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Towards a d-bar reconstruction method for three-dimensional EIT

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Abstract — Three-dimensional electrical impedance tomography (EIT) is considered. Both uniqueness proofs and theoretical reconstruction algorithms available for this problem rely on the use of exponentially growing solutions to the governing conductivity equation. The study of those solutions is continued here. It is shown that exponentially growing solutions exist for low complex frequencies without imposing any regularity assumption on the conductivity. Further, a reconstruction method for conductivities close to a constant is given. In this method the complex frequency is taken to zero instead of infinity. Since this approach involves only moderately oscillatory boundary data, it enables a new class of three-dimensional EIT algorithms, free from the usual high frequency instabilities.

1. INTRODUCTION

Is it possible to reconstruct the electric conductivity inside a physical body from static electrical measurements on the surface of the body? This is the inverse conductivity problem having many practical applications, e. g. the medical imaging technique called electrical impedance tomography (EIT), see [12].

The problem was first formulated mathematically by Calderón [10] as follows. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a simply connected, bounded domain with smooth boundary $\partial\Omega$. Define the open subset

$$L^\infty_+(\Omega) = \{\gamma \in L^\infty(\Omega) \ : \ \gamma > 0 \text{ and } \gamma^{-1} \in L^\infty(\Omega)\}$$

of $L^\infty(\Omega)$ and assume henceforth that the conductivity $\gamma \in L^\infty_+(\Omega)$. The

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Dirichlet-to-Neumann (or voltage-to-current) map Λ_γ is defined by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}, \tag{1.1}$$

where u is the electric potential given as the unique solution to

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega, \qquad u = f \quad \text{on } \partial\Omega. \tag{1.2}$$

Thus Λ_γ represents static electrical measurements: it maps an applied voltage distribution on the boundary to the resulting current through the boundary. Calderón posed two questions: is γ uniquely determined from the knowledge of Λ_γ? If so, how to reconstruct γ from Λ_γ?

The inverse conductivity problem has been studied extensively after Calderón's seminal paper. The problem is very challenging since the map $\Lambda_\gamma \mapsto \gamma$ is nonlinear and the recovery of γ from Λ_γ is ill-posed in the sense of Hadamard. For a thorough review of the history of the problem see [12, 36, 43] and references therein.

Exponentially growing solutions (or complex geometrical optics solutions) to the conductivity equation are central objects in uniqueness and reconstruction results for the inverse conductivity problem. To define these solutions we introduce the set $\mathcal{V} = \{\mathbb{C}^n \setminus \{0\} \mid \zeta \cdot \zeta = 0\}$, where $\zeta \cdot \zeta = \sum_{j=1}^n \zeta_j^2$. Note that $e^{ix\cdot\zeta}$ is harmonic for all $x \in \mathbb{R}^n$ if and only if $\zeta \in \mathcal{V}$. Given $\gamma \in L^\infty_+(\Omega)$ we use the extension $\gamma = 1$ in $\mathbb{R}^n \setminus \overline{\Omega}$ and define the exponentially growing solution $\phi(x,\zeta)$ as any solution to

$$\nabla \cdot \gamma \nabla \phi = 0 \ \text{ in } \mathbb{R}^n, \tag{1.3}$$

which asymptotically behaves like $e^{ix\cdot\zeta}$. More precisely, ϕ has the form

$$\phi(x,\zeta) = e^{ix\cdot\zeta}(1 + \omega(x,\zeta)), \tag{1.4}$$

where $\omega(\cdot,\zeta) \in L^2_\delta(\mathbb{R}^n) = \{u \mid (1+|x|^2)^{\delta/2} u \in L^2(\mathbb{R}^n)\}$ for some $-1 < \delta < 0$.

It turns out that the existence and uniqueness of an exponentially growing solution depends on the choice of parameter $\zeta \in \mathcal{V}$.

Definition 1.1. Let $\gamma \in L^\infty_+(\Omega)$. A vector $\zeta \in \mathcal{V}$ is called an *exceptional point* for γ if there is no unique exponentially growing solution of (1.3) with parameter ζ.

The study of exponentially growing solutions goes back to Faddeev's work on scattering theory for Schrödinger operators [15]. The relevance of such solutions for the inverse conductivity problem was first pointed out by Sylvester and Uhlmann in [42], where they gave the uniqueness question a positive answer in dimensions $n \geq 3$ for C^∞ conductivities. Later the uniqueness was generalized for less regular conductivities in dimensions $n \geq 3$ by a number of authors [2, 7, 11, 18, 34, 35]. Currently the sharpest result seems to require essentially 3/2 derivatives of the conductivity to be either bounded [39] or in L^p, $p > 2n$ [8]. The reconstruction issue for $n \geq 3$ was solved in [35] and [38] independently. In dimension two, Nachman solved both uniqueness and reconstruction problems for conductivities $\gamma \in W^{2,p}(\Omega)$ with $p > 2$ in [36]. This result was improved in [9,26] for conductivities having essentially one derivative. Recently Calderón's problem in dimension two was settled in the affirmative for conductivities $\gamma \in L^\infty(\Omega)$ by Astala and Päivärinta [1]. All the above results are based on the use of exponentially growing solutions.

In addition to theoretical research on the inverse conductivity problem, much effort has been devoted to the design of practical EIT algorithms applicable to noisy measurement data. Existing algorithms can be roughly divided into the following classes: (i) methods solving the linearized problem, (ii) iterative methods, such as regularized output least squares algorithms, (iii) statistical inversion, (iv) layer stripping, and (v) $\bar{\partial}$-methods using exponentially growing solutions. (Note that we discuss here only algorithms attempting point-wise recovery of the conductivity coefficient inside Ω, as opposed to algorithms recovering partial information on γ.) For more details on EIT algorithms see the introduction of [32] and the review paper [6]. While many EIT algorithms produce useful reconstructions, only few of them are mathematically justified. More precisely, of the algorithms solving the full nonlinear problem (excluding linearized methods), so far only methods of class (v) allow rigorous mathematical analysis *and* tolerate significant noise in measurement data.

In practice most conductivity distributions of interest are three-dimensional. In particular, this is the case in medical EIT. Most of the EIT algorithms in the literature are for the two-dimensional problem; the only truly three-dimensional computational reconstruction methods are [5, 17, 28–31, 44–47]. All of these methods belong to class (i) or (ii). Our motivation for this study is laying a theoretical foundation for a mathematically justified three-dimensional EIT algorithm of class (v).

In dimension two exponentially growing solutions exist for any ζ. Moreover, γ can be reconstructed by taking ζ to zero, a fact that enabled two-dimensional EIT algorithms of class (v), see [21, 24, 25, 32, 40, 41]. In dimension three the situation is more difficult due to possible exceptional points. Nachman [35] and Novikov [38] present three-dimensional reconstruction procedures based on determining first the exponentially growing solutions for large enough $|\zeta|$ (known not to be exceptional) and then applying the $\bar{\partial}$-method of inverse scattering. However, since $e^{ix\cdot\zeta}$ (and therefore the exponentially growing solution) is exponentially growing and rapidly oscillating for certain $x \in \partial\Omega$ when $|\zeta|$ increases, it seems impractical and unstable from an applied point of view to use only the high complex frequency information.

In this paper we continue the study of exponentially growing solutions in dimension $n \geq 3$. We will show that such solutions exist and are unique when $\gamma \in L^\infty_+(\Omega)$ and the parameter $\zeta \in \mathcal{V}$ is small. *Note that no smoothness in the conductivity is assumed.* Moreover, we will show that also in dimensions $n \geq 3$ the conductivity can be obtained from the exponentially growing solutions by taking the parameter ζ to zero. We hope that these two results will lead to a new global reconstruction algorithm for rough conductivities. We will at the end show how these results in combination with ideas from the $\bar{\partial}$-method can provide a new and hopefully practical algorithm for the reconstruction of smooth conductivities close to constant.

The paper is organized as follows: in Section 2 we characterize exponentially growing solutions by a certain boundary integral equation on $\partial\Omega$. In Section 3 we show that there is a neighborhood of zero in $\mathcal{V}$ with no exceptional points for $\gamma \in L^\infty_+(\Omega)$. Further, we prove that the conductivity is obtained as the low-frequency limit of the exponentially growing solutions. Then in Section 4 we introduce an important function, the so-called scattering transform, and we consider the $\bar\partial$-equations satisfied by this function and exponentially growing solutions. In Section 5 we first review the $\bar\partial$-method, which gives a method for finding the exponentially growing solutions from boundary data, and then show how this method in combination with our results gives a local reconstruction algorithm. Conclusions are drawn in Section 6.

2. THE BOUNDARY INTEGRAL EQUATION

Here and throughout the rest of the paper we take $n \geq 3$ unless otherwise stated. Let $\gamma \in L^\infty_+(\Omega)$ be twice continuously differentiable. Suppose $\gamma \equiv 1$ near the boundary $\partial\Omega$, and extend γ to $\mathbb{R}^n \setminus \Omega$ by $\gamma = 1$. Then the conductivity problem can be transformed into a Schrödinger problem, i. e. if u solves (1.3) then the well-known substitution $\tilde u = \gamma^{1/2} u$ defines a solution to the problem

$$(-\Delta + q)\tilde u = 0 \quad \text{in } \mathbb{R}^n, \tag{2.1}$$

where

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \quad \text{in } \mathbb{R}^n. \tag{2.2}$$

Note that q is compactly supported inside Ω. Motivated by scattering theory for this type of equations we look for solutions $\psi(x,\zeta)$, $\zeta \in \mathcal{V}$, to (2.1) with

$$\psi(x,\zeta) = e^{ix\cdot\zeta}(1 + \omega(x,\zeta)) \tag{2.3}$$

and $\lim_{|x|\to\infty} \omega(x,\zeta) = 0$. If we introduce the Faddeev Green's functions

$$G_\zeta(x) = e^{ix\cdot\zeta} g_\zeta(x), \qquad g_\zeta(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\xi\cdot\zeta}\, d\xi, \tag{2.4}$$

where the integral in the definition of g_ζ should be understood in the sense of Fourier transform defined on the space of tempered distributions, then the exponentially growing solutions to (2.1) are solutions of the Faddeev–Lippmann–Schwinger equation

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\mathbb{R}^n} G_\zeta(x-y) q(y) \psi(y,\zeta)\, dy, \tag{2.5}$$

or equivalently

$$(I + g_\zeta * (q\,\cdot))\omega(\,\cdot\,,\zeta) = g_\zeta * q. \tag{2.6}$$

Equation (2.6) is usually the starting point in the construction of exponentially growing solutions. For sufficiently large $|\zeta|$ this equation can be solved in the space $L^2_\delta(\mathbb{R}^n)$ for $-1 < \delta < 0$.

For a coefficient $q \in L^p(\Omega)$ with $p > n/2$ we assume that zero is not a Dirichlet eigenvalue of $(-\Delta + q)$ in $L^2(\Omega)$. Then we define $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_q f = \frac{\partial \tilde{u}}{\partial \nu}\Big|_{\partial\Omega},$$

where $\tilde{u}$ is the unique $H^1(\Omega)$ solution to

$$(-\Delta + q)\tilde{u} = 0 \quad \text{in } \Omega, \qquad \tilde{u} = f \quad \text{on } \partial\Omega. \tag{2.7}$$

Note that we abuse notation in the definition of Λ_q and Λ_γ. Further, in what follows, Λ_1 is always the Dirichlet-to-Neumann map corresponding to the constant conductivity 1 and Λ_0 is the Dirichlet-to-Neumann map corresponding to $q = 0$. Of course, $\Lambda_0 = \Lambda_1$.

Assume that q is supported in Ω. For fixed $x \in \mathbb{R}^n \setminus \overline{\Omega}$ the function $G_\zeta(x-y)$ is harmonic in Ω and hence Green's second identity together with the symmetry of the Dirichlet-to-Neumann maps show that

$$\int_\Omega G_\zeta(x-y)q(y)\psi(y,\zeta)\,dy = \int_{\partial\Omega} G_\zeta(x-y)(\Lambda_q - \Lambda_0)\psi(y,\zeta)\,dS(y).$$

Hence taking the trace of (2.5) shows that $\psi(\cdot,\zeta)|_{\partial\Omega}$ satisfies the boundary integral equation

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - S_\zeta(\Lambda_q - \Lambda_0)\psi(x,\zeta), \qquad x \in \partial\Omega, \tag{2.8}$$

in $H^{1/2}(\partial\Omega)$, where $S_\zeta : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is the single layer potential corresponding to Faddeev's Green's function:

$$S_\zeta f(x) = \int_{\partial\Omega} G_\zeta(x-y)f(y)\,d\sigma(y), \qquad x \in \partial\Omega.$$

In [35, 36] it is shown that (2.8) characterizes exponentially growing solutions in the sense that the trace on $\partial\Omega$ of an exponentially growing solution satisfies (2.8), and conversely any solution to (2.8) can be extended to an exponentially growing solution in the entire space $\mathbb{R}^n$.

It is well-known that if $\gamma \in W^{2,p}(\Omega)$, $p > n/2$, and $\gamma = 1$ near $\partial\Omega$, then $\Lambda_\gamma = \Lambda_q$. Note then that if ψ is an exponentially growing solution to (2.1), then $\phi = \gamma^{-1/2}\psi$ is an exponentially growing solution to (1.2) and $\phi = \psi$ on $\partial\Omega$. Hence equation (2.8) reads

$$\phi = e^{ix\cdot\zeta} - S_\zeta(\Lambda_\gamma - \Lambda_1)\phi. \tag{2.9}$$

We will show that also for rough conductivities, a solution to (2.9) can be extended into $\mathbb{R}^n$ to an exponentially growing solution to (1.2).

We will need the following properties of the single layer operator having Faddeev's Green's function as kernel. Denote by G_0 the usual Green's function for the Laplacian

$$G_0(x) = \frac{1}{n\omega_n |x|^{n-2}}, \qquad n \geq 3,$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Since G_ζ is also a Green's function for the Laplacian it differs from G_0 by a harmonic function H_ζ, i.e.

$$H_\zeta(x) = G_\zeta(x) - G_0(x). \tag{2.10}$$

Hence the single layer potential S_ζ can be decomposed as

$$S_\zeta = S_0 + \mathcal{H}_\zeta, \tag{2.11}$$

where S_0 is the usual single layer potential

$$S_0\phi(x) = \int_{\partial\Omega} G_0(x-y)\phi(y)\, d\sigma(y), \tag{2.12}$$

and

$$\mathcal{H}_\zeta\phi(x) = \int_{\partial\Omega} H_\zeta(x-y)\phi(y)\, d\sigma(y). \tag{2.13}$$

To fix notation denote by $\mathcal{B}(X,Y)$ the Banach space consisting of bounded linear maps from the Banach space X into the Banach space Y. From the well-known mapping properties of S_0 (see [23]) it is now straightforward to derive the following proposition:

Proposition 2.1. *Assume that $\partial\Omega$ is smooth and let $s \in \mathbb{R}$. Then $S_\zeta \in \mathcal{B}(H^s(\partial\Omega), H^{s+1}(\partial\Omega))$. Define for $\phi \in H^s(\partial\Omega)$ the function*

$$u(x) = \int_{\partial\Omega} G_\zeta(x-y)\phi(y)\, d\sigma(y), \qquad x \in \mathbb{R}^n. \tag{2.14}$$

Then $u \in H^{s+3/2}(\mathbb{R}^n)$ and

1. *$\Delta u = 0$ in $\mathbb{R}^n \setminus \partial\Omega$.*

2. *The trace of u is given by*

$$u\big|_{\partial\Omega} = S_\zeta\phi. \tag{2.15}$$

3. *Let ∂_{ν^+}, ∂_{ν^-} denote the normal exterior derivative at the boundary taken from outside and inside respectively. Then*

$$\partial_{\nu^\pm} u(x) = \mp\phi(x)/2 + K'_\zeta\phi(x), \tag{2.16}$$

where

$$K'_\zeta\phi(x) = p.v. \int_{\partial\Omega} \frac{\partial G_\zeta(x-y)}{\partial\nu(x)}\phi(y)\, d\sigma(y), \tag{2.17}$$

where p.v. means that the integral is evaluated in the sense of principal value. In particular

$$(\partial_{\nu^-} - \partial_{\nu^+})u(x) = \phi(x), \qquad x \in \partial\Omega. \tag{2.18}$$

Due to (2.15) we will in the sequel not distinguish between $S_\zeta \phi$ defined on $\partial\Omega$ and the function u defined by (2.14) in $\mathbb{R}^n$.

Proposition 2.2. *Let $\gamma \in L^\infty_+(\mathbb{R}^n)$ such that $\gamma - 1$ is compactly supported inside Ω.*

Let $f_\zeta \in H^{1/2}(\partial\Omega)$ be a solution to (2.9). Then f_ζ can be extended to a function ϕ in $\mathbb{R}^n$, which solves (1.3) and satisfies $\omega(\cdot,\zeta) = \phi(x,\zeta)e^{-ix\cdot\zeta} - 1 \in L^2_\delta(\mathbb{R}^n)$, $-1 < \delta < 0$.

Proof. Define $\phi(\,\cdot\,,\zeta)$ on Ω to be the solution to $\nabla\cdot\gamma\nabla\phi = 0$ with boundary value $\phi|_{\partial\Omega} = f_\zeta$, and define ϕ on $\mathbb{R}^n \setminus \overline{\Omega}$ by

$$\phi(x,\zeta) = e^{ix\cdot\zeta} - S_\zeta(\Lambda_\gamma - \Lambda_1)f_\zeta, \qquad x \text{ in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{2.19}$$

By assumption on f_ζ,

$$\phi\big|_{\partial\Omega} = f_\zeta.$$

Furthermore, since $e^{ix\cdot\zeta} - S_\zeta(\Lambda_\gamma - \Lambda_1)f_\zeta$ is harmonic in Ω and equals f_ζ on $\partial\Omega$, we see that

$$\partial_{\nu^-}(e^{ix\cdot\zeta} - S_\zeta(\Lambda_\gamma - \Lambda_1)f_\zeta) = \Lambda_1 f_\zeta.$$

The jump relation for single layer potential (2.18) now implies that

$$(\partial_{\nu^+} - \partial_{\nu^-})(e^{ix\cdot\zeta} - S_\zeta(\Lambda_\gamma - \Lambda_1)f_\zeta) = (\Lambda_\gamma - \Lambda_1)f_\zeta.$$

Hence we conclude that

$$\partial_{\nu^+}\phi(\,\cdot\,,\zeta) = \Lambda_\gamma f_\zeta.$$

This shows that $\phi \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in fact a weak solution to the conductivity equation (1.3) everywhere in $\mathbb{R}^n$. To show that ϕ is exponentially growing we note that

$$\begin{aligned} e^{ix\cdot\zeta} - \phi(x,\zeta) &= S_\zeta(\Lambda_\gamma - \Lambda_1)\phi(\,\cdot\,,\zeta) \\ &= \langle G_\zeta(x - \,\cdot\,), (\Lambda_\gamma - \Lambda_1)\phi(\,\cdot\,,\zeta)\rangle \\ &= \int_\Omega (\gamma(y) - 1)\nabla_y G_\zeta(x-y)\cdot\nabla\phi(y)\,dy \end{aligned}$$

for $x \in \mathbb{R}^n \setminus \overline{\Omega}$. Multiplying this equation by $-e^{-ix\cdot\zeta}$ and replacing G_ζ by g_ζ gives

$$\begin{aligned} \phi(x,\zeta)e^{-ix\cdot\zeta} - 1 &= -\int_\Omega (\gamma(y) - 1)\nabla_y(e^{-iy\cdot\zeta}g_\zeta(x-y))\cdot\nabla\phi(y)\,dy \\ &= -\int_\Omega (\gamma(y) - 1)(\nabla_y - i\zeta)g_\zeta(x-y)\cdot e^{-iy\cdot\zeta}\nabla\phi(y)\,dy \\ &= (\nabla_x + i\zeta)\cdot\int_\Omega g_\zeta(x-y)(\gamma(y)-1)e^{-iy\cdot\zeta}\nabla\phi(y)dy \\ &= (\nabla_x + i\zeta)\cdot(g_\zeta * (\gamma - 1)e^{-iy\cdot\zeta}\nabla\phi). \end{aligned} \tag{2.20}$$

For the convolution operator $g_\zeta *$ we have the estimates

$$\|g_\zeta * u\|_{L^2_\delta(\mathbb{R}^n)} \leq \frac{C}{|\zeta|}\|u\|_{L^2_{\delta+1}(\mathbb{R}^n)},$$
$$\|g_\zeta * u\|_{H^1_\delta(\mathbb{R}^n)} \leq C\|u\|_{L^2_{\delta+1}(\mathbb{R}^n)},$$

for any δ, $-1 < \delta < 0$, see [7, 24, 42]. Hence we conclude from (2.20) that

$$\|\phi(x,\zeta)e^{-ix\cdot\zeta} - 1\|_{L^2_\delta(\mathbb{R}^n)} \leq C\|\gamma - 1\|_{L^\infty(\Omega)}\|e^{-iy\cdot\zeta}\nabla\phi\|_{L^2_{\delta+1}(\Omega)}, \tag{2.21}$$

where the constant C is independent of ζ. □

3. EXPONENTIALLY GROWING SOLUTIONS FOR ζ NEAR ZERO

In this section we show that the boundary integral equation (2.9) in dimensions $n \geq 3$ is solvable in $H^{1/2}(\partial\Omega)$ for any $\zeta \in \mathcal{V}$ sufficiently small, and hence that such a ζ is not exceptional for γ. Moreover, we prove for arbitrary fixed $R > 0$ that if γ is sufficiently close to 1, then there are no exceptional points satisfying $|\zeta| \leq R$. We emphasize here that the main achievement is that we only assume that $\gamma \in L^\infty_+(\Omega)$, i. e. no further regularity is assumed for the conductivity. The technique of proof is analogous to the two-dimensional case analyzed in [40, 41]. Further, we show that a conductivity $\gamma \in W^{2,p}(\Omega)$, $p > n/2$, can be recovered from the exponentially growing solutions given for ζ near zero:

$$\gamma^{1/2}(x) = \lim_{\zeta\to 0}\psi(x,\zeta). \tag{3.1}$$

For the solution of the boundary integral equation we need to show that $(\Lambda_\gamma - \Lambda_1)$ is infinitely smoothing. Although this may be considered well-known, due to the lack of a suitable reference we present a full proof for the reader's convenience.

Lemma 3.1. *Assume $\gamma \in L^\infty_+(\Omega)$ satisfies $\gamma = 1$ near $\partial\Omega$. Then $(\Lambda_\gamma - \Lambda_1) \in \mathcal{B}(H^{1/2}(\partial\Omega), H^s(\partial\Omega))$ for any $s \in \mathbb{R}$, and we have the estimate*

$$\|\Lambda_\gamma - \Lambda_1\|_{\mathcal{B}(H^{1/2}(\partial\Omega),H^s(\partial\Omega))} \leq C\|\gamma - 1\|_{L^\infty(\Omega)}, \tag{3.2}$$

where C depends on s, Ω, the ellipticity constant for γ, and the support of $(\gamma - 1)$.

Proof. We will prove that $(\Lambda_\gamma - \Lambda_1) \in \mathcal{B}(H^{1/2}(\partial\Omega), H^{m+1/2}(\partial\Omega))$ for any $m \in \mathbb{Z}_+$. Take an open domain $\Omega' \subset \Omega$ such that $\gamma = 1$ on $\Omega_0 = \Omega \setminus \overline{\Omega'}$. Take further a set of smooth cut-off functions $\{\phi_k\}_{k=0}^m$ supported near $\partial\Omega$ such that

(1) $0 \leq \phi_k \leq 1$, $k = 0, \ldots, m$;

(2) $\phi_m = 1$ near $\partial\Omega$;

(3) $\phi_0 = 0$ near $\partial\Omega'$;

(4) $\phi_k = 1$ on $\Omega_{k+1} := \operatorname{supp}(\phi_{k+1})$ for $k = 0, \ldots, m-1$.

Let u be the unique solution to (1.2) with $f \in H^{1/2}(\partial\Omega)$ and let v be a harmonic function in Ω with $v|_{\partial\Omega} = f$. Then since $\phi_k(u-v) \in H_0^1(\Omega_k)$ satisfies

$$\Delta(\phi_k(u-v)) = \Delta\phi_k(u-v) + 2\nabla\phi_k \cdot \nabla(u-v) \text{ in } \Omega_k,$$

we can estimate

$$\begin{aligned}
\|\phi_k(u-v)\|_{H^{k+2}(\Omega)} &\le C\|\Delta\phi_k(u-v) + 2\nabla\phi_k \cdot \nabla(u-v)\|_{H^k(\Omega_k)} \\
&\le C\|u-v\|_{H^{k+1}(\Omega_k)} \\
&\le C\|\phi_{k-1}(u-v)\|_{H^{k+1}(\Omega)}, \quad k = 1, 2, \ldots, m,
\end{aligned}$$

where C depends on the cut-off function ϕ_k. By induction we then get

$$\|\phi_m(u-v)\|_{H^{m+2}(\Omega)} \le C\|\phi_0(u-v)\|_{H^2(\Omega)}. \tag{3.3}$$

A similar argument shows that

$$\|\phi_0(u-v)\|_{H^2(\Omega)} \le C\|u-v\|_{H^1(\Omega)}. \tag{3.4}$$

Now since $(u-v) \in H_0^1(\Omega)$ satisfies

$$\nabla \cdot \gamma\nabla(u-v) = -\nabla \cdot (\gamma - 1)\nabla v \ \text{ in } \Omega,$$

standard elliptic estimates show that

$$\begin{aligned}
\|u-v\|_{H^1(\Omega)} &\le C\|\nabla \cdot (\gamma-1)\nabla v\|_{H^{-1}(\Omega)} \\
&\le C\|\gamma - 1\|_{L^\infty(\Omega)}\|v\|_{H^1(\Omega)} \\
&\le C\|\gamma - 1\|_{L^\infty(\Omega)}\|f\|_{H^{1/2}(\partial\Omega)},
\end{aligned} \tag{3.5}$$

where C now depends on the ellipticity constant for γ and Ω. Combining (3.3), (3.4) and (3.5) gives

$$\begin{aligned}
\|(\Lambda_\gamma - \Lambda_1)f\|_{H^{m+1/2}(\partial\Omega)} &\le \|\partial_\nu(\phi_m(u-v))\|_{H^{m+1/2}(\partial\Omega)} \\
&\le C\|\phi_m(u-v)\|_{H^{m+2}(\Omega)} \\
&\le C\|\phi_0(u-v)\|_{H^2(\Omega)} \\
&\le C\|u-v\|_{H^1(\Omega)} \\
&\le C\|\gamma-1\|_{L^\infty(\Omega)}\|f\|_{H^{1/2}(\partial\Omega)},
\end{aligned}$$

which proves the claim. □

As a consequence we obtain the following result, which in particular shows that (2.9) is a Fredholm equation of the second kind.

Corollary 3.2. *Assume $\gamma \in L_+^\infty(\Omega)$ satisfies $\gamma = 1$ near $\partial\Omega$. Then the operators $S_\zeta(\Lambda_\gamma - \Lambda_1)$ and $S_0(\Lambda_\gamma - \Lambda_1)$ are compact in $H^{1/2}(\partial\Omega)$.*

Proof. Since $S_\zeta \in \mathcal{B}(L^2(\partial\Omega), H^1(\partial\Omega))$ (see Proposition 2.1) and the inclusion $H^s(\partial\Omega) \subset H^{s-\epsilon}(\partial\Omega)$ is compact for $s \in \mathbb{R}$, $\epsilon > 0$, the claim follows directly from Lemma 3.1. $\square$

We assume in the following without loss of generality that $\Omega = B_a$, the ball of radius a centered at the origin, and that $\gamma \in L^\infty_+(B_a)$ with $\gamma = 1$ near ∂B_a. The decomposition (2.11) motivates a splitting of the operator

$$(I + S_\zeta(\Lambda_\gamma - \Lambda_1)) = (I + S_0(\Lambda_\gamma - \Lambda_1)) + \mathcal{H}_\zeta(\Lambda_\gamma - \Lambda_1) \tag{3.6}$$

into a free part independent of ζ and a perturbation. For the free part we have the following result.

Lemma 3.3. *Assume $\gamma \in L^\infty_+(B_a)$ satisfies $\gamma = 1$ near ∂B_a. Then the boundary integral operator $I + S_0(\Lambda_\gamma - \Lambda_1)$ is invertible in $H^{1/2}(\partial B_a)$.*

Proof. The operator $S_0(\Lambda_\gamma - \Lambda_1)$ is compact in $H^{1/2}(\partial B_a)$ by Corollary 3.2, and hence injectivity of $I + S_0(\Lambda_\gamma - \Lambda_1)$ implies invertibility by Fredholm's alternative.

Assume now that $h \in H^{1/2}(\partial B_a)$ and

$$(I + S_0(\Lambda_\gamma - \Lambda_1))h = 0. \tag{3.7}$$

Let v be the unique harmonic function in Ω with $v|_{\partial\Omega} = h$ and let

$$w(x) = \int_{\partial\Omega} G_\zeta(x-y)(\Lambda_\gamma - \Lambda_1)h(y)\, d\sigma(y) \in H^1(B_a),$$

Then $v+w$ is harmonic in B_a and has trace $h + S_0(\Lambda_\gamma - \Lambda_1)h = 0$. It follows that $v + w = 0$ in B_a. Hence from (2.16) it follows by taking the trace of $\partial_\nu(v+w)$ from inside Ω that

$$\begin{aligned} 0 &= \Lambda_1 h + \frac{1}{2}(\Lambda_\gamma - \Lambda_1)h + K_0'(\Lambda_\gamma - \Lambda_1)h \\ &= \frac{1}{2}(\Lambda_\gamma + \Lambda_1)h + K_0'(\Lambda_\gamma - \Lambda_1)h. \end{aligned} \tag{3.8}$$

Since on ∂B_a

$$\begin{aligned} \partial_{\nu_x} G_0(x-y) &= \frac{x}{|x|} \nabla_x \frac{1}{n\omega_n |x-y|^{n-2}} \\ &= \frac{x}{|x|} \cdot -\frac{(n-2)(x-y)}{n\omega_n |x-y|^n} \\ &= -\frac{n-2}{2a} \frac{|x|^2 + |y|^2 - 2x\cdot y}{n\omega_n |x-y|^n} \\ &= -\frac{n-2}{2a} G_0(x-y), \end{aligned}$$

we deduce that

$$K_0' = -\frac{n-2}{2a} S_0$$

and hence (3.7) implies

$$K_0'(\Lambda_\gamma - \Lambda_1)h = -\frac{n-2}{2a}\, S_0(\Lambda_\gamma - \Lambda_1)h = \frac{n-2}{2a}\, h. \tag{3.9}$$

Inserting (3.9) into (3.8) gives

$$0 = (\Lambda_\gamma + \Lambda_1)h + \frac{n-2}{a}\, h$$

which implies $h = 0$, since

$$0 = \left\langle (\Lambda_\gamma + \Lambda_1)h + \frac{n-2}{a}\, h, h \right\rangle \geq \frac{n-2}{a}\, \|h\|^2_{L^2(\partial B_a)}.$$

□

Before we consider the full operator (3.6) we analyze the dependency of the kernel H_ζ on the parameter ζ. We start by collecting a few facts about g_ζ. Note that $\zeta = \mathrm{Re}\,(\zeta) + i\,\mathrm{Im}\,(\zeta) \in \mathcal{V}$ satisfies $\zeta \cdot \zeta = \mathrm{Re}\,(\zeta)^2 - \mathrm{Im}\,(\zeta)^2 + 2i\,\mathrm{Re}\,(\zeta) \cdot \mathrm{Im}\,(\zeta) = 0$, which implies that

$$|\,\mathrm{Re}\,(\zeta)| = |\,\mathrm{Im}\,(\zeta)|, \qquad \mathrm{Re}\,(\zeta) \cdot \mathrm{Im}\,(\zeta) = 0.$$

Hence ζ has the form

$$\zeta = \kappa(k_\perp + ik), \tag{3.10}$$

where $k_\perp, k \in \mathbb{R}^n$, $|k_\perp| = |k| = 1/\sqrt{2}$, $k \cdot k_\perp = 0$, and $|\zeta| = \kappa$. We have now the following result.

Proposition 3.1. *Let $\zeta \in \mathcal{V}$ be decomposed as* (3.10)*. Then*

$$g_\zeta(x) = \kappa^{n-2} g_{k_\perp + ik}(\kappa x). \tag{3.11}$$

Let further $\mathbf{R}$ be a real orthogonal $n \times n$ matrix with $\det(\mathbf{R}) = 1$*. Then*

$$g_\zeta(x) = g_{\mathbf{R}\zeta}(\mathbf{R}x). \tag{3.12}$$

Proof. We prove the results by manipulating the formal integral (2.4) (a rigorous but tedious proof by distribution theory can also be given). The substitution $\tilde{\xi} = \xi/\kappa$ implies

$$\begin{aligned} g_\zeta(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2\zeta \cdot \xi}\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i\kappa x\cdot\tilde{\xi}}}{\kappa^2(|\tilde{\xi}|^2 + 2\frac{\zeta}{\kappa} \cdot \tilde{\xi})}\, \kappa^n\, d\tilde{\xi} \\ &= \kappa^{n-2} g_{k_\perp + ik}(\kappa x), \end{aligned}$$

which shows (3.11).

To prove (3.12), write $\xi' = \mathbf{R}^T\xi$ and $d\xi' = d\xi$ to get

$$\begin{aligned} g_\zeta(\mathbf{R}x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i(\mathbf{R}x)\cdot\xi}}{|\xi|^2 + 2\zeta\cdot\xi}\,d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot(\mathbf{R}^T\xi)}}{|\xi|^2 + 2\zeta\cdot\xi}\,d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi'}}{|(\mathbf{R}^T)^{-1}\xi'|^2 + 2\zeta\cdot((\mathbf{R}^T)^{-1}\xi')}\,d\xi' \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{ix\cdot\xi'}}{|\xi'|^2 + 2(\mathbf{R}^{-1}\zeta)\cdot\xi'}\,d\xi' \\ &= g_{\mathbf{R}^{-1}\zeta}(x). \end{aligned}$$

□

Next we will investigate the dependence of the perturbation in (3.6) on ζ. We need the following lemma concerning properties of the harmonic function H_ζ:

Lemma 3.4. *Let $\zeta = \kappa(k_\perp + ik) \in \mathcal{V}$ with $\kappa \in \mathbb{R}$, $k_\perp, k \in \mathbb{R}^n$, $k_\perp \cdot k = 0$ (cf. (3.10)), and let $\mathbf{R}$ be a real orthogonal $n\times n$ matrix with* $\det(\mathbf{R}) = 1$. *Then the harmonic function $H_\zeta = G_\zeta - G_0$ satisfies*

$$H_\zeta(x) = \kappa^{n-2}H_{k_\perp+ik}(\kappa x), \tag{3.13}$$

$$H_\zeta(x) = H_{\mathbf{R}^T\zeta}(\mathbf{R}^Tx). \tag{3.14}$$

Proof. From (3.11) it follows that

$$\begin{aligned} H_\zeta(x) = G_\zeta(x) - G_0(x) &= e^{ix\cdot\zeta}g_\zeta(x) - G_0(x) \\ &= e^{i\kappa x\cdot(k_\perp+ik)}\kappa^{n-2}g_{(k_\perp+ik)}(\kappa x) - \kappa^{n-2}G_0(\kappa x) \\ &= \kappa^{n-2}H_{(k_\perp+ik)}(\kappa x). \end{aligned}$$

Further,

$$\begin{aligned} H_\zeta(\mathbf{R}x) &= G_\zeta(\mathbf{R}x) - G_0(\mathbf{R}x) \\ &= e^{i(\mathbf{R}x)\cdot\zeta}g_\zeta(\mathbf{R}x) - G_0(|\mathbf{R}x|) \\ &= e^{ix\cdot(\mathbf{R}^T\zeta)}g_{\mathbf{R}^{-1}\zeta}(x) - G_0(|x|) \\ &= H_{\mathbf{R}^T\zeta}(x). \end{aligned}$$

The result now follows since $\det(\mathbf{R}) = 1$ together with the real-valuedness of $\mathbf{R}$ implies $\mathbf{R}^T = \mathbf{R}^{-1}$. □

From the properties of H_ζ it is now straightforward to derive estimates for $\mathcal{H}_\zeta$ defined by (2.13).

Lemma 3.5. *The integral operator $\mathcal{H}_\zeta$ satisfies for any $\zeta \in \mathcal{V}$ and $0 < |\zeta| \le 1$ the estimates*

$$\|\mathcal{H}_\zeta\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \le C|\zeta|^{n-2}, \tag{3.15}$$

where the constant C depends only on the radius a.

Proof. From (3.14) we can without loss of generality assume that $\zeta = \kappa(e_1 + ie_2)$ with $e_1 = (1, 0, \ldots, 0)/\sqrt{2}$, $e_2 = (0, 1, 0, \ldots, 0)/\sqrt{2}$ and $\kappa \geq 0$. Then an application of (3.13) gives

$$\begin{aligned}\mathcal{H}_{\kappa(e_1+ie_2)} f(x) &= \int_{\partial B_a} H_{\kappa(e_1+ie_2)}(x-y) f(y)\, d\sigma(y) \\ &= \kappa^{n-2} \int_{\partial B_a} H_{e_1+ie_2}(\kappa(x-y)) f(y)\, d\sigma(y).\end{aligned} \tag{3.16}$$

Define the smooth function $H(x) = H_{e_1+ie_2}(x) - H_{e_1+ie_2}(0)$ and set

$$\mathcal{K}_\kappa f(x) = \int_{\partial B_a} H(\kappa(x-y)) f(y)\, d\sigma(y), \qquad x \in \partial B_a. \tag{3.17}$$

Then

$$\mathcal{H}_{\kappa(e_1+ie_2)} f(x) = \kappa^{n-2} \mathcal{K}_\kappa f(x) + \kappa^{n-2} H_{e_1+ie_2}(0) \int_{\partial B_a} f(y)\, d\sigma(y).$$

Since $H(0) = 0$ and H is smooth, there is a constant C depending only on a such that $\operatorname{supp}_{|x|\leq 2a} |H(x)| \leq C|x|$. Then for $0 \leq \kappa \leq 1$

$$\int_{\partial B_a} \int_{\partial B_a} |H(\kappa(x-y))|^2\, d\sigma(y)\, d\sigma(x) \leq C\kappa^2,$$

which implies

$$\|\mathcal{K}_\kappa\|_{\mathcal{B}(L^2(\partial B_a))} \leq C\kappa. \tag{3.18}$$

Furthermore, since H is a smooth function, $\operatorname{supp}_{|x|\leq 2a} |\nabla H(x)| \leq C$ for some constant $C > 0$. Hence by taking derivatives in (3.17) we find that

$$\|\mathcal{K}_\kappa\|_{\mathcal{B}(H^1(\partial B_a))} \leq C\kappa. \tag{3.19}$$

Thus the estimate

$$\|\mathcal{K}_\kappa\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \leq C\kappa \tag{3.20}$$

can be obtained for $0 \leq \kappa \leq 1$ from (3.18) and (3.19) by interpolation. $\square$

The solvability of the boundary integral equation for small ζ can now be proved using a standard perturbation argument.

Theorem 3.6. *Assume $\gamma \in L^\infty_+(B_a)$ satisfies $\gamma = 1$ near ∂B_a. Then the boundary integral equation (2.9) is uniquely solvable in $H^{1/2}(\partial B_a)$ for $\zeta \in \mathcal{V}$ sufficiently small. Furthermore, assume given $R > 0$. Then, if $\|\gamma - 1\|_{L^\infty(\mathbb{R}^n)}$ is sufficiently small, solvability holds for all $|\zeta| \leq R$.*

Proof. Let

$$A_{\gamma,0} = S_0(\Lambda_\gamma - \Lambda_1), \qquad A_{\gamma,\zeta} = S_\zeta(\Lambda_\gamma - \Lambda_1).$$

Then by Lemma 3.3 the operator $I + A_{\gamma,0}$ is invertible in $H^{1/2}(\partial\Omega)$. Thus

$$\begin{aligned} I + A_{\gamma,\zeta} &= I + A_{\gamma,0} + (A_{\gamma,\zeta} - A_{\gamma,0}) \\ &= (I + A_{\gamma,0})(I + (I + A_{\gamma,0})^{-1}(A_{\gamma,\zeta} - A_{\gamma,0})) \end{aligned}$$

is invertible if

$$\|(I + A_{\gamma,0})^{-1}(A_{\gamma,\zeta} - A_{\gamma,0})\|_{\mathcal{B}(H^{1/2}(\partial B_a))} < 1.$$

From the decomposition $S_\zeta = S_0 + \mathcal{H}_\zeta$ it now follows from (3.15) that for a fixed $\gamma \in L^\infty_+(B_a)$

$$\begin{aligned} &\|(I + A_{\gamma,0})^{-1}(A_{\gamma,\zeta} - A_{\gamma,0})\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \\ &= \|(I + A_{\gamma,0})^{-1}\mathcal{H}_\zeta(\Lambda_\gamma - \Lambda_1)\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \\ &\leq C|\zeta|^{n-2}\|(I + A_{\gamma,0})^{-1}\|_{\mathcal{B}(H^{1/2}(\partial B_a))}\|\Lambda_\gamma - \Lambda_1\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \\ &< 1 \end{aligned} \tag{3.21}$$

for ζ sufficiently small.

If $\gamma - 1$ is sufficiently small then by 3.2 it follows that $I + A_{\gamma,0}$ can be inverted by a Neumann series. This gives the estimate

$$\|(I + A_{\gamma,0})^{-1}\|_{\mathcal{B}(H^{1/2}(\partial B_a))} \leq \frac{1}{1 - \|S_0(\Lambda_\gamma - \Lambda_1)\|_{\mathcal{B}(H^{1/2}(\partial B_a))}}.$$

Hence from (3.21) and Lemma 3.2 we conclude that if ζ and γ satisfy the estimate

$$0 < |\zeta|^{n-2}\frac{C\|\gamma - 1\|_{L^\infty(B_a)}}{1 - C\|S_0\|_{\mathcal{B}(H^{1/2}(\partial B_a))}\|\gamma - 1\|_{L^\infty(B_a)}} < 1,$$

with C being the constant from Lemma 3.2, then ζ is not exceptional for γ. Furthermore, we conclude that if $\|\gamma - 1\|_{L^\infty(B_a)}$ is sufficiently small, then there are no exceptional points $\zeta \in \mathcal{V}$ with $|\zeta| \leq R$. $\square$

From the invertibility of the boundary integral equation we get the following estimate of the exponentially growing solutions for small ζ.

Lemma 3.7. *Assume $\gamma \in L^\infty_+(B_a)$ satisfies $\gamma = 1$ near ∂B_a. Then for $\zeta \in \mathcal{V}$ sufficiently small*

$$\|\phi(\cdot, \zeta) - 1\|_{H^{3/2}(\partial B_a)} \leq C|\zeta|. \tag{3.22}$$

Proof. Let $A_{\gamma,\zeta} = S_\zeta(\Lambda_\gamma - \Lambda_1)$ and note that ϕ satisfies

$$\phi(x, \zeta) - 1 = (e^{ix\cdot\zeta} - 1) - A_{\gamma,\zeta}(\phi(x, \zeta) - 1) \tag{3.23}$$

since $(\Lambda_\gamma - \Lambda_1)1 = 0$. Then by Theorem 3.6 the operator $I + A_{\gamma,\zeta}$ is invertible for ζ near zero and hence

$$\phi(x, \zeta) - 1 = (I + A_{\gamma,\zeta})^{-1}(e^{ix\cdot\zeta} - 1). \tag{3.24}$$

Note then that

$$\|e^{ix\cdot\zeta} - 1\|_{H^{1/2}(\partial B_a)} \leq C|\zeta|,$$
$$\|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial B_a)} \leq C|\zeta|,$$

which is proved by writing the Taylor expansion for $e^{i\zeta\cdot x}$ around zero and then using interpolation similarly to the proof of Lemma 3.5.

Now since $\|(I + A_{\gamma,\zeta})^{-1}\|_{\mathcal{B}(H^{1/2}(\partial B_a))}$ is uniformly bounded for small $|\zeta|$ we find

$$\|\psi(\cdot,\zeta) - 1\|_{H^{1/2}(\partial B_a)} \leq C\|e^{ix\cdot\zeta} - 1\|_{H^{1/2}(\partial B_a)} \leq C|\zeta|.$$

Further, since $A_{\gamma,\zeta}$ is uniformly bounded from $H^{1/2}(\partial B_a)$ to $H^{3/2}(\partial B_a)$ for $|\zeta| < 1$, equation (3.23) implies that

$$\begin{aligned}
&\|\phi(\cdot,\zeta) - 1\|_{H^{3/2}(\partial B_a)} \\
&\leq \|e^{ix\cdot\zeta} - 1\|_{H^{3/2}(\partial B_a)} + \|A_{\gamma,\zeta}\|_{\mathcal{B}(H^{1/2}(\partial B_a),H^{3/2}(\partial B_a))}\|\phi(\cdot,\zeta) - 1\|_{H^{1/2}(\partial B_a)} \\
&\leq C|\zeta|,
\end{aligned}$$

which gives the desired estimate. □

Finally, we prove our result on reconstructing the conductivity at the zero complex frequency limit.

Theorem 3.8. *Let $\zeta \in \mathcal{V}$ be sufficiently small and let $\psi(x,\zeta)$ be the unique exponentially growing solution to the Schrödinger equation* (2.1) *with q defined by* (2.2) *in $L^p(\Omega)$, $p > n/2$. Then*

$$\gamma^{1/2}(x) = \lim_{|\zeta|\to 0} \psi(x,\zeta) \tag{3.25}$$

with convergence in $H^2(\mathcal{B}_a)$.

Proof. Since $\gamma = 1$ near $\partial\Omega$ we have by Lemma 3.7 that for ζ near zero

$$\|\psi(x,\zeta) - \gamma^{1/2}\|_{H^{3/2}(\partial B_a)} = \|\psi(x,\zeta) - 1\|_{H^{3/2}(\partial B_a)} \leq C|\zeta|.$$

Since $\psi(x,\zeta) - \gamma^{1/2}(x)$ is the unique solution to $(-\Delta + q)(\psi(x,\zeta) - \gamma^{1/2}) = 0$ in B_a elliptic regularity estimates imply that

$$\|\psi(\,\cdot\,,\zeta) - \gamma^{1/2}(\,\cdot\,)\|_{H^2(B_a)} \leq C\|\psi(\,\cdot\,,\zeta) - \gamma^{1/2}(\,\cdot\,)\|_{H^{3/2}(\partial B_a)} \leq C|\zeta|,$$

which gives the result. □

4. SCATTERING TRANSFORM AND THE $\overline{\partial}_\zeta$-EQUATION

In this section we assume the conductivity $\gamma \in W^{2,\infty}(\Omega)$ and define q by (2.2).

Let $\zeta \in \mathcal{V}$ be non-exceptional. Define for $\zeta \in \mathcal{V}$ and $\xi \in \mathbb{R}^n$ the non-physical scattering transform of q by

$$\mathbf{t}(\xi,\zeta) = \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x,\zeta)\,dx, \tag{4.1}$$

where $\psi(x,\zeta)$ is the unique solution to (2.5). To turn (4.1) into an integral on the boundary we define for any $\xi \in \mathbb{R}^n$

$$\mathcal{V}_\xi = \{\zeta \in \mathcal{V} \,:\, (\xi+\zeta)^2 = |\xi|^2 + 2\xi\cdot\zeta = 0\}.$$

Then $e^{-ix\cdot(\xi+\zeta)}$ is harmonic if and only if $\zeta \in \mathcal{V}_\xi$, and when $\zeta \in \mathcal{V}_\xi$ is not exceptional we get by Green's second identity

$$\mathbf{t}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}(\Lambda_q - \Lambda_0)\psi(\cdot,\zeta)\, d\sigma(x), \qquad \xi \in \mathbb{R}^n, \quad \zeta \in \mathcal{V}_\xi. \tag{4.2}$$

We note that for fixed $\xi \in \mathbb{R}^n$, the set $\mathcal{V}_\xi$ consists of those $\zeta \in \mathcal{V}$ with real part in the hyperplane $|\operatorname{Re}(\zeta)+\xi| = |\operatorname{Re}(\zeta)|$ and imaginary part in the hyperplane $\operatorname{Im}(\zeta)\cdot\xi = 0$. Hence $\mathcal{V}_\xi$ is a variety of complex dimensions $n-2$.

The usefulness of the scattering transform in relation to the inverse boundary value problem is that it can be computed from the Dirichlet-to-Neumann map, i.e. for $\xi \in \mathbb{R}^n$ and $\zeta \in \mathcal{V}_\xi$ not exceptional, we can first solve (2.9) to find $\psi(\cdot,\zeta)|_{\partial\Omega}$, and then compute $\mathbf{t}(\xi,\zeta)$ from Λ_q by (4.2).

Consider now for fixed $x \in \mathbb{R}^n$, $\mu(x,\cdot)$ as a function on $\mathcal{V}$. Since $\mathcal{V}$ is a variety of $n-1$ complex dimension, $\mu(x,\cdot)$ is locally a function of $n-1$ complex variables. Moreover, it has singularities at exceptional points. Away from the singularities μ is however differentiable: let $\zeta \in \mathcal{V}$ and let $\mathcal{W}_\zeta = \{w \in \mathbb{C}^n : w\cdot\overline{\zeta} = 0\}$. Using a local coordinate chart on $\mathcal{V}$ it can be seen that $\mathcal{W}_\zeta$ is the tangent space to $\mathcal{V}$ in the point ζ. Then for $f \in C^1(\mathcal{V})$ we define the $\overline{\partial}_\zeta$-derivative at the point ζ and direction $w \in \mathcal{W}_\zeta$ by

$$w\cdot\overline{\partial}_\zeta f(\zeta) = \sum_{j=1}^n w_j \overline{\partial}_{\zeta_j} f(\zeta).$$

Similarly, for $\xi \in \mathbb{R}^n, \zeta \in \mathcal{V}_\xi$ we denote by $\mathcal{W}_{\xi,\zeta} = \{w \in \mathbb{C}^n : w\cdot\overline{\zeta} = w\cdot\xi = 0\}$ the tangent space to $\mathcal{V}_\xi$ at ζ, and then we define for $\xi \in \mathbb{R}^n$ and $f \in C^1(\mathcal{V}_\xi)$ the derivative at the point $\zeta \in \mathcal{V}_\xi$ in direction $w \in \mathcal{W}_{\xi,\zeta}$ by

$$w\cdot\overline{\partial}_\zeta f(\zeta) = \sum_{j=1}^n w_j \overline{\partial}_{\zeta_j} f(\zeta).$$

For μ and $\mathbf{t}$ it is now possible to state the following $\overline{\partial}_\zeta$-equations.

Lemma 4.1. *Assume that $\zeta \in \mathcal{V}$ is not exceptional. Then for $w \in \mathcal{V}_\zeta$ we have*

$$w\cdot\overline{\partial}_\zeta \mu(x,\zeta) = -\frac{1}{(2\pi)^{n-1}} \int_{B_\zeta} e^{ix\cdot\xi}\mathbf{t}(\xi,\zeta)\mu(x,\zeta+\xi)(w\cdot\xi)\, d\sigma(\xi), \tag{4.3}$$

where $B_\zeta = \{\xi \in \mathbb{R}^n : (\xi+\zeta)^2 = 0\}$ is the ball in the plane $\xi\cdot\operatorname{Im}(\zeta) = 0$ centred at $c = -\operatorname{Re}(\zeta)$ with radius $r = |\operatorname{Re}(\zeta)|$.

Furthermore, for $\xi \in \mathbb{R}^n$, $\zeta \in \mathcal{V}_\xi$ and $w \in \mathcal{W}_{\xi,\zeta}$ we have

$$w \cdot \overline{\partial}_\zeta \mathbf{t}(\xi,\zeta) = -\frac{1}{(2\pi)^{n-1}} \int_{B_\zeta} \mathbf{t}(\xi-\eta,\zeta+\eta)\mathbf{t}(\eta,\zeta)(w\cdot\eta)d\sigma(\eta). \tag{4.4}$$

for any $\xi \in \mathbb{R}^n$, $\zeta \in \mathcal{V}_\xi$.

Proof. See any of the references [4, 20, 33, 35] for the higher dimensional $\overline{\partial}_\zeta$-equations; the formulation here is taken from [35]. □

Note that integrating by parts in (2.4) shows that $G_{\zeta+\xi}(x) = G_\zeta(x)$ for $\xi \in \mathbb{R}^n$ with $(\xi+\zeta)^2 = 0$, so from (2.5) we have that ζ is exceptional if and only if $\zeta+\xi \in \mathcal{V}$ is exceptional for $\xi \in \mathbb{R}^n$. Hence $\mu(x,\zeta+\xi)$ in the right hand side of (4.3) is well-defined on the domain of integration. The same argument shows that the scattering transform in the right hand side of (4.3) and (4.4) is well-defined and can be evaluated from boundary measurements by (4.2).

The $\overline{\partial}$-equation (4.4) for $\mathbf{t}$ is in fact an implicit characterization of the admissible set of scattering data, i. e. $\mathbf{t}$ is the scattering transform of some q if and only if it solves (4.4) [4]. In presence of noisy data it could be useful to project the scattering transform onto the admissible set defined by that equation before solving the inverse scattering problem of computing q from $\mathbf{t}$.

5. RECONSTRUCTION METHODS

In this section we will focus on the reconstruction issue of the inverse conductivity problem in three dimensions. The conductivity γ is assumed to have enough smoothness for the function q defined by (2.2) to be well-behaved. We will review in Subsection 5.1 two theoretical reconstruction formulae available in the literature. Also, we give in Subsection 5.2 a new reconstruction algorithm for smooth conductivities close to a constant. The key point is the absence of exceptional points for such conductivities, allowing a high-dimensional analogue of Nachman's two-dimensional reconstruction method [36].

5.1. High complex frequency methods

There are two methods available for reconstruction of q from its scattering transform $\mathbf{t}$ using only $\zeta \in \mathcal{V}$ near infinity. For these methods, existence of exceptional points is not a problem if they belong to a bounded subset of $\mathbb{C}^n$. However, these reconstruction procedures seem impractical since high complex frequency information in Λ_γ is related to extremely oscillatory boundary data that are difficult to approximate practically, e.g. with electrodes.

First, by [42] we know that there are no large exceptional points for $q \in L^\infty(\Omega)$ and that

$$|\hat{q}(\xi) - \mathbf{t}(\xi,\zeta)| \leq \frac{C}{|\zeta|}. \tag{5.1}$$

We can thus reconstruct a conductivity $\gamma \in W^{2,\infty}(\Omega)$ with the following steps:

(i) Solve the boundary integral equation (2.9) for $\psi(\cdot,\zeta)|_{\partial\Omega}$ for $\zeta \in \mathcal{V}$ large.

(ii) Calculate the scattering transform $\mathbf{t}$ by (4.2) for $\xi \in \mathbb{R}^3$ and large $\zeta \in \mathcal{V}_\xi$.

(iii) Calculate $\hat{q}(\xi) = \lim_{|\zeta|\to\infty} \mathbf{t}(\xi,\zeta)$ and then q by inverse Fourier transform.

(iv) Solve the Schrödinger equation (2.2) with $\gamma^{1/2}|_{\partial\Omega} = 1$ for $\gamma^{1/2}$.

This method is already implicitly in [42] and given explicitly in [34, 38].

The second approach is based on the $\overline{\partial}_\zeta$-equation (4.4). This idea was suggested in a different context in [27, 33]; a rigorous treatment was given in [35] for $\gamma \in C^{1,1}(\overline{\Omega})$. The method is based on a higher dimensional version of the generalized Cauchy formula, which gives a formula for $\hat{q}$ in terms of $\mathbf{t}$, $\overline{\partial}_\zeta \mathbf{t}$ and the asymptotic value $\lim_{|\zeta|\to\infty} \mathbf{t}(\xi,\zeta)$. A suitable version due to Hatziafratis [19] of this so-called Bochner–Martinelli formula gives the existence of a differential form K on $\mathcal{V}_\xi$ such that for $\xi \in \mathbb{R}^3$, $R > 0$, $f \in C^1(\mathcal{V}_\xi \setminus B(0,R))$ and $R' > R$

$$f(\zeta) = \int_{z\in\mathcal{V}_\xi,|z|=R'} f(z)K(z,\zeta) - \int_{z\in\mathcal{V}_\xi,|z|=R} f(z)K(z,\zeta) + \int_{z\in\mathcal{V}_\xi,R<|z|<R'} \bar{\partial}_z f(z) \wedge K(z,\zeta). \quad (5.2)$$

For the exact definition of the form $K(z,\zeta)$ we refer to [35] (note that in the definition of $K(z,\zeta)$, formula (3.30) in [35], the $\beta(z)$ should have been $\beta(\zeta)$).

Choose $R > 0$ such that there are no exceptional points ζ with $|\zeta| \geq R$, and consider formula (5.2) with $f(\zeta) = \mathbf{t}(\xi,\zeta)$. For $R' \to \infty$ the estimate (5.1) implies that

$$\lim_{R'\to\infty} \int_{z\in\mathcal{V}_\xi,|z|=R'} \mathbf{t}(\xi,z)K(z,\zeta) = \hat{q}(\xi).$$

The resulting formula is then

$$\hat{q}(\xi) = \mathbf{t}(\xi,\zeta) + \int_{z\in\mathcal{V}_\xi,|z|=R} \mathbf{t}(\xi,z)K(z,\zeta) - \int_{z\in\mathcal{V}_\xi,|z|>R} \bar{\partial}_z \mathbf{t}(\xi,z) \wedge K(z,\zeta), \quad (5.3)$$

and since the right hand side is known from the boundary measurements, this formula can be interpreted as a reconstruction formula for $\hat{q}$. A conductivity $\gamma \in C^{1,1}(\overline{\Omega})$ can now be recovered by replacing steps (ii) and (iii) above with

(ii′) Calculate the scattering transform $\mathbf{t}$ by (4.2) and $\overline{\partial}_\zeta \mathbf{t}$ by (4.4) for $\xi \in \mathbb{R}^3$, $\zeta \in \mathcal{V}_\xi$.

(iii′) Calculate $\hat{q}$ by (5.3) and then q by inverting the Fourier transform.

5.2. A medium complex frequency method

We propose an algorithm for the reconstruction of a $W^{1,\infty}(\Omega)$ conductivity close to a constant from boundary data. The interesting point here is that the algorithm to be given is a direct analogue of the reconstruction algorithm for two-dimensional conductivities in [36]. We will in this section without loss of generality assume that $\Omega = B_a$ and $\gamma = 1$ near $\partial\Omega$.

We use exponentially growing solutions for *all* $\zeta \in \mathcal{V}$, and thus the method applies only to conductivities with no exceptional points. We assume that $\gamma \in W^{1,\infty}(\Omega)$ is sufficiently close to a constant and prove the absence of exceptional points in this case. The idea is first to compute the scattering transform from boundary measurements, then to solve the $\overline{\partial}$-equation (4.3) for μ (again using the smallness assumption), and finally to reconstruct the conductivity from μ by taking $\zeta \to 0$.

It is known by [11, 22] that C^2 conductivities near enough to 1 do not have exceptional points. Combining our Theorem 3.6 with [39, Theorem 1.1] enables us to generalize this result for $W^{1,\infty}$ conductivities.

Theorem 5.1. *Let $\gamma \in W^{1,\infty}(B_a)$ with $\gamma = 1$ near ∂B_a. Then there is a $\delta > 0$, depending only on the radius a, such that if*

$$\|\gamma - 1\|_{W^{1,\infty}(B_a)} < \delta,$$

then there are no exceptional points for γ.

Proof. By [39, Theorem 1.1] we can take $\delta_1 > 0$ and $R > 0$ such that there are no exceptional points in $\{\zeta \in \mathcal{V} : |\zeta| > R\}$ for any $\gamma \in W^{1,\infty}(B_a)$ satisfying $\|\gamma - 1\|_{W^{1,\infty}(B_a)} < \delta_1$. Further, by Theorem 3.6 we can take δ_2 such that there are no exceptional points in $\{\zeta \in \mathcal{V} : |\zeta| \leq R\}$ for any $\gamma \in W^{1,\infty}(B_a)$ satisfying $\|\gamma - 1\|_{L^\infty(B_a)} < \delta_2$. The result now follows by choosing $\delta = \min(\delta_1, \delta_2)$. □

The next result concerns the solvability of (4.3).

Lemma 5.2. *For $q \in C_0^\infty(\mathbb{R}^3)$ sufficiently small and $x \in \Omega$ fixed, there is a unique solution $\mu(x, \cdot) \in C(\mathcal{V})$ to (4.3) with*

$$\lim_{|\zeta| \to \infty} \mu(x, \zeta) = 1. \tag{5.4}$$

Proof. We refer to [4, Theorem 4]. □

The proof of this theorem is based on the fact that smoothness in q is equivalent to decay of $\mathbf{t}$ and that smallness in q implies smallness of $\mathbf{t}$. Beals and Coifman do actually give a method for obtaining the unique solution μ to (4.3) subject to the asymptotic condition (5.4). Hence from the scattering transform $\mathbf{t}(\xi, \zeta)$, the exponentially growing solution $\psi(x, \zeta) = e^{ix\cdot\zeta}\mu(x, \zeta)$ can be computed.

Now we can propose the following algorithm for the reconstruction of smooth conductivities close to constants:

(1) Solve (2.9) for $\psi(\cdot,\zeta)$, $\zeta \in \mathcal{V}$.

(2) Calculate the scattering transform $\mathbf{t}$ by (4.2) for $\xi \in \mathbb{R}^3$, $\zeta \in \mathcal{V}_\xi$.

(3) Solve (4.3) for $\mu(x,\cdot)$, $x \in \Omega$.

(4) Reconstruct γ from the formula (3.25) in the $H^2(\Omega)$ topology.

Let us make two remarks about practical applicability of the above method. First, a generalization of this algorithm to conductivities deviating significantly from a constant value requires (a) showing the absence of exceptional points for this class of conductivities and (b) proving the solvability of (4.3) with large kernel. Showing (a) would require a decay estimate for S_ζ when $|\zeta| \to \infty$ which does not seem to be available.

Second, the starting point of our method is exact knowledge of the voltage-to-current map Λ_γ. However, in practice we are given a finite number of noisy measurement data. Analogously to the two-dimensional case (see [21]) we expect that $\mathbf{t}(\xi,\zeta)$ can be recovered approximately and stably from practical measurements for $\xi \in \mathbb{R}^3$, $\zeta \in \mathcal{V}_\xi$ satisfying $|\xi| \le M$ and $|\zeta| \le M$ for some positive constant M depending on the noise level. Then, instead of (4.3), we would solve the truncated $\bar\partial$ equation

$$w\cdot\bar\partial_\zeta \mu_M(x,\zeta) = -\frac{1}{(2\pi)^2}\int_{B_\zeta \cap \{|\xi|\le M\}} e^{ix\cdot\xi}\mathbf{t}(\xi,\zeta)\mu_M(x,\zeta+\xi)(w\cdot\xi)\,d\sigma(\xi), \quad (5.5)$$

and consider μ_M as an approximation to μ. This explains the term *medium frequency* used in the title of this section. Of course, the use of equation (5.5) should be justified by a three-dimensional analogue of [32, Theorem 4.2].

6. CONCLUSION

In this paper we have investigated the usability of exponentially growing solutions in the solution of the EIT problem in three dimensions. The results are two-fold. First we have shown that such solutions exist for a broad class of conductivities provided that the complex frequency is sufficiently small. As a corollary we have obtained the absence of exceptional points for small conductivities having essentially one derivative. Second, we have shown that similarly to the two-dimensional case, the conductivity can be reconstructed in a low frequency limit from the exponentially growing solutions. This result combined with the $\bar\partial$-method of inverse scattering enables a new reconstruction algorithm, which is a direct generalization of a practical and mathematically justified reconstruction algorithm for two-dimensional EIT.

To our best knowledge, all currently available rigorous reconstruction algorithms are based on exponentially growing solutions. Such solutions seem inevitable when one tries to solve an EIT inverse problem, but they have the drawback of causing severe instabilities in the high frequency regime. One way to get around the exponential instability is by avoiding the use of large complex frequencies. In this paper we have obtained results in that direction, which we hope will be another step towards an exact, mathematically sound and practical reconstruction algorithm for three-dimensional EIT.

REFERENCES

1. K. Astala and L. Päivärinta, *Calderón's Inverse Conductivity Problem in the Plane. Ann. of Math.* (2006) **163**, 265–299.

2. G. Alessandrini, Stable determination of conductivity by boundary measurements. *Appl. Anal.* (1988) **27**, 153–172.

3. R. Beals and R. R. Coifman, Scattering, transformations spectrales et équations d'évolution non linéaire. I–II. In: *Goulaouic–Meyer–Schwartz Seminar, 1981/1982*. Exp. No. XXI, 9. École Polytech, 1982.

4. R. Beals and R. R. Coifman, Multidimensional inverse scattering and nonlinear partial differential equations. AMS, 1985, **43**, 45–70.

5. R. Blue, *Real-Time Three-Dimensional Electrical Impedance Tomography.* PhD thesis. R.P.I., Troy, NY, 1997.

6. L. Borcea, Electrical impedance tomography. *Inverse Problems* (2002) **18**, R99–R136; Addendum *Inverse Problems* **19**, 997–998.

7. R. M. Brown, Global uniqueness in the impedance-imaging problem for less regular conductivities. *SIAM J. Math. Anal.* (1996) **27**, 1049–1056.

8. R. M. Brown and R. H. Torres, Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in L^p; $p > 2n$. *J. Fourier Anal. Appl.* (2003) **9**, 563–574.

9. R. M. Brown and G. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. *Comm. Partial Differential Equations* (1997) **22**, 1009–1027.

10. A. P. Calderón, On an inverse boundary value problem. In: *Seminar on Numerical Analysis and its Applications to Continuum Physics.* Society Brasileira de Matemàtica, 1980, 65–73.

11. S. Chanillo, A problem in electrical prospection and an n-dimensional Borg–Levinson theorem. *Proc. Amer. Math. Soc.* (1990) **108**, 761–767.

12. M. Cheney, D. Isaacson, and J. C. Newell, Electrical impedance tomography. *SIAM Review* (1999) **41**, 85–101.

13. D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory.* Wiley, 1983.

14. L. C. Evans, *Partial Differential Equations.* AMS, Providence, RI, 1998.

15. L. D. Faddeev, Growing solutions of the Schrödinger equation. *Dokl. Akad. Nauk SSSR* (1965) **165**, 514–517.

16. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer, 1977.

17. J. Goble, M. Cheney, and D. Isaacson, Electrical impedance tomography in three dimensions. *Appl. Comput. Electromagn. Soc. J.* (1992) **7**, 128–147.

18. A. Greenleaf, M. Lassas, and G. Uhlmann, The Calderón problem for conormal potentials I: Global uniqueness and reconstruction. *Comm. Pure Appl. Math.* (2003) **56**, 328–352.

19. T. E. Hatziafratis, An explicit Koppelman type integral formula on analytic varieties. *Michigan Math. J.* (1986) **33**, 335–341.

20. G. M. Henkin and R. G. Novikov, $\bar{\partial}$-equation in the multidimensional inverse scattering problem. *Uspekhi Math. Nauk* (1987) **42**, 93–152 (in Russian); *Russian Math. Surveys* (1987) **42**, 109–180 (in English).

21. D. Isaacson, J. L. Mueller, J. C. Newell, and S. Siltanen, Reconstructions of chest phantoms by the d-bar method for electrical impedance tomography. *IEEE Transactions on Medical Imaging* (2004) **23**, 821–828.

22. C. E. Kenig, A. Ruiz, and C. D. Sogge, Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.* (1987) **55**, 329–347.

23. A. Kirsch, Surface gradients and continuity properties for some integral operators in classical scattering theory. *Math. Methods Appl. Sci.* (1989) **11**, 789–804.

24. K. Knudsen, *On the Inverse Conductivity Problem.* Ph.D. thesis. Aalborg University, 2002.

25. K. Knudsen, A new direct method for reconstructing isotropic conductivities in the plane. *Physiological Measurement* (2003) **24**, 391–401.

26. K. Knudsen and A. Tamasan, Reconstruction of less regular conductivities in the plane. *Comm. Partial Differential Equations* (2004) **29**, 361–381.

27. R. B. Lavine and A. I. Nachman, On the inverse scattering transform for the n-dimensional Schrödinger operator. In: *Nonlinear Evolution Equations, Solitons and the Inverse Scattering Transform. Oberwolfach July 27–August 8, 1986.* World Scientific Co., 1986.

28. P. Metherall, D. C. Barber, and R. M. Smallwood, Three dimensional electrical impedance tomography. In: *Proc. IX Intern. Conf. Electrical Bio-Impedance.* Heidelberg, 1995, 510–511.

29. P. Metherall, D.C. Barber, R.H. Smallwood, and B.H. Brown, Three-dimensional electrical impedance tomography. *Nature* (1996) **380**, 509–512.

30. P. Metherall, R.H. Smallwood, and D.C. Barber, Three dimensional electrical impedance tomography of the human thorax. In: *Proc 18th Intern. Conf.* IEEE Engng Med. Biol. Society, 1996.

31. J.P. Morucci, M. Granie, M. Lei, *et al.* 3D reconstruction in electrical impedance imaging using a direct sensitivity matrix approach. *Physiol. Meas.* (1995) **16**, A123–A128.

32. J.L. Mueller and S. Siltanen, Direct reconstructions of conductivities from boundary measurements. *SIAM J. Scientific Computation* (2003) **24**, 1232–1266.

33. A.I. Nachman and M.J. Ablowitz, A multidimensional inverse scattering method. *Stud. Appl. Math.* (1984) **71**, 243–250.

34. A.I. Nachman, J. Sylvester, and G. Uhlmann, An n-dimensional Borg–Levinson theorem. *Comm. Math. Phys.* (1988) **115**, 595–605.

35. A.I. Nachman, Reconstructions from boundary measurements. *Ann. of Math.* (1988) **128**, 531–576.

36. A.I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math.* (1996) **143**, 71–96.

37. R.G. Newton, *Inverse Schrödinger Scattering in Three Dimensions.* Springer-Verlag, Berlin, 1989.

38. R.G. Novikov, A multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. *Funktsional. Anal. i Prilozhen.* (1988) **22**, 11–22 (in Russian); *Funct. Anal. Appl.* (1988) **22**, 263–272 (in English).

39. L. Päivärinta, A. Panchenko, and G. Uhlmann, Complex geometrical optics solutions for Lipschitz conductivities. *Rev. Mat. Iberoamericana* (2003) **19**, 56–72.

40. S. Siltanen, J.L. Mueller, and D. Isaacson, An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem. *Inverse Problems* (2000) **16**, 681–699.

41. S. Siltanen, J.L. Mueller, and D. Isaacson, Erratum: An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem. *Inverse problems* (2001) **17**, 1561–1563.

42. J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math.* (1987) **125**, 153–169.

43. G. Uhlmann, Developments in inverse problems since Calderón's foundational paper. In: *Harmonic Analysis and partial Differential Equations.* Univ. Chicago Press, Chicago, 1996, 295–345.

44. P. J. Vauhkonen, *Image Reconstruction in Three-Dimensional Electrical Impedance Tomography.* Ph.D. thesis. Kuopio University publications, 2004, C166.

45. P. J. Vauhkonen, M. Vauhkonen, T. Savolainen, and J. P. Kaipio, Static three-dimensional electrical impedance tomography. *Ann. New York Acad. Sci.* (1999) **873**, 472–481.

46. P. J. Vauhkonen, M. Vauhkonen, T. Savolainen, and J. P. Kaipio, Three-dimensional electrical impedance tomography based on the complete electrode model. *IEEE Trans. Biomed. Eng.* (1999) **46**, 1150–1160.

47. A. Wexler, Electrical impedance imaging in two and three dimensions. *Clin Phys. Physiol. Meas., Suppl. A* (1988) **9**, 29–33.