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# PROBING FOR INCLUSIONS IN HEAT CONDUCTIVE BODIES

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Abstract. This work deals with an inverse boundary value problem arising from the equation of heat conduction. Mathematical theory and algorithm is described in dimensions 1–3 for probing the discontinuous part of the conductivity from local temperature and heat flow measurements at the boundary. The approach is based on the use of complex spherical waves, and no knowledge is needed about the initial temperature distribution. In dimension two we show how conformal transformations can be used for probing deeper than is possible with discs. Results from numerical experiments in the one-dimensional case are reported, suggesting that the method is capable of recovering locations of discontinuities approximately from noisy data.

## 1. Introduction.

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, and consider the following boundary value problem

$$\left\{\begin{array}{rcll} \partial_t v - \nabla\cdot(\gamma(x)\nabla v) &=& 0 & \text{in} \quad (0,T)\times\Omega, \\ v &=& f & \text{on} \quad (0,T)\times\partial\Omega, \\ v\big|_{t=0} &=& v_0 & \text{in} \quad \Omega \end{array}\right. \tag{1}$$

where $\gamma(x)\in L^\infty(\Omega)$ such that $\gamma(x) > c$ for a constant $c>0$. Let $v^f_{v_0,\gamma} = v$ be the unique solution of the above equation. The time-dependent Dirichlet-to-Neumann (DN) map $\Lambda_{v_0,\gamma}$ is then defined by

$$\Lambda_{v_0,\gamma} : f \to \gamma(x)\frac{\partial v^f_{v_0,\gamma}}{\partial\nu}\Big|_{\partial\Omega}, \tag{2}$$

where $\nu$ is the outer unit normal to $\partial\Omega$.

Physically, we consider a heat-conducting body modelled by the set $\overline{\Omega}$ and the strictly positive heat conductivity distribution $\gamma$ inside the body. The function $v_0$ is the initial temperature distribution in $\Omega$. Sometimes, it is natural to deal with

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$v_0$ as unknown, since for instance when we do a series of experiments, we do not have control of the system after the previous experiments.

We perform boundary measurements by applying the temperature $f(x,t)$ at the boundary $\partial\Omega$ during the time $0 < t < T$ and measuring the resulting heat flux $\gamma(x)\frac{\partial v^f_{v_0,\gamma}}{\partial\nu}|_{\partial\Omega}$ through the boundary. The DN map $\Lambda_{v_0,\gamma}$ defined in (2) is an ideal model of all possible infinite-precision measurements of the above type.

We study the inverse problem of detecting conductivity inclusions inside $\Omega$ from the *local* knowledge of $\Lambda_{v_0,\gamma}$. Namely, we assume that only a part $\Gamma \subset \partial\Omega$ is available for measurements and take as data the restrictions $\gamma(x)\frac{\partial v^f_{v_0,\gamma}}{\partial\nu}|_{\Gamma}$ of heat fluxes corresponding to functions $f$ that are supported in $\Gamma$: the temperature on $\partial\Omega \setminus \Gamma$ is assumed to be zero.

An inclusion is defined as a non-empty subdomain $\Omega_1 \subset \Omega$ such that $\gamma(x)$ is perturbed on $\Omega_1$ from some known $\gamma_0(x)$. Our aim is to find the location of $\partial\Omega_1$ from the local measurements. The typical situation is drawn in the figure 1.

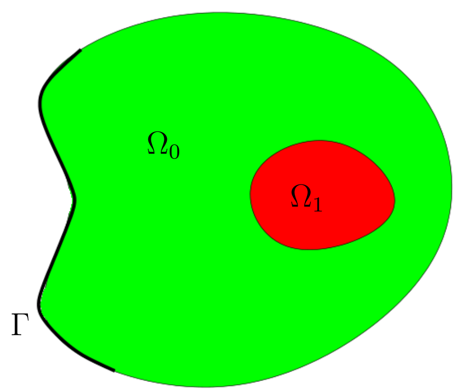


Figure 1.

The above inverse boundary value problem is related to nondestructive testing where one looks for anomalous materials inside a known material. One such example is monitoring a blast furnace used in ironmaking: the corroded thickness of the accreted refractory wall based on temperature and heat flux measurement on the accessible part of the furnace wall [13].

We remark that although $\Lambda_{v_0,\gamma}$ depends on both $f$ and $v_0$, our method of detecting $\partial\Omega_1$ uses no information of the initial data $v_0$.

1.1. **Main theorems.** Suppose we are given $\gamma_0(x) \in C^\infty(\overline{\Omega})$, $\gamma(x) \in L^\infty(\Omega)$ and a non-empty open subset $\Omega_1 \subset\subset \Omega$, i.e. $\overline{\Omega_1} \subset \Omega$, such that for some constant $C_0 > 0$,

$$C_0 < \gamma_0(x) < C_0^{-1}, \quad C_0 < \gamma(x) < C_0^{-1}, \quad \text{on} \quad \Omega,$$

and

$$\gamma(x) = \gamma_0(x), \quad \text{on} \quad \Omega_0 = \Omega \setminus \overline{\Omega_1}.$$

Moreover, we shall assume that $\gamma(x) - \gamma_0(x)$ has a constant sign on $\Omega_1$. Our main purpose is to study discontinuous perturbations, however, we allow $\gamma(x)$ to be continuous. Hence, we impose the following assumption.

**(A)** $\inf_{x\in K} |\gamma(x) - \gamma_0(x)| > 0$ *for any compact set* $K \subset \Omega_1$.

We define (formal) differential operators by

$$\mathcal{A}_0 = -\nabla \cdot (\gamma_0 \nabla), \quad \mathcal{A} = -\nabla \cdot (\gamma \nabla).$$

We extend $\gamma_0(x)$ smoothly outside $\Omega$ so that $\gamma_0(x) = 1$ for large $|x|$.

We shall consider a large parameter $\lambda > 0$ and allow the initial data $v_0(x)$ to depend on $\lambda$, under the following condition:

**(C-0)** *There exist $\lambda$-independent positive constants $C, T_0$ with $T_0 < T$ such that $\|v_0\|_{L^2(\Omega)} \le Ce^{\lambda T_0}$, for all $\lambda$.*

First, we consider the one-dimensional problem. Namely, for the heat equation on $(a,b)$, we try to detect $\mathrm{dist}(a, \partial\Omega_1)$ from the measurement at $a$.

**Theorem 1.1.** *Let $d = 1$, $\Omega = (a,b)$, $\Omega_1 = (a_1, b_1)$ with $a < a_1 < b_1 < b$. Assume* **(C-0)** *and take $x_0 \in \mathbb{R}$ arbitrarily. We define $y(x)$ by*

$$y(x) = \int_{x_0}^{x} \frac{dt}{\sqrt{\gamma_0(t)}}. \tag{3}$$

*Then there exists a real function $\varphi_\lambda(x) \in C^\infty(\mathbb{R})$ depending on a large parameter $\lambda > 0$ having the following properties.*
*(1) It satisfies*

$$(\mathcal{A}_0 + \lambda)\,\varphi_\lambda(x) = 0, \quad x \in (a,b), \quad \lambda >> 1. \tag{4}$$

*(2) There exists a constants $C > 0$ such that for $\lambda > C$*

$$|\varphi_\lambda(x)| \ge Ce^{\sqrt{\lambda}|y(x)|}, \quad |\varphi_\lambda'(x)| \ge Ce^{\sqrt{\lambda}|y(x)|}, \quad \text{if} \quad x > x_0,$$
$$|\varphi_\lambda(x)| \le Ce^{-\sqrt{\lambda}|y(x)|}, \quad |\varphi_\lambda'(x)| \le Ce^{-\sqrt{\lambda}|y(x)|}, \quad \text{if} \quad x < x_0,$$

*(3) Take $T_0 < T_1 < T$ arbitrarily, and let*

$$h_\lambda(a) = \varphi_\lambda(a), \quad h_\lambda(b) = 0,$$
$$f_\lambda(t,x) = e^{\lambda t} h_\lambda(x), \ \textit{for } x = a, b,$$

$$I(\lambda) = e^{-\lambda T_1} \left(\Lambda_{v_0,\gamma} f_\lambda\right)(T_1, a) h_\lambda(a) - \gamma_0(a) \frac{dh_\lambda}{dx}(a) h_\lambda(a). \tag{5}$$

*Then if $\pm(\gamma - \gamma_0) > 0$ on $\Omega_1$, we have*

$$\lim_{\lambda \to \infty} (2\sqrt{\lambda})^{-1} \log(\pm I(\lambda)) = -y(a_1). \tag{6}$$

The higher dimensional case is more involved. By $\mathrm{dist}(x_0, A)$, we mean the distance between a point $x_0 \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$. We denote by $B(x_0, R)$ the open ball $\{x \in \mathbb{R}^d\,; |x - x_0| < R\}$.

We construct complex spherical waves in the following way.

**Theorem 1.2.** *Let $d = 2, 3$ and take two different points $x_0 \in \mathbb{R}^d$, $x^* \in \mathbb{R}^d \setminus \overline{\Omega}$. Take $\mu > 0$ arbitrarily and $y^*, y^*_\perp \in \mathbb{R}^d$ such that*

$$y^* = \frac{x^* - x_0}{|x^* - x_0|}, \quad y^* \cdot y^*_\perp = 0, \quad |y^*_\perp| = 1. \tag{7}$$

*Let $\zeta \in \mathbf{C}^d$ be a complex vector such that*

$$\zeta = \frac{\mu\lambda}{\sqrt{2}}(y^* + iy^*_\perp). \tag{8}$$

*Let* $y = y(x)$ *be the inversion defined by*

$$y = y^* + 2R\frac{x - x^*}{|x - x^*|^2}, \quad R = |x_0 - x^*| > 0. \tag{9}$$

*Then for large* $\lambda > 0$*, there exists a solution* $\varphi_{\lambda,\mu}(x)$ *to the equation*

$$(\mathcal{A}_0 + \lambda)\varphi = 0 \tag{10}$$

*in the region* $0 < |x - x^*| < 2\sqrt{\lambda}R$ *having the following form:*

$$\varphi_{\lambda,\mu}(x) = |x - x^*|^{2-d}\gamma_0(x)^{-1/2}(1 + \phi_{\lambda,\mu}(x))e^{-\zeta\cdot y(x)}, \tag{11}$$

*where for some* $0 < \delta < 1$*,* $\phi_{\lambda,\mu}(x)$ *satisfies*

$$\|\phi_{\lambda,\mu}(x)\|_{L^\infty(\Omega)} \le C_\delta\, \mu^{-1}\lambda^{-\delta/2}, \tag{12}$$

$$\|\nabla_x\phi_{\lambda,\mu}(x)\|_{L^\infty(\Omega)} \le C_\delta\, \mu^{-1}\lambda^{(1-\delta)/2}, \tag{13}$$

*for a constant* $C_\delta > 0$*, which also depends on* $x^*$ *and* $x_0$ *but is independent of* $\mu$ *and large* $\lambda$*. The function* $\varphi_{\lambda,\mu}(x)$ *is exponentially growing in the ball* $B(x_0, R)$*, and exponentially decaying outside the ball* $\overline{B(x_0, R)}$ *because of the following properties:*

$$\begin{aligned} &\mathrm{Re}\,\zeta\cdot y(x) < 0 \Longleftrightarrow |x - x_0| < R,\\ &\mathrm{Re}\,\zeta\cdot y(x) = 0 \Longleftrightarrow |x - x_0| = R,\\ &\mathrm{Re}\,\zeta\cdot y(x) > 0 \Longleftrightarrow |x - x_0| > R. \end{aligned} \tag{14}$$

Take $x^* \in \mathbb{R}^d \setminus \overline{\Omega}$, $e^* \in \mathbb{S}^{d-1}$, $R > 0$ so that the ball $B = B(x_0, R)$ with $x_0 = x^* + Re^*$ satisfies

**(C-1a)** $\emptyset \neq \overline{B} \cap \partial\Omega \subset \Gamma$,

**(C-1b)** $B \cap \Omega \neq \emptyset$.

For a set $\mathcal{O} \subset \mathbb{R}^d \setminus \{x^*\}$, let

$$a_B(\mathcal{O}) = \begin{cases} \sup_{x\in\mathcal{O}} \frac{R^2 - |x-x_0|^2}{|x-x^*|^2} & \text{if} \quad \mathcal{O} \neq \emptyset,\\ -\infty & \text{if} \quad \mathcal{O} = \emptyset. \end{cases} \tag{15}$$

We set

$$\frac{1}{\mu_B} = \begin{cases} \sqrt{2}a_B(\Omega) & \text{if} \quad \Omega \subset B,\\ \sqrt{2}\left(a_B(\Omega) + \sup_{x\in\Omega\setminus B} \frac{|x-x_0|^2 - R^2}{|x-x^*|^2}\right) & \text{if} \quad \Omega \not\subset B. \end{cases} \tag{16}$$

Observe that **(C-1b)** is equivalent to $a_B(\Omega) > 0$, which implies $\mu_B > 0$.

Let us fix $\chi \in C^\infty(\mathbb{R}^d)$, $0 \le \chi \le 1$, $\chi = 1$ in some neighbourhood of $\overline{B}$, $\chi = 0$ on $\partial\Omega \setminus \Gamma$. If $\Gamma = \partial\Omega$ we take $\chi = 1$.

The tool of the probing procedure from the open subset $\Gamma \subset \partial\Omega$ is as follows.

**Theorem 1.3.** *Fix positive constants* $T_1, \mu$ *such that*

$$T_0 < T_1 \le T, \tag{17}$$

$$0 < \mu < \frac{(T_1 - T_0)\mu_B}{a_B(\Omega)\mu_B/\sqrt{2} + 1}. \tag{18}$$

*Let* $\varphi_\lambda = \varphi_{\lambda,\mu}$ *as in Theorem 1.2 with this restriction on* $\mu$*. We put*

$$h_\lambda(x) = \varphi_\lambda(x)\big|_{\partial\Omega}, \quad f_\lambda(t, x) = e^{\lambda t}h_\lambda(x).$$

*For $\chi$ as above, we set*

$$I(\lambda) = e^{-\lambda T_1}\int_\Gamma (\Lambda_{v_0,\gamma}(\chi f_\lambda))(T_1,x)\chi\overline{h_\lambda(x)}d\sigma(x) - \int_\Gamma \gamma_0(x)\frac{\partial\varphi_\lambda(x)}{\partial\nu}\chi\overline{h_\lambda(x)}d\sigma(x), \tag{19}$$

*$d\sigma(x)$ being the induced measure on $\partial\Omega$.*

*Then under the condition* **(C-0)** *we have the following alternative.*

*(1) If $a_B(\{\chi\neq 0\}\cap\Omega_1)\le a_B(\{\chi\neq 1\})$, then $I(\lambda)$ tends exponentially to 0 as $\lambda\to\infty$ and*

$$\limsup_{\lambda\to\infty}(\sqrt{2}\mu\lambda)^{-1}\log(|I(\lambda)|)\le a_B(\{\chi\neq 1\})<0.$$

*(2) If $a_B(\{\chi\neq 0\}\cap\Omega_1)> a_B(\{\chi\neq 1\})$, then*

$$\lim_{\lambda\to\infty}(\sqrt{2}\mu\lambda)^{-1}\log(\pm I(\lambda)) = a_B(\Omega_1).$$

*corresponding to $\pm(\gamma-\gamma_0)>0$.*

By Theorem 1.3 we can see whether $B$ touches $\Omega_1$ or not using only the knowledge of $\gamma_0(x)$ under the condition **(C-0)**.

**Corollary 1.** *(1) If $R<\mathrm{dist}(x_0,\partial\Omega_1)$, then $I(\lambda)$ tends exponentially to 0 as $\lambda\to\infty$.*
*(2) If $R>\mathrm{dist}(x_0,\partial\Omega_1)$, then $|I(\lambda)|$ tends exponentially to $\infty$.*
*(3) If $R=\mathrm{dist}(x_0,\partial\Omega_1)$, then*

$$\lim_{\lambda\to\infty}(\sqrt{2}\mu\lambda)^{-1}\log(\pm I(\lambda)) = 0.$$

*Proof.* This is a consequence of Theorem 1.3 and the following relations:

$$\begin{array}{lllll} a_B(\Omega_1)>0 & \iff & B\cap\Omega_1\neq\emptyset & \iff & R>\mathrm{dist}(x_0,\partial\Omega_1),\\ a_B(\Omega_1)<0 & \iff & \overline{B}\cap\overline{\Omega_1}=\emptyset & \iff & R<\mathrm{dist}(x_0,\partial\Omega_1). \end{array}$$

□

Let us make some remarks on Theorem 1.3 and Corollary 1.

1. We don't assume any smoothness for $\Omega_1$.
2. Since the function $\varphi_\lambda(x)$ is singular at $x=x^*$, we have to take $x^*$ outside $\overline{\Omega}$ in Theorem 1.3 and Corollary 1.
3. The conditions $a_B(\{\chi\neq 0\}\cap\Omega_1)>a_B(\{\chi\neq 1\})$ and $a_B(\Omega_1)>a_B(\{\chi\neq 1\})$ are equivalent. If they, hold then $a_B(\{\chi\neq 0\}\cap\Omega_1)=a_B(\Omega_1)$. See Lemma 4.2.
4. Our results also hold for $4\le d\le 7$. The limitation of the dimension only comes from the proof of Lemma 3.2.
5. In Theorems 1.2 and 1.3, $\varphi_\lambda=\varphi_{\lambda,\mu}$ is complex-valued. For *real* experiments we have to deal with real data. In fact Theorems 1.2 and 1.3 hold with $(h_\lambda,\varphi_\lambda,f_\lambda)$ replaced by $(\Re e(h_\lambda),\Re e(\varphi_\lambda),\Re e(f_\lambda))$. See Remark 1 after Lemma 4.2, and Remark 2 after Lemma 4.4.

1.2. **Detection algorithm.** Suppose for the sake of simplicity that $\gamma_0\equiv 1$. The one-dimensional case is rather easy. We have only to use (6) to compute $a_1$. The detection algorithm for $d=2,3$ is as follows.

1. Take $x^*$ outside $\overline{\Omega}$, but close to $\Gamma$ and $e^*\in\mathbb{S}^{d-1}$ pointing towards $\Omega$.
2. Take $R>0$ as large as possible so that the ball $B=B(x_0,R)$ with $x_0=x^*+Re^*$ satisfies **(C-1a)** and **(C-1b)**.

3. Compute $\mu_B$ and set for example $T_1 = T$, $\mu = \frac{(T-T_0)\mu_B}{\sqrt{2}a_B(\Omega)\mu_B+2}$.
4. For large $\lambda > 0$, compute $I(\lambda)$. The function
$$e^{\lambda t}|x - x^*|^{2-d}e^{-\zeta \cdot y(x)},$$
$$\text{where} \quad y(x) = y^* + 2R\frac{x - x^*}{|x - x^*|^2}, \quad \zeta = \frac{\mu\lambda}{\sqrt{2}}(y^* + iy^*_\perp),$$
$$y^* = \frac{x^* - x_0}{|x^* - x_0|}, \quad y^* \cdot y^*_\perp = 0, \quad |y^*_\perp| = 1,$$
will give an approximation of the probing data.
5. Set $c = \limsup_{\lambda\to\infty}(\sqrt{2}\mu\lambda)^{-1}\log(|I(\lambda)|)$. If $c < 0$ we infer that $B$ does not intersect the inclusion.
If $c \geq 0$ then $a_B(\Omega_1) = c$ and we infer that $B$ intersects the inclusion and $\gamma(x) > \gamma_0(x)$ if $I(\lambda) > 0$ or $\gamma(x) < \gamma_0(x)$ if $I(\lambda) < 0$ in the inclusion. Moreover setting
$$x_c = \frac{x_0 + cx^*}{1 + c}, \quad R_c = \frac{R}{|1 + c|},$$
then $B(x_c, R_c)$ touches $\Omega_1$, that is, $\overline{\Omega_1} \cap \overline{B(x_c, R_c)} \neq \emptyset$ and $\Omega_1 \cap B(x_c, R_c) = \emptyset$. See Figure 2.

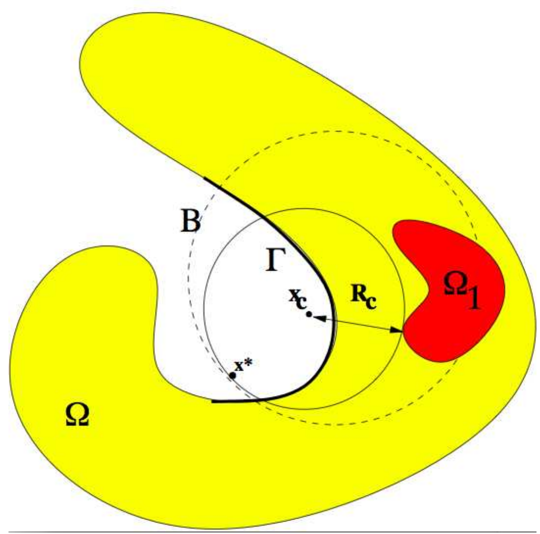


Figure 2.

1.3. **Transformation of exponentially growing solutions.** The main idea of the proof (for $d > 1$) consists in using the probing data of the form $e^{\lambda t}\varphi_\lambda(x)$, where $\varphi_\lambda(x)$ is exponentially growing as $\lambda \to \infty$ in the ball $B(x_0, R)$ and exponentially decaying outside $\overline{B(x_0, R)}$. Such a method was found and used in [5] for the case of the elliptic problem: $\nabla \cdot (\gamma(x)\nabla u) = 0$ by passing through the hyperbolic space. The essential feature of this idea is to use conformal transformation which maps the sphere $B(x_0, R)$ to a plane. In the present paper, we use the same idea but work entirely in the Euclidean space. The difference is that in [5] the center $x_0$ is taken outside the closed convex hull of $\Omega$, while in the present paper $x^*$ is taken outside $\overline{\Omega}$. This makes it possible to probe regions deeper than that of [5].

We are then led to consider the equation of the form

$$(-\Delta + 2\zeta \cdot \nabla + q_\lambda)u = f,$$

where $\zeta \in \mathbf{C}^d$. In [5], $q_\lambda$ does not depend on $\lambda$ and the result of Sylvester-Uhlmann [11] can be applied directly. To compute the behaviour of the DN map, it was enough to use the property that these complex spherical waves grow up exponentially in any compact set in the ball, and decay exponentially in any compact set outside the ball.

In our case, $q_\lambda(x)$ grows up linearly in $\lambda$. However suitable change of variables enables us to reduce the construction of solutions to now standard method of Sylvester-Uhlmann. The next step is to compare the DN map for the parabolic equation with that for the elliptic equation with energy $-\lambda$. This requires us to study the growth (or decay) order of complex spherical waves (and consequently that of $\log|I(\lambda)|$) more precisely and to introduce a new small positive parameter $\mu$. See Lemma 4.6 and 4.7.

It is interesting to find other conformal transformations mapping planes (lines) to some surfaces (curves) which are useful for the probing problem. Putting trivial transformations such as translation and orthogonal transformation aside, for $d = 3$, the spherical inversion is essentially the unique conformal transformation for the Euclidean Laplacian. It can be seen by embedding the problem into the hyperbolic space, as was done in [5]. However, in two dimensions, the complex function theory provides us with lots of conformal transformations. We shall give in this paper one of such examples and use it for the inclusion detection problem in §5.

1.4. **Literature review.** There is an extensive literature on inverse problem of heat conductivity. Among them, our work is closely related with Ikehata's enclosure method applied to the heat conductivity problem. See [6] for the 1-dimensional case. In Theorem 1.2 of [8], they computed $\sup_{x\in D} x \cdot \omega$ ($\omega \in S^2$), from which the convex hull of $D$ is recovered. In [8], [7], they also derived other geometric quantities which are related with the depth of $D$ from the surface of $\Omega$.

Let us also mention the work of Zhou [15] dealing with the inclusion detection problem for the Maxwell equation applying the spherical inversion to construct the complex geometrical optics solution.

In [3] the interior boundary curve of an arbitrary-shaped annulus is reconstructed from overdetermined Cauchy-data on the exterior boundary curve, by assuming $\gamma = 1$ and using the Newton method to a boundary integral equation approach. In [2] the shape of an inaccessible portion of the boundary is reconstructed by linearisation. In [4] the position of the boundary is reconstructed as a function of time, using the sideways heat equation. The proposed method of the present paper differs from these; it is a direct reconstruction method, and it finds inclusions in non-constant background conductivities. For other related results, see [13, 14, 10] and the survey article [12].

The paper is organized as follows. In §2 and §3, we shall constuct the probing data $\varphi_{\lambda,\mu}(x)$. Theorems 1.1 and 1.2 are proved in §4. The use of conformal mapping in two dimensions is explained in §5. In §6, we give some numerical results for the one dimensional case.

2. **One-dimensional trial function.** First we explain the one-dimensional case, although it is standard. Take $x_0 \in \mathbb{R}$ arbitrarily, and make the change of variable by (3). Letting $\varphi_\lambda = \psi_1(y)e^{-\sqrt{\lambda}y}$, we transform the equation (4) into

$$\psi_1'' - (2\sqrt{\lambda} - p(y))\psi_1' + \lambda p(y)\psi_1 = 0, \quad ' = \frac{d}{dy},$$

where $p(y) = (2\sqrt{\gamma_0(x(y))})^{-1}\frac{d\gamma_0}{dx}(x(y)) = \frac{d}{dy}\log\left(\gamma_0(x(y))^{1/2}\right)$. Putting $P(y) = \gamma_0(x(y))^{-1/4}$, we then have $2\frac{d}{dy}P(y) = -P(y)p(y)$. Therefore, letting $\psi_1 = P(y)\psi_2$, we have

$$\psi_2'' - 2\sqrt{\lambda}\psi_2' - Q(y)\psi_2 = 0, \tag{20}$$

where $Q(y) = \frac{p'(y)}{2} + \frac{p(y)^2}{4} \in C_0^\infty(\mathbb{R})$. Putting $\psi_2 = 1 + \phi$, we then have

$$\phi'' - 2\sqrt{\lambda}\phi' - Q(y)\phi = Q. \tag{21}$$

We consider the integral operator

$$(Su)(y) = \frac{1}{2\sqrt{\lambda}}\int_y^\infty (1 - e^{-2\sqrt{\lambda}(t-y)})u(t)dt.$$

Take $A > 0$ sufficiently large. If $u \in C^2((-\infty, A))$ satisfies $u, u', u'' \in L^1((-\infty, A))$, we have

$$(Su)''(y) = u(y) + 2\sqrt{\lambda}(Su)'(y).$$

Therefore, $\phi \in C^2((-\infty, A))$ is a solution of (21) satisfying $\phi, \phi' \in L^1((-\infty, A))$ if and only if $\phi$ is a bounded solution of

$$\phi = S(Q\phi) + SQ.$$

Since $Q(s)$ is compactly supported, we have

$$\|SQ\|_{\mathbf{B}(L^\infty(I);L^\infty(I))} \leq \frac{C}{\sqrt{\lambda}}, \quad I = (-\infty, A),$$

which implies the existence of a solution $\psi_2 \in C^2((-\infty, A))$ of (20) such that

$$|\partial_x^n(\psi_2(y) - 1)| \leq C(\sqrt{\lambda})^{-1+n}, \quad n = 0, 1, 2.$$

Putting $\varphi_\lambda(x) = P(y)e^{\sqrt{\lambda}y}\psi_2(y)$, we obtain the following Proposition.

**Proposition 1.** *There exists a solution $\varphi_\lambda(x)$ to the equation (4) such that*

$$\varphi_\lambda(x) = \gamma_0(x)^{-1/4}e^{-\sqrt{\lambda}y(x)}(1 + \phi_\lambda(x)), \tag{22}$$

*where $y(x)$ is defined by (3) and $\phi_\lambda(x)$ satisfies*

$$|\partial_x^n\phi_\lambda(x)| \leq C(\sqrt{\lambda})^{-1+n}, \quad n = 0, 1, 2, \quad a \leq x \leq b.$$

## 3. Multi-dimensional trial function.

### 3.1. Schrödinger equation and spherical inversion.

To prove Theorem 1.2, we need two changes of (in)dependent variables. For a solution $\varphi$ to (10), we put $v = \sqrt{\gamma_0}\varphi$. Then $v$ satisfies

$$-\Delta v + \left( \frac{\Delta \sqrt{\gamma_0}}{\sqrt{\gamma_0}} + \frac{\lambda}{\gamma_0} \right) v = 0. \tag{23}$$

Next we pass to the inversion (9). Letting $y' = y - (y \cdot y^*)y^*$, and using $x^* = x_0 + Ry^*$, we have

$$y \cdot y^* = \frac{|x - x_0|^2 - R^2}{|x - x^*|^2}, \quad y' = \frac{2R(x - x_0)'}{|x - x^*|^2}. \tag{24}$$

The inverse map : $y \to x$ is given by

$$x = x^* + 2R \frac{y - y^*}{|y - y^*|^2}. \tag{25}$$

Direct computation yields the following lemma.

**Lemma 3.1.** *(1) The above transformation $x \to y$ maps*

*the ball $\{|x - x_0| < R\}$ to the half-space $\{y \cdot y^* < 0\}$,*
*the sphere $\{|x - x_0| = R\}$ to the plane $\{y \cdot y^* = 0\}$,*
*the outer region $\{|x - x_0| > R\}$ to the half-space $\{y \cdot y^* > 0\}$,*

*where $y^*$ is defined by (7).*
*(2) For any $f(y) \in C^\infty(\mathbb{R}^d)$,*

$$4R^2 \Delta_x f(y(x)) = |y - y^*|^{2+d} \Delta_y \left( |y - y^*|^{2-d} f(y) \right) \Big|_{y=y(x)}.$$

*Here $\Delta_x$ means $\sum_{j=1}^d (\partial/\partial x_j)^2$.*

For a solution $v(x)$ to (23), we put $w(y) = |y - y^*|^{2-d} v(x(y))$, where $x(y)$ is given by (25). Then by the above lemma, $w(y)$ satisfies

$$(-\Delta_y + q_\lambda(y))\, w(y) = 0, \tag{26}$$

$$q_\lambda(y) = \frac{4R^2}{|y - y^*|^4} \left( \frac{\Delta \sqrt{\gamma_0}}{\sqrt{\gamma_0}} + \frac{\lambda}{\gamma_0} \right) \Big|_{x = x(y)}.$$

### 3.2. Proof of Theorem 1.2.

Taking $\zeta$ from (8), we look for a solution $w(y)$ of (26) of the form

$$w(y) = (1 + \phi(y)) e^{-\zeta \cdot y}.$$

Since $\zeta \cdot \zeta = 0$, $\phi$ satisfies

$$-\Delta \phi + 2\zeta \cdot \nabla \phi + q_\lambda \phi = -q_\lambda. \tag{27}$$

This is the equation treated in [11], with the difference that the potential $q_\lambda$ grows up linearly in $\lambda$. The remedy is to make the following change of variables and parameters :

$$\begin{aligned} Y &= \sqrt{\lambda}(y - y^*), \\ \eta &= (\sqrt{\lambda})^{-1} \zeta. \end{aligned} \tag{28}$$

Letting $\Phi(Y) = \phi(y)$, we then have

$$-\Delta_Y \Phi + 2\eta \cdot \nabla_Y \Phi + Q_\lambda \Phi = -Q_\lambda,$$

$$Q_\lambda(Y) = \frac{q_\lambda(y)}{\lambda} = \frac{4R^2}{|Y|^4}\left(\frac{\Delta\sqrt{\gamma_0}}{\lambda\sqrt{\gamma_0}} + \frac{1}{\gamma_0}\right)\Big|_{x=x(y)}.$$

The potential $Q_\lambda(Y)$ is now bounded in $\lambda$, however, it has a singularity at $Y = 0$. We take $\chi_\infty \in C^\infty(\mathbb{R}^d)$ such that $\chi_\infty(Y) = 1$ if $|Y| \geq 1/2$, $\chi_\infty(Y) = 0$ if $|Y| \leq 1/4$, and define $V_\lambda(Y)$ by

$$V_\lambda(Y) = \chi_\infty(Y)Q_\lambda(Y).$$

The following lemma can be proved easily, and the proof is omitted.

**Lemma 3.2.** *If $\sigma < (6-d)/2$, $V_\lambda$ satisfies the following inequalities:*

$$\|(1+|Y|)^4 V_\lambda\|_{L^\infty(\mathbb{R}^d)} \leq C,$$
$$\|V_\lambda\|_{L^{2,\sigma+1}} + \|\nabla_Y V_\lambda\|_{L^{2,\sigma+1}} \leq C,$$
$$\|(1+|Y|)^4 \nabla_Y V_\lambda\|_{L^\infty(\mathbb{R}^d)} \leq C,$$

*where the constant $C$ does not depend on $\lambda \geq 1$.*

Let us introduce the weighted $L^2$-space $L^{2,\sigma}$ by

$$L^{2,\sigma} = \{u(Y)\,;\,(1+|Y|^2)^{\sigma/2}, u \in L^2(\mathbb{R}^d)\}$$

with obvious norm. If $d \leq 7$, then $(-1,-1/2) \subset (-1,(6-d)/2)$.

Then, thanks to [11, Theorem 2.3] and Lemma 3.2, there exists a positive constant $c_\sigma$ such that if $|\eta| = \mu\sqrt{\lambda} > c_\sigma$, the equation

$$(-\Delta_Y + 2\eta\cdot\nabla_Y + V_\lambda)\Psi = -V_\lambda, \tag{29}$$

admits a unique solution $\Psi_{\lambda,\eta} \in L^{2,\sigma}$, for all $\sigma \in (-1,-1/2)$. Moreover, for all $|\eta| > c_\sigma$,

$$\|\Psi\|_{L^{2,\sigma}} \leq C|\eta|^{-1}.$$

Letting $' = \partial/\partial Y_j$ and differentiating (29), we obtain

$$(-\Delta_Y + 2\eta\cdot\nabla_Y + V_\lambda)\Psi' = -V_\lambda'(1+\Psi).$$

Lemma 3.2 again implies $\|V_\lambda'(1+\Psi)\|_{L^{2,\sigma+1}} \leq C$, hence

$$\|\nabla_Y\Psi\|_{L^{2,\sigma}} \leq C|\eta|^{-1}.$$

Let us recall the correspondence

$$|Y| > 1 \Longleftrightarrow |y - y^*| > \frac{1}{\sqrt{\lambda}} \Longleftrightarrow |x - x^*| < 2\sqrt{\lambda}R.$$

Therefore, for $\sigma \in (-1,-1/2)$, there exists a unique solution $\phi(y) \in L^{2,\sigma}$ of (27) in the region $|y-y^*| > 1/\sqrt{\lambda}$. We set a bounded open set $\Omega'$ with smooth boundary, such that $\Omega' \supset \{y(x); x\in\Omega\}$, $y^* \notin \overline{\Omega'}$. Noticing that $|\eta| = \mu\sqrt{\lambda}$, we then have

$$\begin{aligned}\|\phi\|_{L^2(\Omega')} &\leq C_\sigma(1+\lambda)^{|\sigma|/2}\,\|(1+\lambda\,|y-y^*|^2)^{\sigma/2}\phi\|_{L^2(\Omega')}\\ &\leq C_\sigma(1+\lambda)^{|\sigma|/2}\,\lambda^{-d/4}\|\Psi(Y)\|_{L^{2,\sigma}}\\ &\leq C_\sigma\mu^{-1}\lambda^{(|\sigma|-d/2-1)/2}.\end{aligned}$$

Similarly, since $\nabla_y\phi(y) = \sqrt{\lambda}\nabla_Y\Psi(Y)$, we obtain

$$\|\nabla_y\phi\|_{L^2(\Omega')} \leq C_\sigma\mu^{-1}\lambda^{(|\sigma|-d/2)/2},$$

and then, generally,

$$\|\phi\|_{H^s(\Omega')} \leq C_{\sigma,s}\mu^{-1}\lambda^{(|\sigma|-d/2+s-1)/2},$$

for all non-negative integer $s$, and consequently for all non-negative $s$ by an interpolation. Thanks to Sobolev's embeddings in $\mathbb{R}^d$, we obtain, if $s > d/2$,

$$\|\phi\|_{L^\infty(\Omega')} \leq C_{\sigma,s}\mu^{-1}\lambda^{(|\sigma|-d/2+s-1)/2},$$
$$\|\nabla\phi\|_{L^\infty(\Omega')} \leq C_{\sigma,s}\mu^{-1}\lambda^{(|\sigma|-d/2+s)/2}.$$

Setting $s \in (d/2, \min(4, 1/2 + d/2))$, $\sigma \in (-1 + s - d/2, -1/2)$, then $\delta = 1 - |\sigma| + d/2 - s > 0$, and if $\mu\sqrt{\lambda} > c_\sigma$, we obtain

$$\|\phi\|_{L^\infty(\Omega)} \leq C_\delta\, \mu^{-1}\lambda^{-\delta/2},$$
$$\|\nabla\phi\|_{L^\infty(\Omega)} \leq C_\delta\, \mu^{-1}\lambda^{(1-\delta)/2}.$$

The formulas (14) follow from Lemma 3.1. □

## 4. Detection of inclusions.

### 4.1. Stationary D-N map.

We define the stationary Dirichlet-Neumann maps $\Lambda_S^0$ and $\Lambda_S \in B(H^{1/2}(\partial\Omega); H^{-1/2}(\partial\Omega))$ as follows:

$$\Lambda_S^0(h) = \gamma_0\frac{\partial u_0}{\partial\nu}, \quad \Lambda_S(h) = \gamma\frac{\partial u}{\partial\nu},$$

where $u_0$ and $u$ are solutions to

$$\left\{\begin{array}{rcll} (\mathcal{A}_0 + \lambda)\, u_0 &=& 0 & \text{on} \quad \Omega, \\ u_0 &=& h & \text{on} \quad \partial\Omega, \end{array}\right. \tag{30}$$

$$\left\{\begin{array}{rcll} (\mathcal{A} + \lambda)\, u &=& 0 & \text{on} \quad \Omega, \\ u &=& h & \text{on} \quad \partial\Omega. \end{array}\right. \tag{31}$$

Let $\langle\,,\,\rangle_{\partial\Omega}$ be the inner product on $L^2(\partial\Omega)$ and put

$$J_S(h) = \langle(\Lambda_S - \Lambda_S^0)h, h\rangle_{\partial\Omega}. \tag{32}$$

The following proposition was initiated by [9, (3.1), (3.6), Lemma 4.3]) and then completely proved in [8] with the Dirichlet-to-Neumann maps replaced respectively by the opposite of the Neumann-to-Dirichlet maps. The proofs in both situations are similar.

**Proposition 2.** *If $\gamma(x) - \gamma_0(x)$ has a constant sign, then $J_S(h)$ has the same sign and there exists a constant $C > 0$ such that*

$$C\int_\Omega |\gamma - \gamma_0||\nabla u_0|^2 dx \leq |J_S(h)| \leq C^{-1}\int_\Omega |\gamma - \gamma_0||\nabla u_0|^2 dx. \tag{33}$$

### 4.2. Uniform estimate for the parabolic equation.

**Lemma 4.1.** *Let $u_0$ be a solution of the equation $(\mathcal{A}_0 + \lambda)u_0 = 0$ in $\Omega$. Construct solution $u$ to (31) with $h = u_0$. Then there exists a constant $C > 0$ independent of $\lambda$ and $u_0$ such that*

$$\|u\|^2_{H^1(\Omega)} \leq C\lambda\|u_0\|^2_{H^1(\Omega)}. \tag{34}$$

*Construct solution $v(t)$ to (1) with $f = e^{\lambda t}u_0$. Then for any $0 < t \leq T$ and $m = 0, 1, 2$, there exists a constant $C_{tm} > 0$ independent of $\lambda$, and $u_0$ such that*

$$\|v(t,\cdot) - e^{\lambda t}u(\cdot)\|_{H^m(\Omega)} \leq C_{tm}\left(\|v_0\|_{L^2(\Omega)} + \|u_0\|_{H^1(\Omega)}\right), \tag{35}$$

$$\|\Delta(v(t,\cdot) - e^{\lambda t}u(\cdot))\|_{L^2(\Omega\setminus\Omega_1)} \leq C_{t2}\left(\|v_0\|_{L^2(\Omega)} + \|u_0\|_{H^1(\Omega)}\right), \tag{36}$$

$$\left\|\tfrac{\partial(v(t,\cdot) - e^{\lambda t}u(\cdot))}{\partial\nu}\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leq C_t\left(\|v_0\|_{L^2(\Omega)} + \|u_0\|_{H^1(\Omega)}\right). \tag{37}$$

*Proof.* Let $\phi$ be the solution of

$$\begin{cases} \mathcal{A}\phi = 0 & \text{on} \quad \Omega, \\ \phi = u_0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then $\|\phi\|_{H^1(\Omega)} \le C\|u_0\|_{H^{1/2}(\partial\Omega)} \le C\|u_0\|_{H^1(\Omega)}$ and $u - \phi$ satisfies

$$\begin{cases} (A+\lambda)(u-\phi) = -\lambda\phi & \text{on} \quad \Omega, \\ u-\phi = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{38}$$

Since $A_D$ is positive definite we have

$$\|u-\phi\|_{L^2(\Omega)} \le C\lambda\|(A_D+\lambda)^{-1}\phi\|_{L^2(\Omega)} \le C\|\phi\|_{L^2(\Omega)} \le C\|u_0\|_{H^1(\Omega)}, \tag{39}$$

where $C$ is independent of $\lambda > 0$. Furthermore to multiply (38) by $u - \phi$, then to integrate by parts shows that

$$\|\nabla(u-\phi)\|_{L^2(\Omega)} \le C\lambda\|u_0\|_{H^1(\Omega)}.$$

Hence we get (34).

Letting $w = v(t) - e^{\lambda t}u$, we have

$$\begin{cases} \partial_t w + \mathcal{A}w = 0 & \text{in} \quad (0,T)\times\Omega, \\ w(t) = 0 & \text{on} \quad (0,T)\times\partial\Omega, \\ w(0) = v_0 - u & \text{on} \quad \Omega. \end{cases}$$

Thus, $w(t) = e^{-tA_D}(v_0 - u)$, where $A_D$ denotes the self-adjoint operator $\mathcal{A}$ with the homogeneous Dirichlet boundary condition. Therefore, for any $t > 0$ and $m \ge 0$, $w(t) \in D((A_D)^m)$. Since $D((A_D)^{\frac{1}{2}}) = H_0^1(\Omega)$ and $D(A_D) \subset \{f \in H^1(\Omega),\ \Delta f \in L^2(\Omega \setminus \Omega_1)\}$ with continuous embedding, then for $m = 0, 1$ we have

$$\|w(t)\|_{H^1(\Omega)} \le C_{t1}\|v_0 - u\|_{L^2(\Omega)}, \tag{40}$$

$$\|\Delta w(t)\|_{L^2(\Omega\setminus\Omega_1)} \le C_{t2}\|v_0 - u\|_{L^2(\Omega)}. \tag{41}$$

By (40) and (41) we have

$$\|\frac{\partial w(t)}{\partial\nu}\|_{H^{-\frac{1}{2}}(\partial\Omega)} \le C_t\|v_0 - u\|_{L^2(\Omega)}. \tag{42}$$

From (40), (41), (42), (39), we get (35), (35), (36). □

4.3. **Proof of Theorems 1.1 and 1.2.** For the sake of simplicity, we restrict the proof to the case $\gamma_1(x) > \gamma_0(x)$ in $\Omega_1$. Let $y(x)$ be defined by (3) for $d = 1$, and by (9) for $d = 2, 3$. Letting $y^*$ be defined by (7), we put

$$y_1(x) = \begin{cases} y(x) & (d=1), \\ y(x)\cdot y^* & (d=2,3), \end{cases}$$

and for $\mathcal{O} \subset \overline{\Omega}$

$$a(\mathcal{O}) = \begin{cases} -\infty, & \text{if} \quad \mathcal{O} = \emptyset \\ \sup\{-y_1(x)\,;\, x \in \mathcal{O}\}, & \text{if} \quad \mathcal{O} \ne \emptyset. \end{cases}$$

Notice that $a(\mathcal{O}) = a_B(\mathcal{O})$ for $d = 2, 3$ by (15) and (24). We give some properties of $a()$ in the following

**Lemma 4.2.** *(1) We have, for all $A, B \subset \overline{\Omega}$*

$$a(A \cup B) = \max\Big(a(A), a(B)\Big), \tag{43}$$

$$a(A) \le a(B) \text{ if } A \subset B. \tag{44}$$

*(2) We have*

$$a(\{\chi \neq 1\}) < 0. \tag{45}$$

*(3) We have*

$$a(\{\chi \neq 0\} \cap \Omega_1) > a(\{\chi \neq 1\}) \Longleftrightarrow a(\Omega_1) > a(\{\chi \neq 1\}). \tag{46}$$

*Moreover if* $a(\Omega_1) > a(\{\chi \neq 1\})$ *then*

$$a(\{\chi \neq 0\} \cap \Omega_1) = a(\Omega_1). \tag{47}$$

*Proof.* (1) Obvious.

(2) Since $\Omega \setminus B$ is a neighbourhood of $\{\chi \neq 1\}$ we get (45).

(3) By (44) we have $a(\Omega_1) \geq a(\{\chi \neq 0\} \cap \Omega_1)$. This proves the straight implication in (46). Since $\{\chi = 0\} \cap \Omega_1 \subset \{\chi = 0\} \subset \{\chi \neq 1\}$ then by (44) we have

$$a(\{\chi = 0\} \cap \Omega_1) \quad \leq \quad a(\{\chi \neq 1\}).$$

So if $a(\{\chi \neq 0\} \cap \Omega_1) \leq a(\{\chi \neq 1\})$, then by (43) we have $a(\Omega_1) \leq a(\{\chi = 0\} \cap \Omega_1)$. Hence we get (46). Moreover we see that if $a(\{\chi \neq 0\} \cap \Omega_1) > a(\{\chi \neq 1\})$, then $a(\{\chi \neq 0\} \cap \Omega_1) > a(\{\chi = 0\} \cap \Omega_1)$ and, by (43),

$$a(\Omega_1) \quad = \quad \max\Big(a(\{\chi \neq 0\} \cap \Omega_1), a(\{\chi = 0\} \cap \Omega_1)\Big) = a(\{\chi \neq 0\} \cap \Omega_1).$$

Hence we get (47). □

We put

$$\kappa = \begin{cases} 2\sqrt{\lambda}, & \text{if} \quad d = 1, \\ \sqrt{2}\mu\lambda, & \text{if} \quad d = 2, 3. \end{cases}$$

Let $\varphi_{\lambda,\mu}(x)$ be as in Theorem 1.2 or Theorem 3.1 and $\tilde{\varphi}_\lambda(x)$ be the solution of (30) with $h = \chi h_\lambda$. Thanks to (22), (11), (12) and (13) we have

$$Ce^{-\kappa y_1(x)} \quad \leq \quad |\varphi_{\lambda,\mu}(x)|^2 \leq C^{-1} e^{-\kappa y_1(x)}, \tag{48}$$

$$C\kappa^2 e^{-\kappa y_1(x)} \quad \leq \quad |\nabla\varphi_{\lambda,\mu}(x)|^2 \leq C^{-1}\kappa^2 e^{-\kappa y_1(x)}. \tag{49}$$

Furthermore, thanks to (48), (4), (10), we have

$$|\Delta\varphi_{\lambda,\mu}(x)|^2 \quad \leq \quad C^{-1}\lambda^2 e^{-\kappa y_1(x)}. \tag{50}$$

**Remark 1.** Inequalities (49), (50) and the second one in (48) hold with $\varphi_\lambda$ replaced by $\Re e(\varphi_\lambda)$. Hence all the following calculations are valid with $\varphi_\lambda$ replaced by $\Re e(\varphi_\lambda)$ and $\tilde{\varphi}_\lambda$ replaced by $\Re e(\tilde{\varphi}_\lambda)$.

We compare $\varphi_\lambda$ and $\tilde{\varphi}_\lambda$ in the following

**Lemma 4.3.** *There exists a positive constant* $C$ *such that for all* $\kappa > 1$ *the following inequalities hold*

$$\|\tilde{\varphi}_\lambda - \chi\varphi_\lambda\|^2_{H^m(\Omega)} \quad \leq \quad C\kappa^{\frac{m}{2}} e^{\kappa a(\{\chi\neq 1\})}, \quad m = 0, 1, \tag{51}$$

$$\|\Delta(\tilde{\varphi}_\lambda - \chi\varphi_\lambda)\|^2_{L^2(\Omega)} \quad \leq \quad C\kappa^2 e^{\kappa a(\{\chi\neq 1\})}, \tag{52}$$

$$\|\frac{\partial(\tilde{\varphi}_\lambda - \chi\varphi_\lambda)}{\partial\nu}\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \quad \leq \quad C\kappa^2 e^{\kappa a(\{\chi\neq 1\})}, \tag{53}$$

$$\|\tilde{\varphi}_\lambda - \varphi_\lambda\|^2_{H^m(\Omega)} \quad \leq \quad C\kappa^{\frac{m}{2}} e^{\kappa a(\{\chi\neq 1\})}, \quad m = 0, 1, \tag{54}$$

$$\|\nabla\tilde{\varphi}_\lambda - \chi\nabla\varphi_\lambda\|^2_{L^2(\Omega)} \quad \leq \quad C\kappa^2 e^{\kappa a(\{\chi\neq 1\})}, \tag{55}$$

$$\|\frac{\partial(\tilde{\varphi}_\lambda - \varphi_\lambda)}{\partial\nu}\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} \quad \leq \quad C\lambda^2 e^{\kappa a(\{\chi\neq 1\})}. \tag{56}$$

*Proof.* Set $\Phi = \tilde{\varphi}_\lambda - \chi\varphi_\lambda$. It satisfies

$$(57)\qquad \begin{aligned} (A_0+\lambda)\Phi &= F := -(\mathcal{A}_0\chi)\varphi_\lambda + 2\gamma_0\nabla\chi\nabla\varphi_\lambda \quad \text{in } \Omega,\\ \Phi &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Multiplying (57) by $\Phi$ and integrating by parts give

$$\int_\Omega (\gamma_0|\nabla\Phi|^2 + \lambda|\Phi|^2) \le \|F\|_{L^2(\Omega)}\|\Phi\|_{L^2(\Omega)}.$$

In view of (48), (49), and since $\{\chi = 1\} \subset \{\nabla\chi = 0\} \cap \{\Delta\chi = 0\}$, we then have for all $\kappa \ge 1$:

$$(58)\qquad \begin{aligned} \|\Phi\|^2_{L^2(\Omega)} &\le \lambda^{-2}\|F\|^2_{L^2(\Omega)} \le C\frac{1}{\lambda^2}\|\nabla\varphi_\lambda\|^2_{L^\infty(\{\chi\neq 1\})}\\ &\le C\frac{\kappa^2}{\lambda^2}e^{\kappa a(\{\chi\neq 1\})} \le Ce^{\kappa a(\{\chi\neq 1\})},\\ \|\nabla\Phi\|^2_{L^2(\Omega)} &\le \lambda^{-1}\|F\|^2_{L^2(\Omega)} \le C\kappa e^{\kappa a(\{\chi\neq 1\})}. \end{aligned}$$

Hence we get (51).

By (57) we have $\Delta\Phi = \gamma_0^{-1}\Big(\lambda\Phi - \nabla\gamma_0\nabla\Phi - F\Big)$ and so, by (58), we get (52).

Estimates (51), (52) imply (53).

Writing $\chi\nabla\varphi_\lambda = \nabla(\chi\varphi_\lambda) - \varphi_\lambda\nabla\chi$ and similarly for $\chi\Delta\varphi_\lambda$, in view of (48), (49) and (51), (52), we get (54), (55).

By (53) we have

$$\|\frac{\partial(\tilde{\varphi}_\lambda - \varphi_\lambda)}{\partial\nu}\|^2_{H^{-\frac12}(\partial\Omega)} \le C\kappa^2 e^{\kappa a(\{\chi\neq 1\})} + \|\frac{\partial((\chi-1)\varphi_\lambda)}{\partial\nu}\|^2_{H^{-\frac12}(\partial\Omega)}.$$

Moreover by (49), (50) we have

$$\begin{aligned} &\|\nabla((\chi-1)\varphi_\lambda)\|^2_{L^2(\Omega)} + \|\Delta((\chi-1)\varphi_\lambda)\|^2_{L^2(\Omega)} \le\\ &\qquad C\|\nabla\varphi_\lambda\|^2_{L^2(\{\chi\neq 1\})} + C\|\Delta\varphi_\lambda\|^2_{L^2(\{\chi\neq 1\})} \quad \le C\lambda^2 e^{\kappa a(\{\chi\neq 1\})}. \end{aligned}$$

Hence we get (56). □

We then put

$$J(\mathcal{O},\kappa) = \int_{\mathcal{O}} e^{-\kappa\, y_1(x)}dx,$$

$$J_\gamma(\kappa) = \int_\Omega (\gamma-\gamma_0)|\nabla\varphi_{\lambda,\mu}(x)|^2 dx,$$

$$J_{\gamma,\chi}(\kappa) = \int_\Omega (\gamma-\gamma_0)\chi^2|\nabla\varphi_{\lambda,\mu}(x)|^2 dx,$$

$$\tilde{J}_\gamma(\kappa) = \int_\Omega (\gamma-\gamma_0)|\nabla\tilde{\varphi}_\lambda(x)|^2 dx.$$

**Lemma 4.4.** *(1) For a non-empty bounded open set $\mathcal{O} \subset \Omega$, we have*

$$(59)\qquad \lim_{\kappa\to\infty} \kappa^{-1}\log J(\mathcal{O},\kappa) = a(\mathcal{O}).$$

*(2) We have*

$$(60)\qquad \lim_{\kappa\to\infty} \kappa^{-1}\log J_\gamma(\kappa) = a(\Omega_1).$$

*(3) We have*

$$(61)\qquad \lim_{\kappa\to\infty} \kappa^{-1}\log J_{\gamma,\chi}(\kappa) = a(\Omega_1 \cap \{\chi \neq 0\}).$$

*(4) We have*

$$\limsup_{\kappa\to\infty} \kappa^{-1} \log \tilde{J}_\gamma(\kappa) \le \max(a(\{\chi \neq 0\} \cap \Omega_1), a(\{\chi \neq 1\})). \tag{62}$$

*Moreover if* $a(\{\chi \neq 0\} \cap \Omega_1) > a(\{\chi \neq 1\})$ *then*

$$\lim_{\kappa\to\infty} \kappa^{-1} \log \tilde{J}_\gamma(\kappa) = a(\Omega_1). \tag{63}$$

*Proof.* (1) - Letting $\epsilon > 0$, $\mathcal{O}_\epsilon = \{x \in \mathcal{O}\,; a(\mathcal{O}) - \epsilon < -y_1(x) < a(\mathcal{O})\}$, we have

$$\int_{\mathcal{O}_\epsilon} e^{\kappa(a(\mathcal{O})-\epsilon)} dx \le \int_{\mathcal{O}} e^{-\kappa y_1(x)} dx \le \int_{\mathcal{O}} e^{\kappa a(\mathcal{O})} dx.$$

Since $y_1$ is continuous and $\mathcal{O}$ is open then the open set $\mathcal{O}_\epsilon$ is non empty and its Lebesgue measure is positive. Then, taking the logarithm in the above inequalities, and letting $\kappa \to \infty$, we get, for all $\epsilon > 0$,

$$a(\mathcal{O}) - \epsilon \le \liminf_{\kappa\to\infty} \kappa^{-1} \log J(\mathcal{O}, \kappa) \le \limsup_{\kappa\to\infty} \kappa^{-1} \log J(\mathcal{O}, \kappa) \le a(\mathcal{O}),$$

and we deduce (59).

(2)- Let $\{\mathcal{O}_n\}_{n\ge 0}$ be a non-decreasing sequence of non-empty open sets such that $\overline{\mathcal{O}_n} \subset \Omega_1$ for all $n$ and $\cup_n \mathcal{O}_n = \Omega_1$. In view of assumption **(A)** (see page 3) and (49), for all $n$ we have $\inf_{x\in\overline{\mathcal{O}_n}}\{\gamma(x) - \gamma_0(x)\} > 0$ and

$$C_n \kappa^2 J(\mathcal{O}_n, \kappa) \le J_\gamma(\kappa) \le C_n^{-1} \kappa^2 J(\Omega_1, \kappa),$$

for some $C_n > 0$. By (59) the two last inequalities imply

$$a(\mathcal{O}_n) \le \liminf_{\kappa\to\infty} \kappa^{-1} \log J_\gamma(\kappa) \le \limsup_{\kappa\to\infty} \kappa^{-1} \log J_\gamma(\kappa) \le a(\Omega_1),\ \forall n.$$

Obviously we have $a(\mathcal{O}_n) \to a(\Omega_1)$ as $n \to \infty$ and so we get (60).

(3)- Since the function $\chi(\gamma(x) - \gamma_0(x))$ satisfies the condition **(A)** with $\Omega_1$ replaced by $\Omega_1 \cap \{\chi \neq 0\}$, we get similarly (61).

(4)- Thanks to the well-known inequalities $\frac{1}{2}\beta^2 - (\alpha-\beta)^2 \le \alpha^2 \le 2\beta^2 + 2(\alpha-\beta)^2$ we have

$$\tilde{J}_\gamma(\kappa) \ge \frac{1}{2} J_{\gamma,\chi}(\kappa) - \int_\Omega (\gamma - \gamma_0)|\nabla\tilde{\varphi}_\lambda(x) - \chi\nabla\varphi_\lambda(x)|^2 dx, \tag{64}$$

$$\tilde{J}_\gamma(\kappa) \le 2J_{\gamma,\chi}(\kappa) + 2\int_\Omega (\gamma - \gamma_0)|\nabla\tilde{\varphi}_\lambda(x) - \chi\nabla\varphi_\lambda(x)|^2 dx. \tag{65}$$

Hence and in view of (55), (61) and (47) we get (62) and (63). □

**Remark 2.** For $d \ge 2$ Estimate (60) also holds with $\varphi_\lambda$ replaced by $\Re e(\varphi_\lambda)$. See Remark 1.

We put

$$h_\lambda(x) = \varphi_{\lambda,\mu}(x)\Big|_{\partial\Omega}.$$

By (33), (60), (61),(62) and (63), we have

**Lemma 4.5.** *Let* $J_S(h)$ *be defined by (32). Then we have*

$$\lim_{\kappa\to\infty} \kappa^{-1} \log J_S(h_\lambda) = a(\Omega_1).$$

*If* $a(\{\chi \neq 0\} \cap \Omega_1) \le a(\{\chi \neq 1\})$ *then*

$$\limsup_{\kappa\to\infty} \kappa^{-1} \log J_S(\chi h_\lambda) \le a(\{\chi \neq 1\}) < 0. \tag{66}$$

*If* $a(\{\chi \neq 0\} \cap \Omega_1) > a(\{\chi \neq 1\})$ *then*

$$\lim_{\kappa \to \infty} \kappa^{-1} \log J_S(\chi h_\lambda) = a(\Omega_1). \tag{67}$$

For $d \geq 2$ the following lemma explains the choice of $\mu_B$ and the conditions (17), (18).

**Lemma 4.6.** *For* $d \geq 2$ *we have*

$$\frac{1}{\sqrt{2}\mu_B} > |a_B(\Omega_1)|, \tag{68}$$

$$a_B(\Omega_1) - a_B(\Omega) + \frac{T_1}{\sqrt{2}\mu} > 0, \tag{69}$$

$$0 < \mu < T_1 \mu_B \leq \frac{T_1}{\sqrt{2}\,(a_B(\Omega) - a_B(\Omega_1))}, \tag{70}$$

$$\frac{1}{\sqrt{2}\mu}(T_1 - T_0) - \frac{1}{2} a_B(\Omega) + a_B(\Omega_1) > 0. \tag{71}$$

*Proof.* If $\Omega \subset B$ then $0 < a_B(\Omega_1) < a_B(\Omega)$ and so

$$\frac{1}{\sqrt{2}\mu_B} = a_B(\Omega) > a_B(\Omega_1) > 0. \tag{72}$$

Hence (68).

If $\Omega \not\subset B$ observe that $0 < \sup_{x \in \Omega \setminus B} y_1(x)$ and $0 \geq \sup_{x \in \Omega \cap B} y_1(x)$ and so

$$\sup_{x \in \Omega \setminus B} y_1(x) = \sup_{x \in \Omega} y_1(x) \geq \inf_{x \in \Omega_1} y_1(x) = -a_B(\Omega_1). \tag{73}$$

Hence, by (16), we have

$$\frac{1}{\sqrt{2}\mu_B} \quad = \quad a_B(\Omega) + \sup_{x \in \Omega \setminus B} y_1(x) > a_B(\Omega) > a_B(\Omega_1), \tag{74}$$

$$\frac{1}{\sqrt{2}\mu_B} \quad \geq \quad a_B(\Omega) - a_B(\Omega_1) > -a_B(\Omega_1). \tag{75}$$

From (74), (75) we get (68) also in this case.

From (18) we have $\mu < T_1 \mu_B$ and then from (16) and (75) we obtain

$$0 < \frac{\mu}{T_1} < \mu_B \leq (\sqrt{2}(a_B(\Omega) - a_B(\Omega_1)))^{-1},$$

Hence (69) and (70). From (18) we have

$$\frac{1}{\sqrt{2}\mu}(T_0 - T_1) < -\frac{1}{2} a_B(\Omega) - \frac{1}{\sqrt{2}\mu_B},$$

and so, thanks to (68), we obtain (71). □

The following lemma gives a relation between the time-dependent measurement $I(\lambda)$ defined by (5) or (20) and the stationary one $J_S(h_\lambda)$. For $d = 1$ set $\chi(x) = 1$ if $x < a_2$, $\chi(x) = 0$ if $x > a_3$ with $a_1 < a_2 < a_3 < b$: hence we are in the case $a(\{\chi \neq 0\} \cap \Omega_1) > a(\{\chi \neq 1\})$.

**Lemma 4.7.** *Assume the conditions of Theorem 1.1 if* $d = 1$ *and of Theorem 1.2 if* $d \geq 2$. *Then there exists* $\delta > 0$ *such that the following estimates hold: If* $a(\{\chi \neq 0\} \cap \Omega_1) \leq a(\{\chi \neq 1\})$.

$$|I(\lambda) - J_S(\chi h_\lambda)| \leq C \kappa \lambda e^{\kappa a(\{\chi \neq 1\})}, \quad \forall \lambda > 1/\delta, \tag{76}$$

*If* $a(\{\chi \neq 0\} \cap \Omega_1) > a(\{\chi \neq 1\})$

$$\left| \frac{I(\lambda) - J_S(\chi h_\lambda)}{J_S(\chi h_\lambda)} \right| \leq e^{-\delta\kappa}, \quad \forall \lambda > 1/\delta. \tag{77}$$

*Proof.* Let $f_\lambda(t,x) = e^{\lambda t}\varphi_{\lambda,\mu}(x)$, and $v(t,x)$ (respectively, $\tilde{v}(t,x)$) the solution of (1) with $f = f_\lambda$ (respectively, with $f = \chi f_\lambda$). Let $u_\lambda$ (respectively, $\tilde{u}(t,x)$) be the solution of (31) with $h = h_\lambda$ (respectively, with $h = \chi h_\lambda$). Let $\tilde{\varphi}_\lambda(x)$ be the solution of (30) with $h = \chi h_\lambda$. Letting $w = \tilde{v} - e^{\lambda t}\tilde{u}_\lambda$, we then have

$$I(\lambda) = J_S(\chi h_\lambda) + R_\Gamma(\lambda) + R_S(\lambda),$$

$$\begin{aligned} R_\Gamma(\lambda) &= e^{-\lambda T_1} \int_\Gamma \gamma \frac{\partial w(T_1,\cdot)}{\partial \nu}\, \chi \overline{h_\lambda} d\sigma, \\ R_S(\lambda) &= \int_{\partial\Omega} \gamma_0 \frac{\partial}{\partial \nu}(\tilde{\varphi}_\lambda - \varphi_\lambda)\, \chi \overline{h_\lambda} d\sigma. \end{aligned}$$

Then, thanks to Lemma 4.1, we have:

$$\begin{aligned} |R_\Gamma| &\leq C e^{-\lambda T_1} \| \frac{\partial w(T_1,\cdot)}{\partial \nu} \|_{H^{1/2}(\partial\Omega)} \, \|\chi h_\lambda\|_{H^{1/2}(\partial\Omega)} \\ &\leq C e^{-\lambda T_1} \Big( \|v_0\|_{L^2(\Omega)} + \|\tilde{u}_\lambda\|_{H^1(\Omega)} \Big) \|\varphi_\lambda\|_{H^1(\Omega)}. \end{aligned}$$

By (C-0) we have $\|v_0\|_{L^2(\Omega)} \leq e^{\lambda T_0}$, by (38) $\|\tilde{u}_\lambda\|_{H^1(\Omega)} \leq C\|\tilde{\varphi}_\lambda\|_{H^1(\Omega)}$, by (54) $\|\tilde{\varphi}_\lambda\|_{H^1(\Omega)} \leq \|\varphi_\lambda\|_{H^1(\Omega)} + C\kappa^{\frac{3}{4}} e^{\frac{1}{2}\kappa a(\{\chi\neq 1\})}$, by (49) $\|\varphi_\lambda\|_{H^1(\Omega)} \leq C\kappa e^{\frac{1}{2}\kappa a(\Omega)}$. Hence we have

$$\begin{aligned} |R_\Gamma| &\leq C\kappa \Big( e^{\kappa(\frac{\lambda}{\kappa}(T_0 - T_1) + \frac{1}{2}a(\Omega))} + \kappa e^{\kappa(-\frac{\lambda}{\kappa}T_1 + a(\Omega))} + \\ &\qquad \kappa^{\frac{3}{4}} e^{\kappa(-\frac{\lambda}{\kappa}T_1 + \frac{1}{2}a(\Omega) + \frac{1}{2}a(\{\chi\neq 1\}))} \Big) \\ &\leq C\kappa \Big( e^{\kappa(\frac{\lambda}{\kappa}(T_0 - T_1) + \frac{1}{2}a(\Omega))} + \kappa e^{\kappa(-\frac{\lambda}{\kappa}T_1 + a(\Omega))} \Big). \end{aligned} \tag{78}$$

If $d = 1$ then $\frac{\lambda}{\kappa} = \frac{\sqrt{\lambda}}{2}$ tends to infinity and so, since $T_0 < T_1$, for all $\epsilon > 0$, there exists $\lambda_0 \geq 1$ such that, for all $\lambda > \lambda_0$,

$$\max\Big( \frac{\lambda}{\kappa}(T_0 - T_1) + \frac{1}{2}a(\Omega), -\frac{\lambda}{\kappa}T_1 + a(\Omega) \Big) < a(\Omega_1) - \epsilon. \tag{79}$$

If $d \geq 2$ then take $\epsilon > 0$ so that

$$\epsilon \in (0, \min\Big( a(\Omega_1) - a(\Omega) + \frac{T_1}{\sqrt{2}\mu}, a(\Omega_1) - \frac{1}{2}a(\Omega) + \frac{T_1 - T_0}{\sqrt{2}\mu} \Big)). \tag{80}$$

This is possible thanks to (69), (71). So (79) holds also in this case. By (78) and (79), we get

$$|R_\Gamma| \leq C e^{\kappa(a(\Omega_1) - \epsilon)}, \ \forall \kappa > \kappa_0. \tag{81}$$

Let us estimate $R_S$. If $\chi|_\Omega = 1$ then $\tilde{\varphi}_\lambda = \varphi_\lambda$ and $R_S = 0$. Assume that $\chi|_\Omega \neq 1$ and $\Gamma \neq \partial\Omega$. We have

$$\begin{aligned} R_S &= \int_{\partial\Omega} \gamma_0 \frac{\partial(\tilde{\varphi}_\lambda - \varphi_\lambda)}{\partial \nu} (\chi - 1)\overline{h_\lambda} d\sigma + \int_{\partial\Omega} \gamma_0 \frac{\partial(\tilde{\varphi}_\lambda - \varphi_\lambda)}{\partial \nu} \overline{h_\lambda} d\sigma \\ &\equiv R_{S1} + R_{S2}. \end{aligned}$$

By (56), (48), (49) we have

$$\begin{aligned}(82) \qquad |R_{S1}| &\leq C\|\frac{\partial(\tilde{\varphi}_\lambda - \varphi_\lambda)}{\partial\nu}\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|(\chi-1)\varphi_\lambda\|_{H^1(\Omega)} \\ &\leq C\kappa\lambda e^{\kappa a(\{\chi\neq 1\})}.\end{aligned}$$

By Integrations by parts and by (48), (49) we have

$$\begin{aligned}|R_{S2}| &= \Big|\int_{\partial\Omega}\gamma_0(\chi-1)h_\lambda\frac{\partial\overline{\varphi_\lambda}}{\partial\nu}\,d\sigma\Big| \\ &\leq C\|\varphi_\lambda\|_{L^\infty(\{\chi\neq 1\})}\,\|\nabla\varphi_\lambda\|_{L^\infty(\{\chi\neq 1\})} \leq C\kappa e^{\kappa a(\{\chi\neq 1\})}.\end{aligned}$$

Hence we get

$$(83) \qquad |R_S| \leq C\kappa\lambda e^{\kappa a(\{\chi\neq 1\})},$$

and so, from (81), (83),

$$(84) \qquad |I(\lambda) - J_S(\chi h_\lambda)| \leq C\Big(e^{\kappa(a(\Omega_1)-\epsilon))} + \kappa\lambda e^{\kappa a(\{\chi\neq 1\})}\Big).$$

Consider the case $a(\{\chi\neq 0\}\cap\Omega_1) \leq a(\{\chi\neq 1\})$. By (46) we get (76).
Consider the case $a(\{\chi\neq 0\}\cap\Omega_1) > a(\{\chi\neq 1\})$. By (47), (67) we get (77). □

Now Theorems 1.1 and 1.2 follow from Lemma 4.5 and Lemma 4.7.

5. **Two-dimensional conformal map.** In this section, we identify $x = (x_1, x_2) \in \mathbb{R}^2$ with a complex number $z = x_1 + ix_2 \in \mathbb{C}$. Consider a univalent holomorphic function $F(z)$ defined in a neighborhood of $\Omega$, and the conformal map : $z \to w = y_1 + iy_2 = F(z)$. As in §3, we look for a solution of (10) in the form $\varphi_\lambda(x) = (1+\phi(y))e^{-\zeta\cdot F(x)}$, where $\phi$ satisfies (27). Note that $\zeta\cdot F(z)$ is identified with $\frac{\mu\lambda}{\sqrt{2}}F(z)$. Assume that $0 \notin \overline{\Omega}$ and let us choose $F(z) = z^{-n} - \rho^{-n}$ with $\rho > 0$ and $n = 1, 2, \ldots$. The case $n = 1$ corresponds to the inversion we have studied in §3, where $R$ corresponds to $2/\rho$.

Consider the curve $\operatorname{Re} F(z) = 0$ that separates the set of $x$ where $\varphi_\lambda(x)$ has an exponential growth from the set of $x$ where $\varphi_\lambda(x)$ has an exponential decay as $\lambda \to \infty$. Writing $z = re^{i\alpha}$, we have that $\operatorname{Re} F(z) = 0 \iff r = \rho(\cos(n\alpha))^{1/n}$. It appears that if $\Omega$ is convex, then $n = 3$ is a better choice than $n = 1, 2$ since the curve $\operatorname{Re} F(z) = 0$ goes more deeply inside $\Omega$ (cf. figure 3).

6. **Numerical tests in the one-dimensional case.** We proceed to test the one-dimensional case numerically. Assume $\gamma_0 = 1$, $a = 0$, $b = 1$, and $\Omega_1 = [a_1, b_1]$ with $0 < a_1 < b_1 < 1$. Then we have

$$\varphi_\lambda(x) = e^{-\sqrt{\lambda}(x-x_0)},$$

and so we have the functions $h_\lambda, f_\lambda$ and $I(\lambda)$ of theorem 1.1. By simple calculation the indicator function gets the form

$$(85) \qquad I(T_1,\lambda) = [e^{-\lambda T_1}(\Lambda_\gamma f_\lambda)(T_1,0) - \sqrt{\lambda}x_0 e^{\sqrt{\lambda}x_0}]e^{\sqrt{\lambda}x_0},$$

and the equation (6) gets the form

$$(86) \qquad a_1 = x_0 - \lim_{\lambda\to\infty}\frac{\log(\pm I(T_1,\lambda))}{2\sqrt{\lambda}}.$$

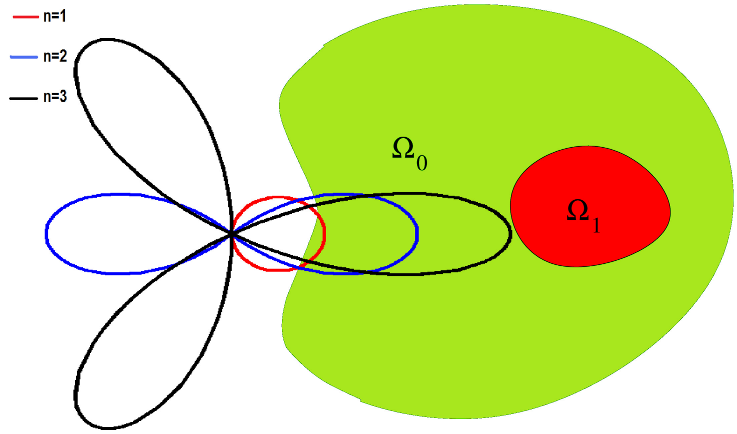


FIGURE 3.

The equation (86) is calculated with a finite $\lambda$, so we have the approximative reconstruction equation

$$a_1 \approx x_0 - \frac{\log(\pm I(T_1, \lambda))}{2\sqrt{\lambda}}, \tag{87}$$

which is the more accurate the larger the parameter $\lambda$ is. In practice large values of $\lambda$ imply very large values of $f_\lambda$ on the boundary possibly causing numerical errors. However for initial testing purposes we proceed by using $\lambda = 1.0, 1.5, 2.0, \ldots, 20.0$. Also we choose $x_0 = 1$ and $T = 1$.

Four test cases of the following heat conductivity are considered:

$$\begin{aligned} \gamma &= \gamma_1, \quad a_1 \le x \le b_1, \\ \gamma &= \gamma_0, \quad \text{otherwise}, \end{aligned}$$

where the values of $a_1, b_1$ and $\gamma_1$ are given in table 1. These test cases are pictured in figure 4. The solutions $v$ in (1) are calculated with FEM so that there are $N_x = 5000$ points of $x \in [0, 1]$ and $N_t = 100$ points of $t \in [0, 1]$. We consider three different initial temperature distributions, $v(x, 0) = 1, v(x, 0) = 1 + 2\sin(\pi x)$ and $v(x, 0) = 1 + 4|\sin(4\pi x)|$. In figure 5 we display the solution function $v(x, t)$, with $v(x, 0) = 1, \lambda = 2$, calculated by FEM for the test case 1. Smaller numbers of $N_x$ and $N_t$ are only used for printing purposes.

| | $a_1$ | $b_1$ | $\gamma_1$ |
|---|---|---|---|
| Case 1 | 0.1 | 0.3 | 0.5 |
| Case 2 | 0.1 | 0.3 | 2 |
| Case 3 | 0.4 | 0.6 | 0.5 |
| Case 4 | 0.4 | 0.6 | 2 |

TABLE 1. Parameters $a_1, b_1$ and $\gamma_1$ for the test cases

To simulate measurement noise we define the noisy measurement vector by

$$L_n = L \cdot (1 + c \cdot r),$$

where $L$ is the measurement vector

$$(\Lambda f_\lambda)(t,0) = \gamma(x)\frac{\partial v^{f_\lambda}}{\partial \nu}\Big|_{x=0} \tag{88}$$
$$= -\frac{\partial v^{f_\lambda}}{\partial x}\Big|_{x=0},$$

$r$ is a random gaussian vector of the same size, and $c$ is adjusted separately for each $\lambda$ so that the relative noise in terms of vector norm is approximately two and a half percent:

$$\frac{||L_n - L||}{||L||} \approx 0.025.$$

Our FEM-parameters $N_t = 100$ and $N_x = 5000$ imply $L \in \mathbb{R}^{100}$ and $v^{f_\lambda} \in \mathbb{R}^{5000}$, the differentiation in (88) is done with the Matlab-function `''pdeval.m''`. Also notice that evaluating the derivative at $x = 0$ means that we take the first component of the corresponding vector.

The logarithm of the indicator, $I(T_1, \lambda)$ defined by (85), is calculated for each $T_1 \in [0, 1]$ and $\lambda = 1.0, 1.5, 2.0, \ldots, 20.0$. This information can be presented in several ways. In figure 6 one can see how $I(T_1, \lambda)$, noisy and non-noisy, behaves as a function of $T_1$, for $\lambda = 1$ and $\lambda = 10$, in test case 1. We conclude that the value $\lambda = 1$ is too small to get accurate indicator values, and that $\lambda = 10$ provides good values throughout the time interval $[T/2, T]$. In figure 7 we have a similar graph for $\lambda = 10$, but for each initial temperature distribution. We conclude that the initial distribution $v(x, 0)$ does not change the indicator values in the interval $[T/2, T]$. In figure 8 one can see how $I(T_1, \lambda)$, noisy and non-noisy, behaves as a function of $\lambda$, for several choices of $T_1$, in test case 1. Note that in these graphs the analytical value based on (87) is also shown.

Based on the observation that the values of the indicator function are good between the time interval $[T/2, T]$ we use the following indicator value in reconstructions:

$$I^\star(\lambda) = \frac{1}{N_t^\star}\sum_{T_1=T/2}^{T} I(T_1, \lambda),$$

where $I(T_1, \lambda)$ is defined by (85) and $N_t^\star$ is the number of FEM -points $t \in [0, 1]$ between the time values $T/2$ and $T$. The value $I^\star(\lambda)$ is then inserted into (87), from which the reconstruction $a_1$ is calculated with $\lambda = 1.0, 1.5, 2.0, \ldots, 20.0$. In figure 9 the reconstructed $a_1$, noisy and non-noisy, is shown as a function of $\lambda$, for each test case.

## REFERENCES


[1] K. Astala and L. Päivärinta, *Calderón's inverse conductivity problem in the plane*, Ann. of Math, **163** (2006), 265–299.

[2] K. Bryan and L. F. Caudill Jr, *Stability and reconstruction for an inverse problem for the heat equation*, Inverse Problems, **14** (1998), 1429–1453.

[3] R. Chapko, R. Kress and J. R. Yoon, *On the numerical solution of an inverse boundary value problem for the heat equation*, Inverse Problems, **14** (1998), 853–867.

[4] T. P. Fredman, *A boundary identification method for an inverse heat conduction problem with an application in ironmaking*, Heat Mass Transfer, **41** (2004), 95–103.

[5] T. Ide, H. Isozaki, S. Nakata, S. Siltanen and G. Uhlmann, *Probing for electrical inclusions with complex spherical waves*, Comm. in Pure and Appl. Math., **60** (2007), 1415–1442.

[6] M. Ikehata, *Extracting discontinuity in a heat conductiong body. One-space dimensional case*, Appl. Anal., **86** (2007), 963–1005.

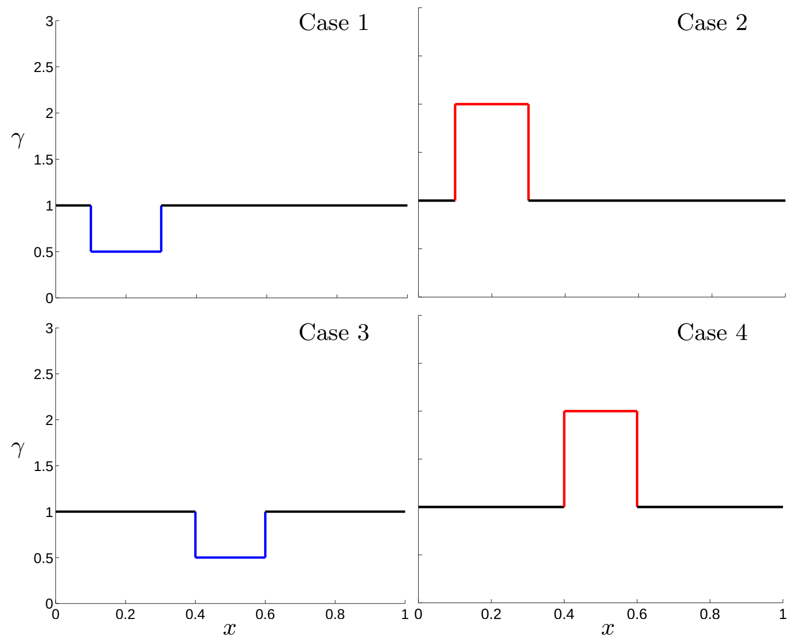


FIGURE 4. The test case heat conductivities

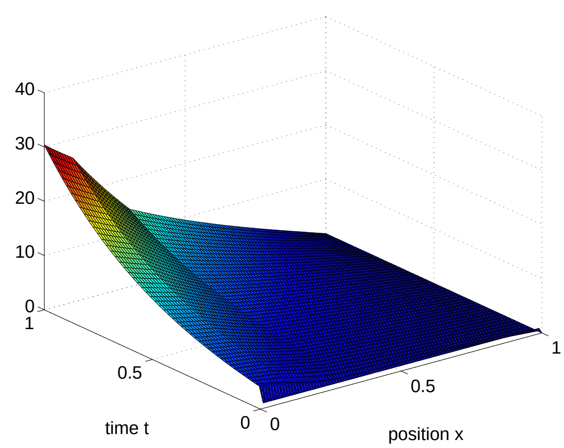


FIGURE 5. The solution function $v(x,t), \lambda = 2$ calculated by FEM for test case 1, $v(x,0) = 1$.

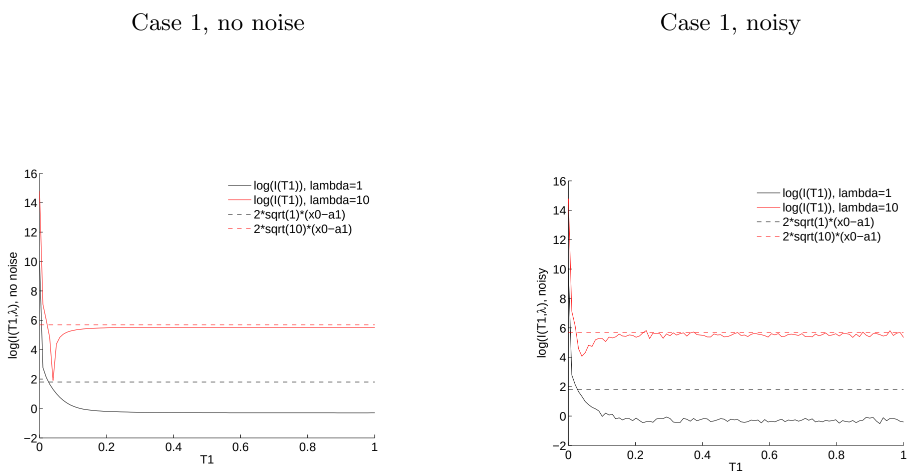


FIGURE 6. The logarithm of the indicator, non-noisy on the left and noisy on the right, as a function of $T_1$ for $\lambda = 1$ and $\lambda = 10$ in test case 1.

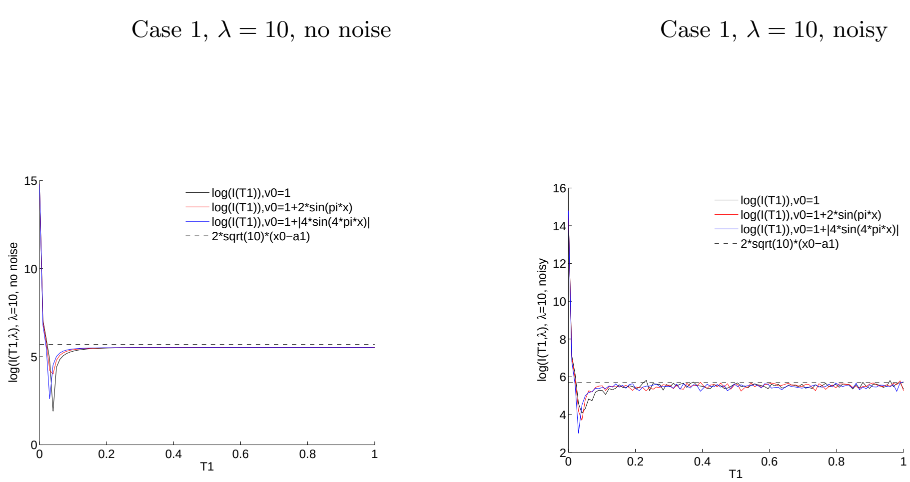


FIGURE 7. The logarithm of the indicator, non-noisy on the left and noisy on the right, as a function of $T_1$ for $\lambda = 10$ in test case 1, with three different initial data $v(x, 0) = 1, v(x, 0) = 1 + 2\sin(\pi x)$ and $v(x, 0) = 1 + 4|\sin(4\pi x)|$.


[7] M. Ikehata and M. Kawashita, *An inverse problem for a three-dimensional heat equation in thermal imaging and the enclosure method*, arXiv:1002.4457.

[8] M. Ikehata and M. Kawashita, *On the reconstruction of inclusions in a heat conductivity body from dynamical boundary data over a finite interval*, Inverse Problems, **26** (2010), 15pp.

[9] H. Kang, J. K. Seo and D. Sheen, *The inverse conductivity problem with one measurement: stability and estimation of size*, SIAM J. Math. Anal., **28** (1997), 1389–1405.

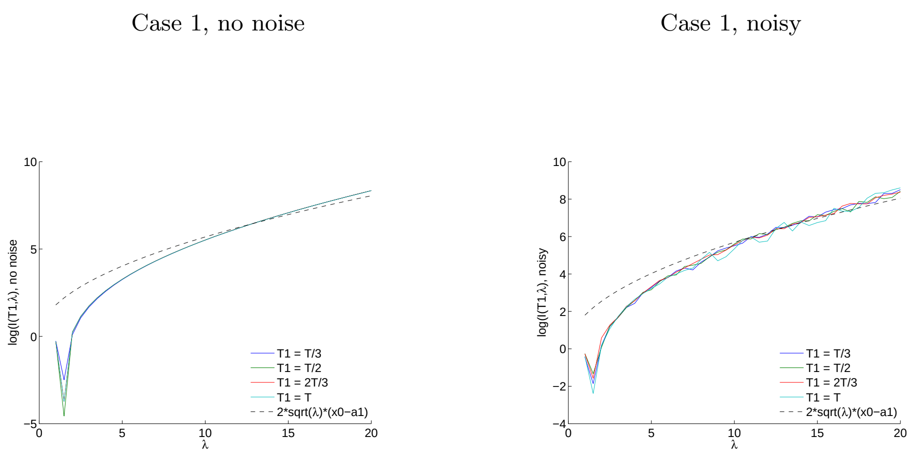


FIGURE 8. The logarithm of the indicator, non-noisy on the left and noisy on the right, as a function of $\lambda$ for several choices of $T_1$ in test case 1.


[10] N. S. Mera, *The method of fundamental solutions for the backward heat conduction problem*, Inverse Probl. Sci. Eng., **13** (2005), 65–78.

[11] J. Sylvester and G. Uhlmann, *A global uniquness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153–169.

[12] S. Vessella, *Quantitative estimates of unique continuation for parabolic equations, determination of time-varying boundaries and optimal stability estimates*, Topical Review, Inverse Problems, **24** (2008), 81pp.

[13] T. Wei and Y. S. Li, *An inverse boundary problem for one-dimensional heat equation with a multilayer domain*, Engineering Analysis with Boundary Elements, **33** (2009), 225–232.

[14] T. Wei and M. Yamamoto, *Reconstruction of a moving boundary from Cauchy data in one-dimensional heat equation*, Inverse Problems in Science and Engineering, **17** (2009), 551–567.

[15] T. Zhou, *Reconstructing electromagnetic obstacles by the enclosure method*, Inverse Problems and Imaging, **4** (2010), 547–569.


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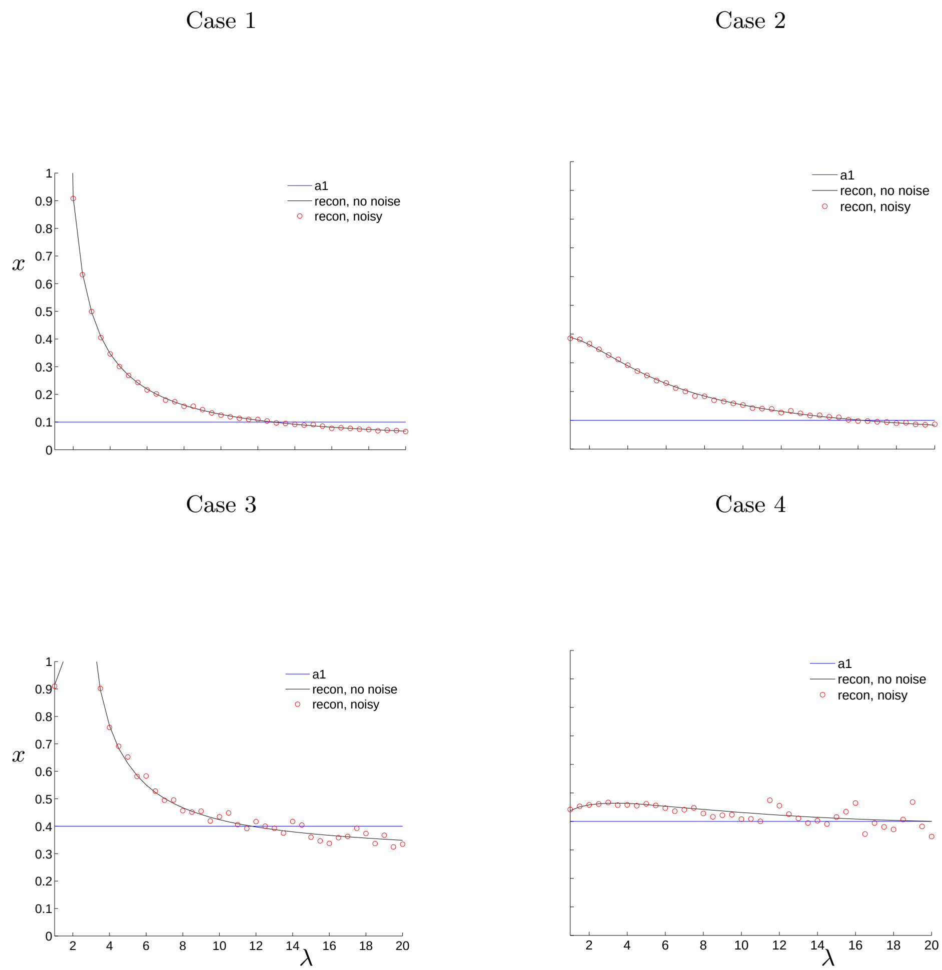


Figure 9. The reconstructed $a_1$, noisy and non-noisy, as a function of $\lambda$.