g^{jk}(x)\nu_k\frac{\partial}{\partial x^j}v(x),$$

with $\nu(x) = (\nu_1, \nu_2, \dots, \nu_m)$ being the normalized, $\sum_{j,k=1}^m g^{jk}\nu_j\nu_k = 1$, interior co-normal vector field of ∂M.

A particular example is the operator

$$A_0 = -c^2(x)\Delta + q(x) \tag{6}$$

for which $\partial_\nu v = c(x)^{-m+1}\partial_n v$, where $\partial_n v$ is the Euclidean normal derivative of v.

We denote the solutions of (4) by

$$u(x,t) = u^f(x,t).$$

For the initial boundary value problem (4) we define the response operator (or the non-stationary Robin-to-Dirichlet map) Λ by setting

$$\Lambda f = u^f|_{\partial M\times\mathbb{R}_+}. \tag{7}$$

We also consider the finite time response operator Λ_T corresponding to the finite observation time $T > 0$,

$$\Lambda_T f = u^f|_{\partial M\times(0,T)}. \tag{8}$$

By [42], the map $\Lambda_T : L^2(\partial M \times (0,T)) \to H^{1/3}(\partial M \times (0,T))$ is bounded, where $H^s(\partial M \times (0,T))$ denotes the Sobolev space on $\partial M \times (0,T)$. Below we consider Λ_T as a bounded operator that maps $L^2(\partial M \times (0,T))$ to itself.

The matrix $g_{jk}(x)$ (the inverse of the matrix $g^{jk}(x)$) is called the *travel time metric*. This is because waves propagate with unit speed with respect to the metric $ds^2 = \sum_{jk} g_{jk}(x)dx^j dx^k$. We denote by $d(x,y)$ the distance function corresponding to $g_{jk}(x)$. For the wave equation we define the space $L^2(M, dV_\mu)$ with inner product

$$\langle u,v\rangle_{L^2(M,dV_\mu)} = \int_M u(x)v(x)\,dV_\mu(x),$$

where $dV_\mu = \mu(x)|g(x)|^{1/2}dx^1dx^2\dots dx^m$.

For $t > 0$ and $\Gamma \subset \partial M$, let

$$M(\Gamma,t) = \{x \in M \ : \ d(x,\Gamma) \le t\}, \tag{9}$$

be the domain of influence of Γ at time t.

For any set $B \subset \partial M \times \mathbb{R}_+$, we denote $L^2(B) = \{f \in L^2(\partial M \times \mathbb{R}_+) : \text{ supp } (f) \subset B\}$, identifying functions and their zero continuations. When $\Gamma \subset \partial M$ is an open

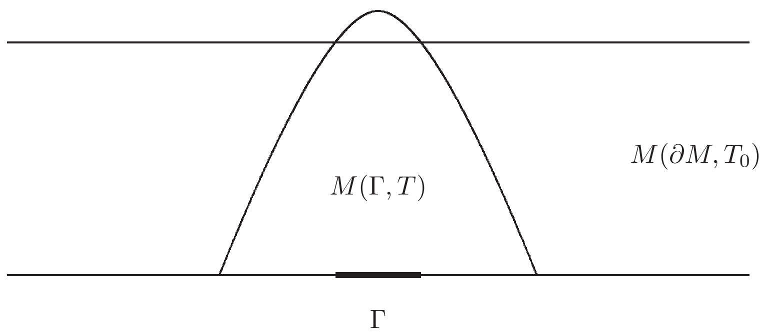

Figure 1. The set $M(\Gamma,T)\setminus M(\partial M,T_0)$. Our goal is to find waves sent from the boundary that focus into this area.

set and $f \in L^2(\Gamma\times\mathbb{R}_+)$, it is well known (see e.g. [19]) that the wave $u^f(t) = u^f(\cdot\,,t)$ is supported in the domain $M(\Gamma,t)$,

$$u^f(t) \in L^2(M(\Gamma,t)) = \{v \in L^2(M):\ \text{supp}\ (v) \subset M(\Gamma,t)\}.$$

2.2. **Definition and convergence results for the ITRC.** Our objective is to find a boundary source h such that the wave $u^h(x,T)$ is localized in a neighbourhood of a single point x_0. Note that in this paper we do not focus the pair $(u^h(x,T), u^h_t(x,T))$, but only the value of the wave, $u^h(x,T)$.

Let $\Gamma \subset \partial M$ be an open set and $f \in L^2(\Gamma\times\mathbb{R}_+)$. Then $\text{supp}(u^f(T)) \subset M(\Gamma,T)$. Let now $0 < T_0 < T$ so that the set $M(\Gamma,T)\setminus M(\partial M,T_0)$ is a subregion in M, see Fig. 1. We will show how to construct boundary sources $h(\alpha) \in L^2(\partial M\times[0,T])$, $\alpha > 0$, such that

$$\lim_{\alpha\to 0} u^{h(\alpha)}(T) = \chi_{M(\partial M,T_0)} u^f(T), \tag{10}$$

where $\chi_N(x)$ is the characteristic function of a set N. Then

$$\lim_{\alpha\to 0} u^{f-h(\alpha)}(T) = (1-\chi_{M(\partial M,T_0)})u^f(T)$$

is the wave $u^f(T)$ cut-off onto $M(\Gamma,T)\setminus M(\partial M,T_0)$, see Fig. 1. Actually, our construction will be valid in a more general setting when, instead of the collar neighborhood $M(\partial M,T_0)$, we will be dealing with a more general set N of the form $N = \bigcup_{j=1}^J M(\Gamma_j,T_j)$.

To proceed further, we define the *time reversal map* R_{2T} and the *time filter* J_{2T}:

$$\begin{aligned} R_{2T}f(x,t) &= f(x,2T-t), \\ J_{2T}h(x,t) &= \int_{[0,2T]} J_{2T}(s,t)h(x,s)ds \end{aligned} \tag{11}$$

where $J_{2T}(s,t) = \frac{1}{2}\chi_L(s,t)$,

$$L = \{(s,t)\in\mathbb{R}_+\times\mathbb{R}_+:\ t+s\le 2T,\quad s>t\}. \tag{12}$$

We note that the doubling of time, from T to $2T$ in (12) is due to the fact that, for wave u^f measured at ∂M to bring information about all points $x \in M(\Gamma, T)$, the measurements' time should be at least $2T$.

For an open set $B \subset \partial M \times [0, 2T]$, we denote by $P = P_B : L^2(\partial M \times [0, 2T]) \to L^2(\partial M \times [0, 2T])$ the multiplication operator

$$P_B f(x,t) = \chi_B(x,t)\, f(x,t).$$

We choose $B = \bigcup_{j=1}^J (\Gamma_j \times [T - T_j, T])$, where $\Gamma_j \subset \partial M$ are open sets and $0 \le T_j \le T$. In the following, we consider Λ_{2T}, R_{2T}, P_B and J_{2T} as operators in $L^2(\partial M \times [0, 2T])$.

Let $\alpha \in (0,1)$ and $\omega > 0$ be a sufficiently large constant. We define a_n, b_n and $h_n = h_n(\alpha) \in L^2(\partial M \times [0, 2T])$ iteratively:

$$\begin{aligned} &a_n := \Lambda_{2T}(h_n), \quad b_n := \Lambda_{2T}(R_{2T} J_{2T} h_n), \\ &h_{n+1} := (1 - \frac{\alpha}{\omega}) h_n - \frac{1}{\omega}(P_B R_{2T} b_n - P_B J_{2T} a_n) + F, \end{aligned} \tag{13}$$

with $a_0 = 0$, $b_0 = 0$, $h_0 = 0$, and

$$F = \frac{1}{\omega} P_B (R_{2T} \Lambda_{2T} R_{2T} J_{2T} - J_{2T} \Lambda_{2T}) f.$$

Here a_n corresponds to the "iterated measurement", b_n to the "time filtered and time-reversed measurement" and h_{n+1} to post-processing of h_n, a_n and b_n using time reversal and time filtering. We say that formula (13) describes iterative time-reversal control (ITRC) with time-interval $[0, 2T]$, projector P_B, and starting point $f \in L^2(\partial M \times [0, 2T])$.

Theorem 1. *Let $T > 0$. Assume we are given ∂M and the response operator Λ_{2T}. Let $\Gamma_j \subset \partial M$, $j = 1, \dots, J$, be non-empty open sets, $0 \le T_j \le T$, and $B = \bigcup_{j=1}^J (\Gamma_j \times [T - T_j, T])$. Let $f \in L^2(\partial M \times \mathbb{R}_+)$ and let, for ω large enough and $\alpha \in (0,1)$, the functions $h_n = h_n(\alpha)$ be defined by the ITRC (13) with projector P_B and starting point $f_0 = f|_{\partial M \times (0,2T)}$. These functions converge in $L^2(\partial M \times \mathbb{R}_+)$,*

$$h(\alpha) = \lim_{n \to \infty} h_n(\alpha)$$

and the limits satisfy

$$\lim_{\alpha \to 0} u^{h(\alpha)}(x, T) = \chi_N(x) u^f(x, T)$$

in $L^2(M)$, where $N = \bigcup_{j=1}^J M(\Gamma_j, T_j) \subset M$.

We note that similar approaches in the case when $N = M$ have been developed in [21, 31].

This theorem is proven in Section 3. Let us now consider some its consequences.

2.3. **Results on focusing of the waves.** Let us consider a geodesic $\gamma_{x,\xi}$ in (M, g) parameterized by the arclength with $\gamma_{x,\xi}(0) = x$, $\dot\gamma_{x,\xi}(0) = \xi$, and $\|\xi\|_g = 1$. Let $\nu = \nu(z)$, $z \in \partial M$ be the interior unit normal vector to ∂M. There is a critical value $\tau(z) \in (0, \infty]$, such that for $t < \tau(z)$ the geodesic $\gamma_{z,\nu}([0,t])$ is the unique shortest geodesic from its endpoint $\gamma_{z,\nu}(t)$ to ∂M, and for $t > \tau(z)$ it is no longer a shortest geodesic.

We say that $\Gamma_j \to \{\hat z\}$ if $\Gamma_{j+1} \subset \Gamma_j$ and $\bigcap_{j=1}^\infty \Gamma_j = \{\hat z\}$.

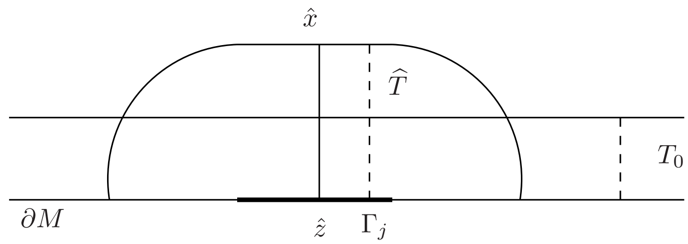

FIGURE 2. Subsets of M used in Corollary 1. Note that the set $M(\Gamma_j, \widehat{T}) \setminus M(\partial M, T_0)$ shrinks to one point $x = \hat{x}$ when $j \to \infty$ and $T_0 \to \hat{T}$.

Theorem 1 yields the following result which provides a method to generate focusing waves, that is, the wave from the boundary that, at a fixed time $t = T$, are supported at a single point.

Corollary 1. *Let $\hat{z} \in \partial M$ and $0 < T_0 < \widehat{T} < T$. Let $\hat{x} = \gamma_{\hat{z},\nu}(\widehat{T})$ and $\Gamma_j \subset \partial M$, $j \in \mathbb{Z}_+$ be open neighbourhoods of $\hat{z} \in \partial M$ such that $\overline{\Gamma}_{j+1} \subset \Gamma_j$ and $\Gamma_j \to \{\hat{z}\}$ when $j \to \infty$. Let $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$ and $h_n(\alpha; T_0, j)$ be the functions obtained by the ITRC (13) with projector P_B, $B = (\Gamma_j \times [T - \widehat{T}, T]) \cup (\partial M \times [T - T_0, T])$ and starting point $f_0 = f|_{\partial M \times (0,2T)}$. Similarly, let $h'_n(\alpha; T_0)$ be the functions obtained by (13) with projector $P_{B'}$, $B' = \partial M \times [T - T_0, T]$ and starting point f_0. Denote*

$$\widetilde{h}_n(\alpha; T_0, j) = h_n(\alpha; T_0, j) - h'_n(\alpha; T_0).$$

If $\widehat{T} < \tau(\hat{z})$ then

(14)
$$\lim_{T_0 \to \hat{T}-} \lim_{j \to \infty} \lim_{\alpha \to 0} \lim_{n \to \infty} \frac{1}{(\widehat{T} - T_0)^{(m+1)/2}} u^{\widetilde{h}_n(\alpha; T_0, j)}(T) = C_0(\hat{x}) u^f(\hat{x}, T) \delta_{\hat{x}}(x)$$

in $\mathcal{D}'(M)$ where $C_0(\hat{x}) > 0$ does not depend on f. If $\widehat{T} > \tau(\hat{z})$ the limit (14) is zero.

Above, $\delta_{\hat{x}}$ is the Dirac delta-distribution on (M, g) such that

$$\int_M \delta_{\hat{x}}(x) \phi(x) \, dV_\mu = \phi(\hat{x}), \quad \phi \in C_0^\infty(M).$$

Note that we can determine $\tau(\hat{z})$ by finding supremum of all those $\hat{T}$ for which (14) is non-zero with some f.

2.4. **Acoustic wave equation.** Before formulating the results on the inverse problems for the general equation (4), (5), let us describe the procedure of finding the unknown wave speed, $c(x)$ for the equation (1). The additional property used for this case is that we can actually find the Euclidian coordinates $\hat{x}^k$, $k = 1, \dots, m$, of the point $\hat{x} = \gamma_{\hat{z},\nu}(\widehat{T})$ described in Corollary 1. Indeed, for any boundary source f, we have two important identities to be proven in section 3. To describe them, we introduce the functionals

$$I^j_{2T} f = \int_0^{2T} \int_0^{t'} \int_{\partial M} (f(x, t'') x^j - \Lambda_{2T} f(x, t'') \partial_\nu x^j) \, dS_g(x) dt'' dt', \tag{15}$$

$$I_{2T}^0 f = \int_0^{2T}\int_0^{t'}\int_{\partial M} f(x,t)\, dS_g(x)\, dt'' dt', \tag{16}$$

where $(x^1,\dots,x^m)$ are the Euclidian coordinates of $x\in M$. Then, defining $\mu(x) = c(x)^{2-n}$, we have

$$\int_M x^j u^f(x,T)\, dV_\mu(x) = I_{2T}^j f, \quad \int_M u^f(x,T)\, dV_\mu(x) = I_{2T}^0 f. \tag{17}$$

As the right-hand side of (15) is written in terms of the response operator Λ_{2T}, this makes it possible to immediately solve the inverse problem for the acoustic equation (1):

Corollary 2. *Let $\hat z\in\partial M$ and $0<\widehat T<T$, $\hat x = \gamma_{\hat z,\nu}(\widehat T)$. Also, let $f\in C_0^\infty(\partial M\times\mathbb{R}_+)$ and $\widetilde h_n(\alpha;T_0,j)$ be the functions obtained from the ITRC as in Corollary 1. Assume that the limit (14) is non-zero. Then*

$$\lim_{T_0\to\widehat T}\lim_{j\to\infty}\lim_{\alpha\to 0}\lim_{n\to\infty}\frac{I_{2T}^l \widetilde h_n(\alpha;T_0,j)}{I_{2T}^0\widetilde h_n(\alpha;T_0,j)} = \hat x^l(\hat z,\widehat T), \quad l=1,2,\dots,m.$$

where $\hat x^l(\hat z,\widehat T)$ are the Euclidian coordinates of $\hat x = \gamma_{\hat z,\nu}(\widehat T)$.

Moreover,

$$c(\hat x)^2 = \sum_{l=1}^m \left(\partial_s \hat x^l(\hat z,s)|_{s=\widehat T}\right)^2.$$

We note that, for equation (17) to be valid, we need $Au=0$ for $u=x^l$ and $u=1$. This is definitely the case for the acoustic operator, $A=c(x)^2\Delta$ but not for a general A of form (5).

2.5. **General hyperbolic equation.** It is well known, see e.g. [23, 24, 25, 28], that the response map Λ can not, in general, determine coefficients μ and g^{jk} of operator A (see (5)) uniquely due to two transformations discussed below.

First, we can introduce a differentiable coordinate transformation $F: M\to M$ such that the boundary value $F|_{\partial M}$ is the identity operator. Then the push forward metric $\widetilde g = F_* g$, that is,

$$\widetilde g_{jk}(y) = \sum_{p,\ell=1}^m \frac{\partial x^p}{\partial y^j}\frac{\partial x^\ell}{\partial y^k}\, g_{p\ell}(x), \quad y = F(x),$$

and the functions $\widetilde\mu = \mu\circ F^{-1}$, $\widetilde q = q\circ F^{-1}$, and $\widetilde\eta=\eta$ determine the operator $\widetilde A$ of form (5) such that the response operators for A and $\widetilde A$ are the same.

Second, we can apply a gauge transformation $u(x,t)\to\kappa(x)u(x,t)$, where $\kappa\in C^\infty(M)$ is a strictly positive function. The operator $\widetilde A$ is then transformed to the operator $\widetilde A_\kappa$,

$$\widetilde A_\kappa w := \kappa\widetilde A(\kappa^{-1}w)$$

and the boundary operator $B_{\nu,\eta}$ – to $B_{\nu,\widehat\eta}$, with

$$\widehat\eta = \eta - \kappa^{-1}\partial_\nu\kappa.$$

Thus, the response operator $\widetilde\Lambda_\kappa$ of $\widetilde A_\kappa$ coincides with the response operator Λ of A.

Note that, making a gauge transformation with $\kappa=\mu^{-1}$, we come to the operator $\widetilde A_\kappa$ of form (5) with $\mu(x)=1$.

Corollary 3. *Assume that $\mu = 1$ and we are given the boundary ∂M and the response operator Λ. Then, using ITRC, we can find, in a constructive way, the manifold (M, g) up to an isometry and the operator A uniquely.*

Moreover, if $M \subset \mathbb{R}^m$, $m \geq 2$, and A is of form (6), given the set M and the response operator Λ we can determine $c(x)$ and $q(x)$ uniquely.

3. Proof of the convergence of the iteration.

3.1. Controllability results. Proof of equation (17).

The seminal result implying controllability is Tataru's unique continuation result, see [41, 43]:

Theorem 2 (Tataru). *Let u be a solution of wave equation*

$$u_{tt}(x,t) + Au(x,t) = 0.$$

Assume that

$$u|_{\Gamma\times(0,2\tau)} = 0,\ \ \partial_\nu u|_{\Gamma\times(0,2\tau)} = 0 \tag{18}$$

where $\Gamma \subset \partial M$, $\Gamma \neq \emptyset$ is open, and $\tau > 0$. Then,

$$u(x,\tau) = 0,\ \ \partial_t u(x,\tau) = 0 \quad \text{for } x \in M(\Gamma,\tau).$$

This result yields the following Tataru's controllability result, see e.g. [24] and references therein.

Theorem 3. *Let $\Gamma \subset \partial M$ be open and $\tau > 0$. Then the linear subspace*

$$\{u^f(\tau) \in L^2(M(\Gamma,\tau)) :\ f \in C_0^\infty(\Gamma\times[0,\tau])\}$$

is dense in $L^2(M(\Gamma,\tau))$.

3.2. Inner products.

Let us consider the Blagovestchenskii identity, which goes back to [10]

Lemma 1. *Let $f, h \in L^2(\partial M \times [0, 2T])$. Then*

$$\int_M u^f(x,T)u^h(x,T)\,dV_\mu(x) = \tag{19}$$

$$\int_{[0,2T]^2}\int_{\partial M} J(t,s)\big[f(t)(\Lambda_{2T}h)(s) - (\Lambda_{2T}f)(t)h(s)\big]\,dS_g(x)dtds,$$

where $J(t,s) = \frac{1}{2}\chi_L(s,t)$, see (12).

The proof in a slightly different context is given e.g. in [24].

Proof. Let $w(t,s) = \int_M u^f(t)u^h(s)\,dV_\mu$. Integrating by parts, we see that

$$\begin{aligned}
&(\partial_t^2 - \partial_s^2)w(t,s) \\
=\ & -\int_M \big[Au^f(t)u^h(s) - u^f(t)Au^h(s)\big]\,dV_\mu(x) \\
& -\int_{\partial M}\big[\partial_\nu u^f(t)u^h(s) - u^f(t)\partial_\nu u^h(s)\big]\,dS_g \\
=\ & \int_{\partial M}\big[(-\partial_\nu u^f(t) + \eta u^f(t))u^h(s) - u^f(t)(-\partial_\nu u^h(s) + \eta u^h(s))\big]\,dS_g \\
=\ & \int_{\partial M}\big[f(t)\Lambda_{2T}h(s) - \Lambda_{2T}f(t)h(s)\big]\,dS_g.
\end{aligned} \tag{20}$$

Moreover, as

$$w|_{t=0} = w|_{s=0} = 0, \quad \partial_t w|_{t=0} = \partial_s w|_{s=0} = 0,$$

we can consider (20) as one dimensional wave equation with known right hand side and vanishing initial and boundary data. Solving this initial boundary value problem, we obtain (19). □

The Schwartz kernel of Λ_{2T} is the Dirichlet boundary value of the Green's function $G(x, x', t-t')$, $t' > 0$, satisfying

$$\begin{aligned}(\partial_t^2 + A)G_{x',t'}(x,t) &= \delta_{x'}(x)\delta(t-t') \quad \text{in } M \times \mathbb{R}_+, \\ G_{x',t'}|_{t=0} = 0, \quad \partial_t G_{x',t'}|_{t=0} &= 0, \quad B_{\nu,\eta} G_{x',t'}|_{\partial M \times \mathbb{R}_+} = 0,\end{aligned} \tag{21}$$

where $G_{x',t'}(x,t) = G(x,x',t-t')$. As

$$G(x,x',t-t') = G(x',x,t-t'),$$

we see that

$$\Lambda_{2T}^* = R_{2T}\Lambda_{2T}R_{2T}$$

where $R_{2T}f(x,t) = f(x, 2T-t)$ is the time reversal map.

Thus, we can rewrite formula (19) in the form

$$\int_M u^f(x,T)u^h(x,T)\,dV_\mu(x) = \int_{\partial M \times [0,2T]} (Kf)(x,t)\,h(x,t)\,dS_g(x)dt \tag{22}$$

where K is defined as

$$K = K_{2T} := R_{2T}\Lambda_{2T}R_{2T}J_{2T} - J_{2T}\Lambda_{2T}. \tag{23}$$

Analyzing (23), we see that the inner product in the left-hand side of (22) can be found by making two measurements, one with the input f and the other with the input $R_{2T}J_{2T}f$, obtained from f by basic operations of the time reversal R_{2T} and the time filtering J_{2T}.

Similar considerations make it possible to prove equations (17) for the acoustic equation (1). Recall that in this case $\mu = c(x)^{2-m}$. Let $p(x) = x^j$, $j = 1, \dots, m$, be coordinate functions and

$$v_j(t) = \int_M p(x)u^f(x,t)\,dV_\mu(x).$$

Note that $\Delta p(x) = 0$. Integrating by parts, we see that

$$\begin{aligned}\partial_t^2 v_j(t) &= -\int_M (-c^2(x)\Delta u^f(x,t))p(x)\,dV_\mu(x) = \int_M \Delta u^f(x,t)p(x)\,dx \\ &= \int_{\partial M} (\partial_n u^f(x,t)p(x) - u^f(x,t)n^j)\,dS_e(x) \\ &= \int_{\partial M} (f(x,t)p(x) - (\Lambda_{2T}f)(x,t)\partial_\nu p(x))\,dS_g(x),\end{aligned}$$

where $n = (n^1, \dots, n^m)$ is the Euclidian unit normal to ∂M. Again, $v_j(0) = v_j'(0) = 0$ and we obtain formula (15) by integration. The formula (17) follows analogously using the constant function $p(x) = 1$.

3.3. **Proof of Theorem 1.** Let us analyze the minimization problem

$$\min_{h\in L^2(B)} \|u^f(T)-u^h(T)\|^2_{L^2(M,dV_\mu)} \tag{24}$$

where $B=\bigcup_{j=1}^J(\Gamma_j\times[T-T_j,T])$, $\Gamma_j\subset\partial M$ are open and $0\le T_j\le T$.

As the solution of this minimization problem does not always exist, we can regularize it and study

$$\min_{h\in L^2(\partial M\times[0,2T])} F(h,\alpha), \tag{25}$$

where $\alpha\in(0,1)$ and

$$F(h,\alpha)=\langle K(Ph-f),Ph-f\rangle_{L^2(\partial M\times[0,2T],dS_g)}+\alpha\|h\|^2_{L^2(\partial M\times[0,2T],dS_g)}.$$

We recall that $P=P_B$ is multiplication with the characteristic function of B, that is, $(P_Bh)(x,t)=\chi_B(x,t)\cdot h(x,t)$.

By (22),

$$F(h,\alpha)=\|u^f(T)-u^{Ph}(T)\|^2_{L^2(M,dV_\mu)}+\alpha\|h\|^2_{L^2(\partial M\times[0,2T])}.$$

The minimization problem (25) is equivalent to (24) when $\alpha=0$.

Lemma 2. *For given $\alpha\in(0,1)$ the problem (25) has a unique minimizer. Moreover, the minimizer is the unique solution of the equation*

$$(PKP+\alpha)h=PKf. \tag{26}$$

Proof. The minimization problem is strictly convex, and the map $h\mapsto u^h(T)$ is continuous $L^2(\partial M\times[0,2T])\to H^{s_1}(M)$, $s_1<3/5$ by [35], see also [34]. Now if $h_j\to h$ weakly in $L^2(\partial M\times[0,2T])$ as $j\to\infty$ then $Ph_j\to Ph$ weakly in $L^2(\partial M\times[0,2T])$. As the embedding $H^{s_1}(M)\to L^2(M)$ is compact, then $u^{Ph_j}(T)\to u^{Ph}(T)$ in $L^2(M)$. Therefore,

$$\begin{aligned}&\lim_{j\to\infty}\langle K(Ph_j-f),Ph_j-f\rangle_{L^2(\partial M\times[0,2T])}\\&\qquad=\lim_{j\to\infty}\|u^{Ph_j}(T)-u^f(T)\|^2_{L^2(M,dV_\mu)}=\|u^{Ph}(T)-u^f(T)\|^2_{L^2(M,dV_\mu)}.\end{aligned}$$

Let h_j be a sequence such that $\lim_{j\to\infty}F(h_j,\alpha)=\inf_h F(h,\alpha)$. As $F(h,\alpha)\ge\alpha\|h\|^2_{L^2}$, the sequence (h_j) is bounded. Thus by choosing a subsequence of h_j, we can assume that h_j converge to h in the weak topology of $L^2(\partial M\times[0,2T])$, Since $\|h\|_{L^2}\le\liminf_{j\to\infty}\|h_j\|_{L^2}$, this implies that $F(h,\alpha)\le\liminf F(h_j,\alpha)$. Thus, the limit h is a global minimizer of $F(\cdot,\alpha)$.

By computing the Fréchet derivative, D_h of the functional $F(h,\alpha)$ in any direction $\widetilde{h}\in L^2(\partial M\times[0,2T])$ at a minimizer h, we see that

$$\begin{aligned}0&=D_h(\langle K(Ph-f),Ph-f\rangle+\alpha\|h\|^2_{L^2})\widetilde{h}\\&=\langle\widetilde{h},P^*K(Ph-f)\rangle+\langle\widetilde{h},P^*K^*(Ph-f)\rangle+2\alpha\langle\widetilde{h},h\rangle\end{aligned}$$

for all $\widetilde{h}$. Since $K^*=K$ and $P^*=P$, this implies that h satisfies equation (26). As K is non-negative, $PKP+\alpha I\ge\alpha I$ so that equation (26) has a unique solution providing the minimizer $h=h(\alpha)$. □

Note that equation (26) also implies that

$$h(\alpha)\in\operatorname{Ran}(P). \tag{27}$$

Next we want to solve equation (26) using iteration. To this end, let $\omega \in \mathbb{R}_+$ be a constant such that

$$\omega > 2(1 + \|PKP\|).$$

Then equation (26) can be written as

$$(I - S)h = \frac{1}{\omega} PKf, \quad \text{where} \quad S = I - \frac{\alpha + PKP}{\omega}. \tag{28}$$

Since PKP is non-negative and

$$\alpha I \le \alpha I + PKP \le \frac{\omega}{2} I,$$

we see that $\|S\| \le 1 - \alpha/\omega < 1$. Thus we can solve h using iterations: Let

$$F := \frac{1}{\omega} PKf = \frac{1}{\omega} P(J_{2T}\Lambda_{2T} - R_{2T}\Lambda_{2T}R_{2T}J_{2T})f,$$

$h_0 = 0$, and consider the iterations

$$h_{n+1} = Sh_n + F, \quad n = 0, 1, \dots.$$

As $h_n = Ph_n$, we can write these iterations in the form (13), and $\lim_{n\to\infty} h_n = h(\alpha)$ in $L^2(\partial M \times [0, 2T])$.

Applying this algorithm can find the minimizers $h = h(\alpha) \in \mathrm{Ran}\,(P)$ of problem (26). The corresponding waves converge by the following lemma:

Lemma 3. *We have*

$$\lim_{\alpha \to 0} u^{h(\alpha)}(x, T) = \chi_N(x) u^f(x, T)$$

in $L^2(M)$, with $N = \bigcup_{j=1}^J M(\Gamma_j, T_j)$ as in Theorem 1.

Proof. As $u^{Ph}(T) = \chi_N u^{Ph}(T)$,

$$F(h, \alpha) = F_0 + \|\chi_N(u^{Ph}(T) - u^f(T))\|^2_{L^2(M)} + \alpha \|h\|^2_{L^2(\partial M \times [0,2T])}, \tag{29}$$

where

$$F_0 = \|(1 - \chi_N) u^f(T)\|^2_{L^2(M)}$$

does not depend on h. By Theorem 3, for any $\epsilon > 0$ there is $h_\epsilon \in L^2(B)$ such that

$$\|u^{Ph_\epsilon}(T) - \chi_N u^f(T)\|^2_{L^2(M)} < \frac{\epsilon}{2}. \tag{30}$$

Thus we have

$$F(h_\epsilon, \alpha) = F_0 + \frac{\epsilon}{2} + \alpha \|h(\epsilon)\|^2_{L^2(\partial M \times [0,2T])}.$$

This shows that if $\alpha < \alpha(\epsilon)$, where

$$\alpha(\epsilon) := \frac{\epsilon}{2\|h(\epsilon)\|^2_{L^2(\partial M \times [0,2T])}},$$

then

$$F(h_\epsilon, \alpha) \le F_0 + \epsilon.$$

Thus the minimizer $h(\alpha)$ of $F(h, \alpha)$ with $\alpha < \alpha(\epsilon)$ satisfies

$$F(h(\alpha), \alpha) \le F_0 + \epsilon.$$

As by (27), $h(\alpha) = Ph(\alpha)$, we get from (29) that, for $\alpha < \alpha(\epsilon)$,

$$\|\chi_N(u^{h(\alpha)}(T) - u^f(T))\|^2_{L^2(M)} \le \epsilon. \tag{31}$$

Using again that $u^{h(\alpha)}(T) = \chi_N u^{h(\alpha)}(T)$, the claim follows. □

Theorem 1 follows from Lemmata 2 and 3. □

4. **Proofs for the focusing of the waves.** In this section we prove Corollaries 1 and 2.

Proof of Corollary 1. Let $(z(x), s(x))$ be the boundary normal coordinates of $x \in M$, that is, $s(x) = d(x, \partial M)$ and $z(x)$ is the closest point of ∂M to x when such a point is unique. When a closest boundary point is not unique, the boundary normal coordinates are not defined.

We consider first the claim of the corollary in the case when $\widehat{T} < \tau(\hat{z})$. Then near $\hat{x} = \gamma_{\hat{z},\nu}(\widehat{T})$ the boundary normal coordinates are well defined and the metric tensor in these coordinates has the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & [g_{\alpha\beta}(z,s)]_{\alpha,\beta=1}^{m-1} \end{pmatrix}, \tag{32}$$

where $[g_{\alpha\beta}(z,s)] \in \mathbb{R}^{(m-1)\times(m-1)}$ is a smooth positive definite matrix-valued function near $(\hat{z}, \widehat{T}) = (z(\hat{x}), s(\hat{x}))$. The distance function $f(z,s) = d(\gamma_{z,\nu}(s), \hat{z})$ is smooth near $(\hat{z}, \hat{s})$, $\hat{s} = s(\hat{x})$, and there is a $C_1 = C_1(\hat{z}, \widehat{T}) > 1$ such that

$$C_1^{-1} I \le [g_{\alpha\beta}(z,s)] \le C_1 I$$

in an open neighborhood of $\{(\hat{z}, t) :\ t \le \widehat{T}\}$. This implies that there is $C_2 > 1$ such that near $(\hat{z}, \hat{s})$

$$C_2^{-1} d_{\partial M}^2(z, \hat{z}) \le f(z,s) - s \le C_2 d_{\partial M}^2(z, \hat{z}). \tag{33}$$

When the smooth surface $f(z,s) = \widehat{T}$ is represented in the form $s = S(z)$, the formula (33) implies that the derivative of $S(z)$ at $\hat{z}$ vanishes and that the Hessian of $S(z)$ at $\hat{z}$ is strictly negative, that is, the surface is strictly convex near $\hat{x}$. Since the set $M(\{\hat{z}\}, \widehat{T}) \setminus M(\partial M, T_0))$ in the boundary normal coordinates is $\{(z,s) : T_0 < s \le S(z)\}$, a straightforward computation shows that the limit

$$C_0(\hat{x}) := \lim_{T_0 \to \widehat{T}-} \frac{\mathrm{vol}_g\big(M(\{\hat{z}\}, \widehat{T}) \setminus M(\partial M, T_0)\big)}{(\widehat{T} - T_0)(\widehat{T}^2 - T_0^2)^{(m-1)/2}} \tag{34}$$

exists and $C_0(\hat{x}) > 0$. As the solution $u^f(x,t)$ is smooth for $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$,

$$\lim_{T_0 \to \widehat{T}-} \frac{\chi_{M(\{\hat{z}\},\widehat{T}) \cup M(\partial M, T_0)} - \chi_{M(\partial M, T_0)}}{\mathrm{vol}_g\big(M(\{\hat{z}\}, \widehat{T}) \setminus M(\partial M, T_0)\big)}\, u^f(T) = u^f(\hat{x}, T)\delta_{\hat{x}} \tag{35}$$

in $\mathcal{D}'(M)$. By multiplying formulae (34) and (35) the claim follows in the case $\widehat{T} < \tau(\hat{z})$.

When $\widehat{T} > \tau(z)$, the claim follows from the fact that, for sufficiently small $\widehat{T} - T_0 > 0$, the set $M(\{\hat{z}\}, \widehat{T}) \setminus M(\partial M, T_0)$ is empty. □

Proof of Corollary 2. The claim follows directly from Corollary 1 if we take into account that the geodesics have speed one with respect to the travel time metric $c(x)^2|dx|^2$ and that almost any point in M lies on a unique shortest normal geodesic to ∂M. □

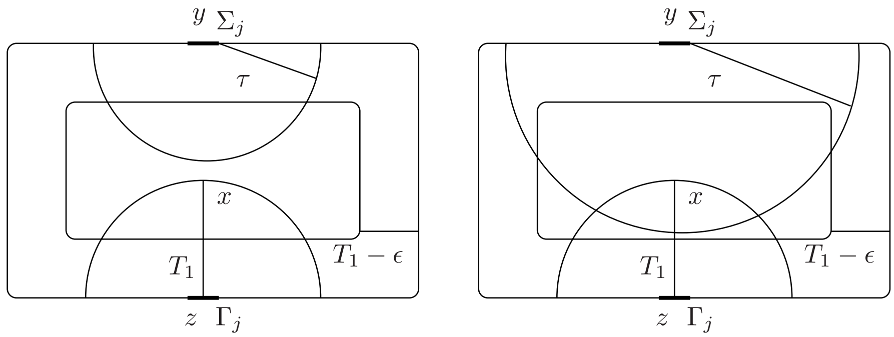

Figure 3. Sketch of the situation in Lemma 4 when $I(j,\epsilon)$ is empty (left) and non-empty (right).

5. **Boundary distance functions and reconstruction of the metric.** Here we present a method for finding the boundary distance functions using ITRC.

Let $z, y \in \partial M$, $0 \le T_1 \le \tau(z)$ and $T > \mathrm{diam}(M)$. Denote $x = \gamma_{z,\nu}(T_1)$. Next we give an algorithm that can be used to determine $d(x, y)$.

To this end, let $\Gamma_j \subset \partial M$ and $\Sigma_j \subset \partial M$ be neighbourhoods of z and y, respectively, such that $\overline{\Gamma}_{j+1} \subset \Gamma_j$ and $\overline{\Sigma}_{j+1} \subset \Sigma_j$, $\Gamma_j \to \{z\}$ and $\Sigma_j \to \{y\}$ when $j \to \infty$.

Let $\epsilon > 0$, $\tau \in [0, T]$, and

$$
\begin{aligned}
N_j^1 &= M(\Gamma_j, T_1) & B_j^1 &= \Gamma_j \times [T - T_1, T]\\
N_j^2 &= M(\Sigma_j, \tau) & B_j^2 &= \Sigma_j \times [T - \tau, T]\\
N_\epsilon^3 &= M(\partial M, T_1 - \epsilon) & B_\epsilon^3 &= \partial M \times [T - (T_1 - \epsilon), T].
\end{aligned}
$$

Lemma 4. *The distance $d(x, y)$ is the infimum of all $\tau \in [0, T]$ that satisfy the condition*

$$
\text{the set } I(j,\epsilon) = (N_j^1 \cap N_j^2) \setminus N_\epsilon^3 \text{ contains a non-empty open set for all } j \in \mathbb{Z}_+,\ \epsilon > 0. \tag{36}
$$

See Figure 3 for a sketch of the cases of empty and non-empty $I(j, \epsilon)$.

Proof. First consider what happens if $d(x, y) < \tau$. Since $T_1 \le \tau(z)$, we see that then $B(x, r) \cap (N_j^1 \setminus N_\epsilon^3)$ contains a non-empty open set for all $r > 0$, where $B(x, r) \subset M$ is a ball of (M, g) with center x and radius r. When $r < \tau - d(x, y)$, we see that $B(x, r) \subset N_j^2$. Thus $I(j, \epsilon)$ contains an open set and (36) is satisfied.

On other hand, if $d(x, y) > \tau$, let $r = d(x, y) - \tau$. When $j \to \infty$ and $\epsilon \to 0$, we see using metric in the boundary normal coordinates (32) that $N_j^1 \setminus N_\epsilon^3 \to \{x\}$ in the Hausdorff metric. Thus when j is large enough and ϵ is small enough, $N_j^1 \setminus N_\epsilon^3 \subset B(x, r/2)$. Then $B(x, r/2) \cap N_j^2 = \emptyset$, and (36) is not satisfied.

Summarizing, condition (36) is satisfied if $d(x, y) < \tau$ and not satisfied if $d(x, y) > \tau$. As $d(x, y) < T$, this yields the claim. □

Next we show that by using ITRC we can test if condition (36) is valid. To this end, denote

$$\begin{aligned}
\widetilde{N}_1(j,\epsilon) &= N_j^1 \cup N_\epsilon^3 & \widetilde{B}_1(j,\epsilon) &= B_j^1 \cup B_\epsilon^3\\
\widetilde{N}_2(j,\epsilon) &= N_j^2 \cup N_\epsilon^3 & \widetilde{B}_2(j,\epsilon) &= B_j^2 \cup B_\epsilon^3\\
\widetilde{N}_3(j,\epsilon) &= N_j^1 \cup N_j^2 \cup N_\epsilon^3 & \widetilde{B}_3(j,\epsilon) &= B_j^1 \cup B_j^2 \cup B_\epsilon^3\\
\widetilde{N}_4(j,\epsilon) &= N_\epsilon^3 & \widetilde{B}_4(j,\epsilon) &= B_\epsilon^3.
\end{aligned}$$

Let $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$. Using ITRC on time interval $[0,2T]$ with projectors P_B corresponding to $B = \widetilde{B}_k(j,\epsilon)$, $k = 1,2,3,4$ and starting point f, we obtain functions $h_n(\alpha;\epsilon,j,k) \in L^2(\partial M \times [0,2T])$. Using them, define

(37)
$$p_n(\alpha,j,\epsilon) = h_n(\alpha;\epsilon,j,1) + h_n(\alpha;\epsilon,j,2) - h_n(\alpha;\epsilon,j,3) - h_n(\alpha;\epsilon,j,4).$$

Lemma 5. *The condition (36) is satisfied if and only if there exists an $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$ such that for any $j \in \mathbb{Z}_+$ and $\epsilon > 0$ the functions $p_n(\alpha,j,\epsilon)$ defined in formula (37) satisfy*

$$\lim_{\alpha\to 0}\lim_{n\to\infty} \langle K_{2T} p_n(\alpha,j,\epsilon), p_n(\alpha,j,\epsilon)\rangle \neq 0. \tag{38}$$

Proof. The functions $h_n(\alpha;\epsilon,j,k)$ defined by ITRC satisfy

$$\chi_{\widetilde{N}_k(j,\epsilon)} u^f(T) = \lim_{\alpha\to 0}\lim_{n\to\infty} u^{h_n(\alpha;\epsilon,j,k)}(T), \quad k = 1,2,3,4.$$

A simple computation gives us

$$\chi_{I(j,\epsilon)}(x) = \chi_{\widetilde{N}_1(j,\epsilon)}(x) + \chi_{\widetilde{N}_2(j,\epsilon)}(x) - \chi_{\widetilde{N}_3(j,\epsilon)}(x) - \chi_{\widetilde{N}_4(j,\epsilon)}(x) \tag{39}$$

for all $x \in M$. Therefore, using (37) we see that in $L^2(M)$

$$\chi_{I(j,\epsilon)} u^f(T) = \lim_{\alpha\to 0}\lim_{n\to\infty} u^{p_n(\alpha,j,\epsilon)}(T).$$

By Theorem 3 we see that the functions $u^f(T)$ with $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$ are smooth and form a dense set in $L^2(M(\partial M,T))$. Thus condition (36) is satisfied if and only if there exists an $f \in C_0^\infty(\partial M \times \mathbb{R}_+)$ such that for any $j \in \mathbb{Z}_+$ and $\epsilon > 0$

$$\begin{aligned}
&\langle \chi_{I(j,\epsilon)} u^f(T), u^f(T)\rangle_{L^2(M)}\\
&= \lim_{\alpha\to 0}\lim_{n\to\infty} \langle K_{2T} p_n(\alpha,j,\epsilon), p_n(\alpha,j,\epsilon)\rangle_{L^2(\partial M\times[0,2T])} \neq 0.
\end{aligned}$$

This proves the claim. □

Lemmata 4 and 5 give an algorithm for the determination of $d(x,y)$ from $x = \gamma_{z,\nu}(T_1) \in M$ to $y \in \partial M$ by using ITRC. Indeed,

$$\begin{aligned}
d(x,y) = \inf\{\tau \in [0,T] :\ &\text{there is an } f \in C_0^\infty(\partial M \times \mathbb{R}_+)\\
&\text{such that (38) holds for all } j \in \mathbb{Z}_+ \text{ and } \epsilon > 0\}.
\end{aligned} \tag{40}$$

Summarizing, we have proven:

Proposition 1. *Assume we are given ∂M and the response operator Λ. Let $z, y \in \partial M$, $T_1 \le \tau(z)$. Then using the algorithm (40) we can compute $d(x,y)$ for $x = \gamma_{z,\nu}(T_1)$.*

Let us consider consequences of the result above. To this end, we define the set of the boundary distance functions. For each $x \in M$, the corresponding boundary distance function, $r_x \in C(\partial M)$ is given by

$$r_x : \partial M \to \mathbb{R}_+, \quad r_x(z) = d(x,z), \quad z \in \partial M.$$

In fact, $r_x \in \mathrm{Lip}\,(\partial M)$ with the Lipschitz constant equal to one. The boundary distance functions define *the boundary distance map* $\mathcal{R} : M \to C(\partial M)$, $\mathcal{R}(x) = r_x$, which is continuous and injective (see [24]). Denote by

$$\mathcal{R}(M) = \{r_x \in C(\partial M) : x \in M\},$$

the image of $\mathcal{R}$. It is known (see [24, 25, 29]) that, given the set $\mathcal{R}(M) \subset C(\partial M)$ we can endow it, in a constructive way, with a differentiable structure and a metric tensor $\widetilde{g}$, so that $(\mathcal{R}(M), \widetilde{g})$ becomes a manifold that is isometric to (M, g),

$$(\mathcal{R}(M), \widetilde{g}) \cong (M, g).$$

A stable procedure of construction of (M, g) as a metric space from the set $\mathcal{R}(M)$ and the corresponding Hölder type stability estimates are given in [22].

Example By the triangle inequality,

$$\|r_x - r_y\|_\infty \le d(x,y). \tag{41}$$

We consider in this example the case when (M, g) is a compact manifold such that all points $x, y \in M$ can be joined with a unique shortest geodesic. This implies that, for any $x, y \in M$, the shortest geodesic $\gamma([0, s])$ from x to y, parameterized by the arclength, can be continued to a maximal geodesic $\gamma([0, L])$ that hits the boundary at a point $z = \gamma(L) \in \partial M$. Then

$$|r_x(z) - r_y(z)| = d(x,y)$$

implying equality in (41). Thus in the case when all geodesics between arbitrary points $x, y \in M$ are unique, the manifold (M, g) is isometric to the manifold $\mathcal{R}(M)$ with the distance function inherited from $C(\partial M)$. In the general case the construction of the metric is more elaborate.

By Theorem 1 we can compute for all $x = \gamma_{z,\nu}(T)$ with $T \le \tau(z)$ the corresponding boundary distance function r_x. Since all points $x \in M$ can be represented in this form (see e.g. [13]) we can find the set $\mathcal{R}(M)$ that can be endowed with a manifold structure isometric to the original manifold (M, g). We have thus proven the following result:

Corollary 4. *Assume we are given ∂M and the response operator Λ. Then using the ITRC we can find the manifold (M, g) up to an isometry.*

Corollary 4 can also be formulated by saying that we can find the metric tensor g in local coordinates. For example, for any point $x_0 \in M$ there are $z_j \in \partial M$, $j = 1, 2, \ldots, m$, such that $x \mapsto (X^1(x), \ldots, X^m(x))$ with $X^j(x) := d(x, z_j)$ define local coordinates near x_0. In these coordinates the distance functions $x \mapsto d(x, z)$, $z \in \partial M$ determine the metric tensor. For details of this construction, see [24].

Next we prove Corollary 3.

Proof of Corollary 3. As the boundary data ∂M and Λ determine (M, g) up to an isometry, we can apply formula (34) to find the function $C_0(x)$, $x \in M$ in Corollary 1. Thus in local coordinates we can find the values of waves $u^f(x, t)$ for all $x \in M$, $t > 0$. By Tataru's controllability theorem, the waves $u^f(x, T)$ with

$f \in C_0^\infty(\partial M \times (0,T))$ form a dense set in $L^2(M(\partial M,T))$. In this dense set we can find the functions

$$Au^f(x,t) = -\partial_t^2 u^f(x,t) = -u^{f_{tt}}(x,t),$$

implying that we can find the values Aw for all $w \in L^2(M(\partial M,T))$. As T is arbitrary, we find Aw for all $w \in L^2(M)$.

Having at hand the values of Aw for all $w \in L^2(M)$, and using (5), we can identify, for $\mu = 1$ and already known in local coordinates $g_{ij}(x)$, the potential $q(x)$.

When $M \subset \mathbb{R}^m$ and A is of the form (6), our further constructions are based on the uniqueness, if any, of the isometric embedding of a Riemannian manifold into $\mathbb{R}^m$ of the same dimension such that the metric becomes isotropic and the boundary ∂M is kept fixed. As we know *a priori* that such embedding exists, we use this to find $c(x)$ in the Euclidian coordinates in M (for the details of this and following constructions, see [24, section 4.5], or [28]). We then find the unique κ so that the gauge transformed operator A_κ takes the form (5) with $\mu = 1$. As the potential q for A_κ is already found, by applying κ^{-1} we recover q. □

6. **Iteration when measurements have errors.** Let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space.

Assume the measurements have a random noise, that is, the measurements give us, for an input f, the output $\Lambda_{2T} f + \epsilon$, where ϵ is random Gaussian noise that has values in $L^2(\partial M \times (0,2T))$. Assume that $\mathbb{E}\epsilon = 0$ and denote the covariance operator of ϵ by C_ϵ. Note that C_ϵ is a compact operator on $L^2(\partial M \times (0,2T))$, e.g. [11], and thus the standard white noise on $\partial M \times (0,2T)$ does not satisfy our assumptions.

Assume that the noise is independent of previous measurements each time when we do a new measurement. When the noise is added to the ITRC, we come to the iteration of the form

$$\widetilde{h}_{n+1} = S\widetilde{h}_n + F + N_n,$$

where $N_n = PJ\epsilon_n^1 - PR\epsilon_n^2$ with the random variables ϵ_n^1 and ϵ_n^2 having the same distribution as ϵ. Thus N_n are independent identically distributed Gaussian random variables satisfying $\mathbb{E}N_n = 0$ and having covariance operator $C_N = PJC_\epsilon J^* P^* + PRC_\epsilon R^* P^*$. Let us consider the averaged results of iterations

$$\widetilde{h}_K^{ave} = \frac{1}{K}\sum_{n=1}^{K} \widetilde{h}_n. \tag{42}$$

Then

$$\begin{aligned}
\frac{1}{K}\sum_{n=1}^{K}\widetilde{h}_n &= \frac{1}{K}\sum_{n=1}^{K} h_n + \frac{1}{K}\sum_{n=1}^{K}\sum_{m=1}^{n}(n-m+1)S^{n-m}N_n \\
&= \left(\frac{1}{K}\sum_{n=1}^{K} h_n\right) + \left(\frac{1}{K}\sum_{n=1}^{K}(I-S)^{-2}N_n\right) \\
&\quad + \left(\frac{1}{K}\sum_{n=1}^{K}(-(I-S)^{-2}S^{n+2} - (n+2)(I-S)^{-1}S^{n+1})N_n\right) \\
&= H_K^1 + H_K^2 + H_K^3,
\end{aligned}$$

where h_n are the results of the ITRC without noise.

Above, the deterministic term H^1_K converges to $\lim_{n\to\infty}\widetilde{h}_n = h(\alpha)$, that is, to the same limit as the ITRC without noise. Now consider H^2_K and H^3_K as random variables in $L^2(\partial M\times(0,T))$. They can also be viewed as random fields on $\partial M\times(0,T)$, see [39]. By the law of large numbers in infinite dimensional spaces [18], we see that

$$\lim_{K\to\infty}\|H^2_K\|_{L^2(\partial M\times(0,T))} = 0. \tag{43}$$

As $\|S\|_{L(L^2(\partial M))}\le \frac{1}{2}$, the last term H^3_K also satisfies an estimate analogous to (43). Thus the averaged ITRC with noise converges to the same limit as ITRC without noise, that is,

$$\lim_{K\to\infty}\widetilde{h}^{ave}_K = h(\alpha) \quad \text{in } L^1(\Omega; L^2(\partial M\times(0,T))).$$

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