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Mapping Properties of the Nonlinear Fourier Transform in Dimension Two

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A class of compactly supported Schrödinger potentials in dimension two is given for which the inverse scattering method related to the Novikov–Veselov evolution equation is well-defined. There is no smallness assumption on the initial potential. Regularity results are proven for the direct and inverse scattering transforms, also called nonlinear Fourier transforms.

Keywords Complex geometrical optics; Evolution equation; Exponentially growing solution; Fourier transform; Inverse scattering; Novikov–Veselov equation; Schrödinger equation.

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1. Introduction

Let $q \in L^p(\mathbb{R}^2)$ for some $1 < p < 2$ and consider the Schrödinger equation

$$(-\Delta + q)\psi(\cdot, k) = 0 \tag{1.1}$$

where $k \in \mathbb{C}\backslash 0$ is a parameter. As in Faddeev (1966) and Nachman (1996), we look for *exponentially growing solutions* ψ of (1.1) with asymptotic behavior $\psi(x, k) \sim e^{ikx}$. More precisely, the solution ψ is characterized by

$$e^{-ikx}\psi(x, k) - 1 \in L^{\tilde{p}} \cap L^\infty(\mathbb{R}^2) \quad \text{for fixed } k \in \mathbb{C}\backslash 0, \tag{1.2}$$

where $1/\tilde{p} = 1/p - 1/2$. Here and throughout the paper a point $x = (x_1, x_2)$ in $\mathbb{R}^2$ will be identified with $x = x_1 + ix_2 \in \mathbb{C}$. So $\exp(ikx) = \exp(i(k_1 + ik_2)(x_1 + ix_2))$ with $k \in \mathbb{C}$ and $x \in \mathbb{R}^2$.

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Exponentially growing solutions do not necessarily exist for all $k \in \mathbb{C}$. A point k is called a *non-exceptional point* of q if there is a unique solution of (1.1) satisfying (1.2). Otherwise k is called an *exceptional point* of q. If a potential q does not have exceptional points, one can define the scattering map $\mathcal{T} : q \mapsto \mathbf{t}$, taking the potential q to its *scattering transform* $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ defined by

$$\mathbf{t}(k) = \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x, k)dx. \tag{1.3}$$

Under suitable assumptions the potential q can be recovered from its scattering transform $\mathbf{t}$ via the inverse scattering map $\mathcal{Q} : \mathbf{t} \mapsto q$ defined by

$$(\mathcal{Q}\mathbf{t})(x) := \frac{i}{\pi^2}\bar{\partial}_x \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e_{-k}(x)\overline{\mu(x, k)}dk, \tag{1.4}$$

where $x \in \mathbb{R}^2$, dk denotes Lebesgue measure, $\bar{\partial}_x = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$ and $e_{-k}(x) = e^{-i(kx+\bar{k}\bar{x})}$. The functions $\mu(x, k)$ in (1.4) are determined by solving the $\bar{\partial}$ equation

$$\bar{\partial}_k \mu(x, k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}} e_{-k}(x)\overline{\mu(x, k)} \tag{1.5}$$

with fixed $x \in \mathbb{R}^2$ and assuming large $|k|$ asymptotics $\mu(x, \cdot) - 1 \in L^\infty \cap L^r(\mathbb{C})$ for some $2 < r < \infty$. We remark that the maps $\mathcal{T}$ and $\mathcal{Q}$ are often called the direct and inverse *nonlinear Fourier transforms*. See the work of Beals and Coifman (1980, 1985, 1986, 1989), Ablowitz and Nachman (1986), Nachman and Ablowitz (1984), and Henkin and Novikov (1988) for early references on the $\bar{\partial}$ method. Note also that a formula equivalent to (1.4) is given by Boiti et al. (1987, formula (4.10)).

Let us define a special class of potentials.

Definition 1.1. A compactly supported potential $q \in C_0^\infty(\mathbb{R}^2)$ is *of conductivity type* if $q = \gamma^{-1/2}\Delta\gamma^{1/2}$ for some real-valued $\gamma \in C^2(\mathbb{R}^2)$ satisfying $\gamma(x) \geq c > 0$ for all $x \in \Omega$ and $\gamma(x) \equiv 1$ for all $x \in \mathbb{R}^2 \backslash \Omega$.

Nachman (1996) shows in that $\mathcal{T}q$ is well-defined for conductivity type potentials. We extend his result in Theorems 2.1 and 2.2 by proving that the apparently singular functions $k^{-1}(\mathcal{T}q)(k)$ and $\bar{k}^{-1}(\mathcal{T}q)(k)$ belong to the Schwartz class $\mathcal{S}(\mathbb{C})$. The crucial point of the proof is to use boundary integral equations arising from electrical impedance tomography to avoid the $\log|k|$ singularity of Faddeev's Green's function. Also, we make use of a relationship found by Barceló et al. (2001) (see also Knudsen, 2002) between $\mathcal{T}$ and the well-known scattering map of the Davey–Stewartson (DS) II equation. Then we can prove the following new result.

Theorem 1.1. *Let $q \in C_0^\infty(\mathbb{R}^2)$ be of conductivity type. Then $q = \mathcal{Q}\mathcal{T}q$.*

Further, we study mapping properties of the inverse scattering map $\mathcal{Q}$.

Theorem 1.2. *Let $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ satisfy $\mathbf{t}(k)/\bar{k} \in \mathcal{S}(\mathbb{C})$ and $\mathbf{t}(k)/k \in \mathcal{S}(\mathbb{C})$. Then the function $\mathcal{Q}\mathbf{t} : \mathbb{R}^2 \to \mathbb{C}$ given by (1.4) is well-defined and continuous. Furthermore,*

$$|(\mathcal{Q}\mathbf{t})(x)| \leq C\langle x\rangle^{-2}. \tag{1.6}$$

Our motivation for the study of the above scattering problem is two-fold. First, the problem is formally related to the Novikov–Veselov evolution equation that generalizes the KdV equation to dimension $2+1$. Second, the scattering transform $\mathbf{t}(k)$ can be used to solve the inverse conductivity problem that has many practical applications. Let us discuss these motivations in more detail.

Novikov and Veselov (1986) and Veselov and Novikov (1984) (see also Nizhnik, 1980) introduce the following evolution equation in a periodic setting:

$$\frac{\partial q_\tau}{\partial \tau} = -\partial_x^3 q_\tau - \bar{\partial}_x^3 q_\tau + 3\partial_x(q_\tau v) + 3\bar{\partial}_x(q_\tau \bar{v}), \tag{1.7}$$

where $\tau \geq 0$ and $\bar{\partial}_x v = \partial_x q_\tau$. Boiti et al. (1987) discuss equation (1.7) in the non-periodic case and show that if $q_\tau : \mathbb{R}^2 \to \mathbb{C}$ evolves in τ according to (1.7) and does not have exceptional points, then the scattering data evolves as

$$(\mathcal{T} q_\tau)(k) = e^{i(k^3+\bar{k}^3)\tau}(\mathcal{T} q_0)(k).$$

Tsai (1993, 1994) assumes the absence of exceptional points and gives a formal derivation of a hierarchy of evolution equations parametrized by $n = 1, 3, 5, \ldots$ and containing the non-periodic version of (1.7) as the case $n = 3$. Thus equation (1.7) can be solved formally using the inverse scattering method:

$$q_\tau = \mathcal{Q}(\exp(i(k^3 + \bar{k}^3)\tau)\, \mathcal{T} q_0). \tag{1.8}$$

However, problems caused by exceptional points prevent the rigorous use of formula (1.8) for general potentials q_0.

We remark that the relationship between the Novikov–Veselov equation (1.7) and a scattering problem analogous to (but different from) the above is studied by Grinevich (1986, 2000), Grinevich and Manakov (1986), and Grinevich and Novikov (1985, 1995).

The inverse conductivity problem is formulated by Calderón (1980) as follows: given a regular domain $\Omega \in \mathbb{R}^d$ with $d > 1$ and a strictly positive function $\gamma \in L^\infty(\Omega)$, define the Dirichlet-to-Neumann (DN) map by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}, \tag{1.9}$$

where u is the unique $H^1(\Omega)$ solution of the Dirichlet problem $\nabla \cdot \gamma \nabla u = 0$ in Ω and $u = f$ on $\partial\Omega$. The DN map represents all static voltage-to-current density measurements on the boundary $\partial\Omega$. Calderón asks whether γ is uniquely determined by Λ_γ and if so, how to reconstruct γ from boundary measurements. This problem has a practical imaging application called electrical impedance tomography (EIT) used for medical imaging, geophysics, nondestructive testing, and process tomography. See Cheney et al (1991), Siltanen et al. (2000), Mueller and Siltanen (2003), and Isaacson et al. (2004) for more details on EIT.

What is the relation between EIT and inverse scattering? Assume that the conductivity $\gamma : \Omega \to \mathbb{R}$ is smooth, strictly positive, and $\gamma \equiv 1$ in a neighborhood of $\partial\Omega$. Then we can extend γ as one outside Ω and define $q := \gamma^{-1/2}\Delta\gamma^{1/2} \in C_0^\infty(\mathbb{R}^2)$. Exponentially growing solutions corresponding to such conductivity type potential q can then be used to solve the inverse conductivity problem, as shown by Sylvester and Uhlmann (1988), Nachman (1988, 1996), and Novikov (1988). Our starting point is Nachman (1996), where Nachman shows that conductivity type potentials in dimension two do not have exceptional points, allowing us to show that formula (1.8) is well-posed for such potentials. Also, an EIT algorithm based on Nachman's proof is presented in Siltanen et al. (2000), Mueller and Siltanen (2003), Knudsen et al. (2004), Isaacson et al. (2004), readily providing numerical inverse scattering algorithms.

The two-dimensional isotropic inverse conductivity problem is solved for once weakly differentiable conductivities by Brown and Uhlmann (1997), and further, for L^∞ conductivities by Astala and Päivärinta (2006). Both approaches are based on the use of a $\bar\partial$ equation. An EIT algorithm based on Brown and Uhlmann (1997) is given in Knudsen (2002, 2003) and Knudsen and Tamasan (2004). We remark that Brown and Uhlmann (1997) is related to the DS II equation, and it is not known if Astala and Päivärinta (2006) is related to some evolution equation.

Choose a positive odd integer n, and set for all $k \in \mathbb{C}$

$$\mathbf{m}_\tau^{(n)}(k) := \exp(-i^n(k^n + \bar{k}^n)\tau). \tag{1.10}$$

Note that $|\mathbf{m}_\tau^{(n)}(k)| = 1$. Given $q \in L^p(\mathbb{R}^2)$ with no exceptional points, one can formally define a function $q_\tau(x)$ at time $\tau > 0$ by multiplying the initial scattering transform $\mathcal{T}q$ by $\mathbf{m}_\tau^{(n)}$ and applying the inverse scattering map $\mathcal{Q}$:

$$\begin{array}{ccc} \mathbf{t} & \xrightarrow{\mathbf{m}_\tau^{(n)}} & \mathbf{t}_\tau \\ \mathcal{T}\big\downarrow & & \big\downarrow\mathcal{Q} \\ q & \rightarrow & q_\tau \end{array} \tag{1.11}$$

The smoothness results for $k^{-1}(\mathcal{T}q)(k)$ and $\bar{k}^{-1}(\mathcal{T}q)(k)$ and Theorem 1.2 can be used to show that the inverse scattering scheme (1.11) is well-defined for conductivity type initial potentials:

Corollary. *Let $q \in C_0^\infty(\mathbb{R}^2)$ be of conductivity type. Fix a positive odd integer n, let $\tau \geq 0$ and define $q_\tau : \mathbb{R}^2 \to \mathbb{C}$ by*

$$q_\tau := \mathcal{Q}(\mathbf{m}_\tau^{(n)}\mathcal{T}q).$$

Then $q_\tau(x)$ is continuous in x and belongs to $L^p(\mathbb{R}^2)$ for any $1 < p < 2$.

Proof. The infinitely smooth function $\mathbf{m}_\tau^{(n)}(k)$ and its derivatives grow at most polynomially at infinity. Thus by Theorems 2.1 and 2.2

$$\frac{\mathbf{m}_\tau^{(n)}(k)\,\mathbf{t}(k)}{\bar{k}} \in \mathcal{S}(\mathbb{C}) \quad \text{and} \quad \frac{\mathbf{m}_\tau^{(n)}(k)\,\mathbf{t}(k)}{k} \in \mathcal{S}(\mathbb{C}),$$

and the claim follows from Theorem 1.2. □

Note that there is no smallness assumption for the initial data in the corollary.

2. The Scattering Map $\mathcal{T}$

This section is devoted to proving that if $q \in C_0^\infty(\mathbb{R}^2)$ is of conductivity type, then $(\mathcal{T}q)(k)$ is a well-defined, smooth function whose derivatives decay when $|k| \to \infty$, and the apparently singular functions $(\mathcal{T}q)(k)/k$ and $(\mathcal{T}q)(k)/\bar{k}$ actually have bounded derivatives of all orders at $k = 0$.

Theorem 2.1. *Let $q \in C_0^\infty(\mathbb{R}^2)$ be of conductivity type. Then the functions $(\mathcal{T}q)(k)$ and $(\mathcal{T}q)(k)/k$ belong to the Schwartz class $\mathcal{S}(\mathbb{C})$.*

Proof. By Nachman (1996, Thm 1.1) the potential q does not have exceptional points. Thus $\mathbf{t}(k) = (\mathcal{T}q)(k)$ is well-defined using formula (1.3) for nonzero $k \in \mathbb{C}$ and setting $\mathbf{t}(0) = 0$.

To analyze derivatives of $\mathbf{t}$, let us discuss scattering data introduced by Beals and Coifman (1988) for the Davey–Stewartson (DS) II equation. By assumption we have $q = \gamma^{-1/2}\Delta\gamma^{1/2}$ for some smooth, strictly positive function γ satisfying $\gamma - 1 \in C_0^\infty(\mathbb{R}^2)$. Define

$$\mathbf{q} := -\frac{\partial_x \gamma^{1/2}}{\gamma^{1/2}}.$$

Consider the equation

$$(D - Q)\Psi(\cdot, k) = 0 \tag{2.1}$$

for a 2×2 matrix $\Psi(x, k)$ depending on $x \in \mathbb{R}^2$ and $k \in \mathbb{C}$, where

$$D = \begin{bmatrix} \bar{\partial}_x & 0 \\ 0 & \partial_x \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & \mathbf{q} \\ \bar{\mathbf{q}} & 0 \end{bmatrix}.$$

Brown and Uhlmann (1997) showed that there are exponentially growing solutions of equation (2.1) of the form

$$\Psi(x, k) = m(x, k) \begin{bmatrix} e^{ikx} & 0 \\ 0 & e^{-i\bar{x}\bar{k}} \end{bmatrix},$$

where m is uniquely specified by the requirement that each element of the matrix $m(\cdot, k) - I$ belongs to $L^r(\mathbb{R}^2)$ for any $r > 2$. The DS scattering data is the 2×2 matrix

$$S(k) = \frac{i}{\pi} \int_{\mathbb{R}^2} \begin{bmatrix} 0 & e^{-i\bar{k}x}\mathbf{q}(x)\Psi_{22}(x, k) \\ -e^{-ik\bar{x}}\bar{\mathbf{q}}(x)\Psi_{11}(x, k) & 0 \end{bmatrix} dx.$$

Based on Barceló et al. (2001), Knudsen (2002, Thm 3.5.2) showed that the scattering transforms $\mathbf{t}$ of q and S of $\mathbf{q}$ are related by

$$\mathbf{t}(k) = -2kS_{21}(k). \tag{2.2}$$

Since $\mathbf{q}$ is compactly supported and N times continuously differentiable, a result of Sung (1994, Thm 4.4) shows that

$$k_1^{\alpha_1} k_2^{\alpha_2} \frac{\partial^{\beta_1+\beta_2}}{\partial k_1^{\beta_1} \partial k_2^{\beta_2}} S_{21}(k) \tag{2.3}$$

is continuous and vanishes at infinity for $\alpha_1 + \alpha_2 \leq N$ and arbitrary $\beta_1, \beta_2 \geq 0$. A combination of (2.2) and (2.3) yields the claim. □

Theorem 2.2. *Let $q \in C_0^\infty(\mathbb{R}^2)$ be of conductivity type. Then $(\mathcal{T} q)(k)/\bar{k} \in C^\infty(\mathbb{R}^2)$.*

Proof. Denote $\mathbf{t}(k) = (\mathcal{T} q)(k)$. By Theorem 2.1 we know that $\mathbf{t}$ is infinitely smooth in $\mathbb{C}$ and all its derivatives vanish at infinity. Thus we only need to prove that all derivatives of $\mathbf{t}(k)/\bar{k}$ are bounded in a neighborhood of 0.

With no loss of generality we can assume that q and $\gamma - 1$ are supported in the open unit disc $\Omega = D(0, 1) \subset \mathbb{R}^2$. Namely, replacing q by $\tilde{q}(x) := \lambda^2 q(\lambda x)$ and γ by $\tilde{\gamma}(x) := \gamma(\lambda x)$ with large enough $\lambda > 0$ yields $\text{supp}(\tilde{q}) \subset \Omega$ while preserving the equality $\tilde{q} = \tilde{\gamma}^{-1/2}\Delta\tilde{\gamma}^{1/2}$. By Theorem 3.3 of Siltanen et al. (2000) we have $\mathbf{t}(k) = \tilde{\mathbf{t}}(\lambda k)$, so the claim holds for $\mathbf{t}(k)/\bar{k}$ if and only if it does for $\tilde{\mathbf{t}}(k)/\bar{k}$.

Let $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be the DN map of γ. Denote by Λ_1 the DN map of the homogeneous conductivity $\gamma \equiv 1$. It is known that Λ_γ is a classical pseudodifferential operator and if $\gamma \equiv 1$ near $\partial\Omega$ then $\Lambda_\gamma - \Lambda_1$ is an infinitely smoothing map, see Sylvester and Uhlmann (1988) and Lee and Uhlmann (1989). We can thus extend $\Lambda_\gamma - \Lambda_1$ as a continuous map $L^2(\partial\Omega) \to C^\infty(\partial\Omega)$ and write

$$(\Lambda_\gamma - \Lambda_1)\phi = \int_{\partial\Omega} \Phi(\cdot, y)\phi(y)d\sigma(y) \tag{2.4}$$

where $d\sigma(y)$ is Lebesgue measure on the unit circle $\partial\Omega$ and Φ is some C^∞ function on the torus $\partial\Omega \times \partial\Omega$. Further, we have

$$\Lambda_\gamma 1 = 0 \quad \text{and} \quad \int_{\partial\Omega} \Lambda_\gamma \phi \, d\sigma = 0 \tag{2.5}$$

for any $\phi \in H^{1/2}(\partial\Omega)$. Of course, (2.5) holds as well when γ is replaced with 1.

Consider the following representation of $\mathbf{t}$ given by Nachman (1996) (originally introduced by Novikov, 1988 and Nachman, 1988):

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k)d\sigma, \quad k \in \mathbb{C}\backslash 0. \tag{2.6}$$

Use equations (2.4)–(2.6) to write for any $k \neq 0$

$$\begin{aligned}
\frac{\mathbf{t}(k)}{\bar{k}} &= \frac{1}{\bar{k}} \int_{\partial\Omega} (e^{i\bar{k}\bar{x}} - 1)(\Lambda_\gamma - \Lambda_1)(\psi(\cdot, k) - 1)d\sigma(x) \\
&= \int_{\partial\Omega} \frac{e^{i\bar{k}\bar{x}} - 1}{\bar{k}} \Big(\int_{\partial\Omega} \Phi(x, y)(\psi(y, k) - 1)d\sigma(y) \Big) d\sigma(x) \\
&= \int_{\partial\Omega} \int_{\partial\Omega} \Phi(x, y)E_k(x)\Psi_k(y) \, d\sigma(x)d\sigma(y),
\end{aligned} \tag{2.7}$$

where we denote

$$E_k(x) := \frac{e^{i\bar{k}\bar{x}} - 1}{\bar{k}}, \quad \Psi_k(y) := \psi(y, k) - 1.$$

The use of Fubini's theorem in (2.7) is justified since $\psi(y, k)$ is continuous in y by Nachman (1996, Thm 1.1) and Sobolev imbedding, and $E_k(x)$ is continuous in x. Use (2.7) to compute formally for any $\alpha_1, \alpha_2 \geq 0$

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial k_1^{\alpha_1} \partial k_2^{\alpha_2}} \left(\frac{\mathbf{t}(k)}{\bar{k}} \right) = \int_{\partial\Omega\times\partial\Omega} \Phi(x, y) \sum_{v_1=0}^{\alpha_1} \sum_{v_2=0}^{\alpha_2} c(v_1, v_2, \alpha_1, \alpha_2) \times E_k^{(v_1, v_2)}(x) \Psi_k^{(\alpha_1 - v_1, \alpha_2 - v_2)}(y) d\sigma(x) d\sigma(y), \tag{2.8}$$

where $c(v_1, v_2, \alpha_1, \alpha_2)$ are constants and we use the following notation:

$$E_k^{(\alpha_1,\alpha_2)}(x) := \frac{\partial^{\alpha_1+\alpha_2} E_k(x)}{\partial k_1^{\alpha_1} \partial k_2^{\alpha_2}}, \quad \Psi_k^{(\alpha_1,\alpha_2)}(x) := \frac{\partial^{\alpha_1+\alpha_2} \Psi_k(y)}{\partial k_1^{\alpha_1} \partial k_2^{\alpha_2}}.$$

We will prove that the maps $k \mapsto E_k^{(\alpha_1,\alpha_2)}$ and $k \mapsto \Psi_k^{(\alpha_1,\alpha_2)}$ are well-defined and continuous from $\mathbb{C}$ to $L^2(\partial\Omega)$ for any $\alpha_1, \alpha_2 \geq 0$ and $k \neq 0$. (We remark that a proof is given also in Astala and Päivärinta, 2006; for the reader's convenience we present a constructive proof here.) Then differentiation under the integral sign in (2.8) is justified by repeated applications of Lebesgue's dominated convergence theorem. Further, we show that

$$\|E_k^{(\alpha_1,\alpha_2)}\|_{L^2(\partial\Omega)} \leq C, \tag{2.9}$$

$$\|\Psi_k^{(\alpha_1,\alpha_2)}\|_{L^2(\partial\Omega)} \leq C \tag{2.10}$$

uniformly for $0 < |k| \leq 1$. Combining (2.9) and (2.10) with (2.8) yields the claim.

The function $E_k(x)$ has the following power series expansion:

$$E_k(x) = \frac{e^{i\bar{k}\bar{x}} - 1}{\bar{k}} = \sum_{n=1}^{\infty} \frac{\bar{k}^{n-1}(i\bar{x})^n}{n!}. \tag{2.11}$$

Thus the map $k \mapsto E_k^{(\alpha_1,\alpha_2)}$ is well-defined and continuous from $\mathbb{C}$ to $L^2(\partial\Omega)$ for any $\alpha_1, \alpha_2 \geq 0$ and $k \neq 0$. Moreover, since $\lim_{k\to 0} E_k(x) = i\bar{x}$, we see that (2.9) holds.

Let us discuss k-smoothness of $\psi(y, k)$. Define a single-layer operator S_k by

$$(S_k\phi)(x) = \int_{\partial\Omega} G_k(x - y)\phi(y) d\sigma(y),$$

where G_k is Faddeev's Green function $G_k(x) = e^{ikx} g_k(x)$ satisfying $-\Delta G_k = \delta$. The function g_k is the following fundamental solution:

$$g_k(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{(\xi_1 + i\xi_2)(\xi_1 - i\xi_2 + 2k)} d\xi. \tag{2.12}$$

The function g_k satisfies $(-\Delta - 4ik\bar{\partial})g_k = \delta$. According to Theorem 5 of Nachman (1996), the operator $[I + S_k(\Lambda_\gamma - \Lambda_1)]$ is invertible in $H^{1/2}(\partial\Omega)$. Since the single layer operator S_k (as well as $\Lambda_\gamma - \Lambda_1$) is a classical pseudodifferential operator, we see by Hörmander (1994, Thm. 19.2.1) that the pseudodifferential operator $[I + S_k(\Lambda_\gamma - \Lambda_1)]$ is invertible also as an operator $L^2(\partial\Omega) \to L^2(\partial\Omega)$. Thus we can write for $k \neq 0$

$$(\psi(y,k) - 1)|_{\partial\Omega} = (I + S_k(\Lambda_\gamma - \Lambda_1))^{-1}(e^{ikx} - 1)|_{\partial\Omega}. \tag{2.13}$$

Let us rewrite (2.13) following Siltanen et al. (2000). Set $G_0 := -(2\pi)^{-1}\log|x|$ and denote

$$H_k(x) := G_k(x) - G_0(x), \quad k \neq 0.$$

Now $\Delta H_k = 0$ and so $H_k(x)$ is infinitely smooth in x. Changes of variables in the integral in (2.12) show that g_k satisfies

$$g_k(x) = g_1(kx) = \overline{g_{\bar{k}}(-\bar{x})} = e_{-k}(x)\overline{g_k(x)}, \tag{2.14}$$

and we see that $G_k(x) = G_1(kx)$ and

$$H_k(x) = H_1(kx) - (2\pi)^{-1}\log|k|, \quad k \neq 0.$$

Write now

$$I + S_k(\Lambda_\gamma - \Lambda_1) = A_\gamma + \mathscr{H}_k(\Lambda_\gamma - \Lambda_1),$$

where $A_\gamma = I + S_0(\Lambda_\gamma - \Lambda_1)$ and the operator $\mathscr{H}_k$ is given by

$$(\mathscr{H}_k\phi)(x) = \int_{\partial\Omega} \widetilde{H}_1(k(x-y))\phi(y)d\sigma(y);$$

here $\widetilde{H}_1(x) := H_1(x) - H_1(0)$ and the constants $H_1(0)$ and $\log|k|$ were annihilated by (2.5). Note that $\widetilde{H}_1 : \mathbb{R}^2 \to \mathbb{R}$ is real-valued (Siltanen et al., 2000), and we abuse notation by using interchangeably $\widetilde{H}_1(x) = \widetilde{H}_1(x_1 + ix_2) = \widetilde{H}_1(x_1, x_2)$. So (2.13) takes the form

$$\Psi_k(y) = (I + A_\gamma^{-1}\mathscr{H}_k(\Lambda_\gamma - \Lambda_1))^{-1}A_\gamma^{-1}(e^{ikx} - 1)|_{\partial\Omega}, \tag{2.15}$$

where A_γ is invertible in $H^{1/2}(\partial\Omega)$ by the proof of Theorem 3.1 in Siltanen et al. (2000). As before, we see using Hörmander (1994, Thm. 19.2.1) that A_γ is invertible also in $L^2(\partial\Omega)$.

For differentiating (2.15) we analyze the k_1-differentiability of $\mathscr{H}_k(\Lambda_\gamma - \Lambda_1)$ in the strong operator topology of $L(L^2(\partial\Omega))$. Denote for any $\alpha_1, \alpha_2 \geq 0$

$$\widetilde{H}_1^{(\alpha_1,\alpha_2)}(x) := \frac{\partial^{\alpha_1+\alpha_2}\widetilde{H}_1(x_1,x_2)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}.$$

Define an operator $D_{(1,0)}(k) \in L(L^2(\partial\Omega))$ by

$$\begin{aligned}(D_{(1,0)}(k)f)(x) = &\int_{\partial\Omega\times\partial\Omega}(x_1 - y_1)\widetilde{H}_1^{(1,0)}(k(x-y))\Phi(y,z)f(z)d\sigma^2\\ &+ \int_{\partial\Omega\times\partial\Omega}(x_2 - y_2)\widetilde{H}_1^{(0,1)}(k(x-y))\Phi(y,z)f(z)d\sigma^2.\end{aligned}$$

Take any $f \in L^2(\partial\Omega) \subset L^1(\partial\Omega)$ and write

$$\lim_{h\to 0}\left\|\left(\frac{\mathscr{H}_{k+h}-\mathscr{H}_k}{h}(\Lambda_\gamma-\Lambda_1)f - D_{(1,0)}(k)\right)f\right\|^2_{L^2(\partial\Omega)}$$
$$= \lim_{h\to 0}\int_{\partial\Omega}\left|\int_{\partial\Omega}\frac{\widetilde{H}_1((k+h)(x-y))-\widetilde{H}_1(k(x-y))}{h}\right.$$
$$\left.\times(\Lambda_\gamma-\Lambda_1)f(y)d\sigma(y) - D_{(1,0)}(k)f(x)\right|^2 d\sigma(x) = 0, \qquad (2.16)$$

where we used Fubini's theorem once and dominated convergence twice; this is possible since $\widetilde{H}_1$ and Φ are smooth.

Thus $k \mapsto \Psi_k$ as a mapping $\mathbb{C}$ to $L^2(\partial\Omega)$ has a differential

$$\Psi_k^{(1,0)}(y) = \frac{\partial}{\partial k_1}\big(I + A_\gamma^{-1}\mathscr{H}_k(\Lambda_\gamma-\Lambda_1)\big)^{-1}A_\gamma^{-1}e^{ikx}|_{\partial\Omega}$$
$$= (I + A_\gamma^{-1}\mathscr{H}_k(\Lambda_\gamma-\Lambda_1))^{-1}A_\gamma^{-1}\frac{\partial\mathscr{H}_k(\Lambda_\gamma-\Lambda_1)}{\partial k_1}$$
$$\times (I + A_\gamma^{-1}\mathscr{H}_k(\Lambda_\gamma-\Lambda_1))^{-1}A_\gamma^{-1}(e^{ikx}-1)|_{\partial\Omega}$$
$$+ (I + A_\gamma^{-1}\mathscr{H}_k(\Lambda_\gamma-\Lambda_1))^{-1}A_\gamma^{-1}\frac{\partial e^{ikx}}{\partial k_1}\bigg|_{\partial\Omega}$$

implying that $k \mapsto \Psi_k$ is in $C^1(\mathbb{C}, L^2(\partial\Omega))$. The above analysis can be extended to any higher partial derivative $\partial^{\alpha_1+\alpha_2}/\partial k_1^{\alpha_1}\partial k_2^{\alpha_2}$ in a similar manner. □

3. Properties of Solutions of the $\bar{\partial}$ Equation

Fix $2 < r < \infty$ and $x \in \mathbb{R}^2$. Assume given a function $\mathbf{t}: \mathbb{C} \to \mathbb{C}$ satisfying $\mathbf{t}(k)/\bar{k} \in \mathscr{S}(\mathbb{C})$ and consider the $\bar{\partial}$ equation

$$\bar{\partial}_k\mu(x,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}}e_{-k}(x)\overline{\mu(x,k)} \qquad (3.1)$$

for μ satisfying the asymptotic condition

$$\mu(x,\cdot) - 1 \in L^r \cap L^\infty. \qquad (3.2)$$

Denote the solid Cauchy transform by

$$\mathscr{C}\varphi(k) := \frac{1}{\pi}\int_{\mathbb{C}}\frac{\varphi(k')}{k-k'}dk', \qquad (3.3)$$

where dk' denotes the Lebesgue measure of $\mathbb{R}^2$. Note that $\mathscr{C}$ and $\bar{\partial}_k$ are inverses of each other (modulo analytic functions). Further, define a real-linear operator

$$T_x\varphi(x,k) := \frac{\mathbf{t}(k)}{4\pi\bar{k}}e_{-x}(k)\overline{\varphi(x,k)}. \qquad (3.4)$$

Nachman (1996) proved that the operator $[I - \mathscr{C}T_x] : L^r \to L^r$ is invertible and that $\mathscr{C}T_x 1 \in L^r$. Now the $\bar{\partial}$ equation (3.1) together with the asymptotic condition (3.2) can be written in the convenient form

$$\mu = 1 + \mathscr{C}T_x\mu \tag{3.5}$$

or in the long form

$$\mu(x, k) = 1 + \frac{1}{4\pi^2}\int_{\mathbb{C}} \frac{\mathbf{t}(k')}{(k-k')\overline{k'}} e_{-x}(k')\overline{\mu(x, k')}dk'. \tag{3.6}$$

The solution of equation (3.5) is given by

$$\mu(x, k) = 1 + [I - \mathscr{C}T_x]^{-1}\mathscr{C}T_x 1. \tag{3.7}$$

The following lemma can be found in Vekua (1962).

Lemma 3.1. *Let $f \in L^p(\mathbb{R}^2) \cap L^{p'}(\mathbb{R}^2)$, where $p \in [1, 2)$ and $1/p + 1/p' = 1$. Then*

$$\int_{\mathbb{C}} \frac{|f(k')|}{|k-k'|}\, dk_1' dk_2' \leq \left[8\pi\|f\|_{L^p}\|f\|_{L^{p'}}(2-p)^{-1/p}(p'-2)^{-1/p'}\right]^{1/2} \quad \forall k \in \mathbb{C}.$$

The following lemma establishes some properties of the function $\mathscr{C}T_x 1$ appearing in (3.7).

Lemma 3.2. *Let $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ satisfy $\mathbf{t}(k)/\bar{k} \in \mathscr{S}(\mathbb{C})$. Define for any $x \in \mathbb{R}^2$*

$$g(x, k) := \frac{1}{4\pi^2}\int_{\mathbb{C}} \frac{\mathbf{t}(k')}{(k-k')\overline{k'}} e_{-x}(k')dk'. \tag{3.8}$$

Then for any $\alpha_1, \alpha_2 \geq 0$ there is a constant $M = M(\phi, \alpha_1, \alpha_2) > 0$, independent of x and k, such that the following estimate holds:

$$\left|\frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} g(x, k)\right| \leq M\min\{\langle x\rangle^{-1}, \langle k\rangle^{-1}\}, \quad \textit{for all } x \in \mathbb{R}^2,\ k \in \mathbb{C}, \tag{3.9}$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$ and $\langle k\rangle = (1 + |k|^2)^{1/2}$. Moreover, the maps

$$x \mapsto g(x, \cdot) \quad \textit{and} \quad x \mapsto \frac{\partial}{\partial x_j} g(x, \cdot)$$

are continuous from $\mathbb{R}^2$ to $L^r(\mathbb{C})$ for any $2 < r < \infty$ and $j = 1, 2$.

Proof. Denote $\phi(k) := \mathbf{t}(k)/\bar{k} \in \mathscr{S}(\mathbb{C})$. Choose some $1 < p < 2$ and note that the norms $\|\phi\|_p$ and $\|\phi\|_{p'}$ are bounded by Schwartz seminorms of ϕ. Thus by Lemma 3.1 we have for any $x \in \mathbb{R}^2$

$$\|g(x, \cdot)\|_\infty \leq \left[8\pi\|\phi\|_p\|\phi\|_{p'}(2-p)^{-1/p}(p'-2)^{-1/p'}\right]^{1/2} =: M_1, \tag{3.10}$$

where $M_1 < \infty$ does not depend on x.

The norm $\|\phi\|_1$ is bounded by Schwartz seminorms of ϕ. Thus we can estimate for $k \neq 0$

$$4\pi^2|g(x,k)| = \left| \int_{\mathbb{C}} \frac{\phi(k+k')}{k} e_{-x}(k+k')dk' \right| \leq \frac{1}{|k|}\int_{\mathbb{C}} |\phi(k+k')|dk' = \frac{\|\phi\|_1}{|k|},$$

leading to

$$|g(x,k)| \leq \frac{\|\phi\|_1}{4\pi^2|k|} = \frac{M_2}{|k|} \quad \text{for } x \in \mathbb{R}^2,\ \ k \in \mathbb{C}\backslash 0, \tag{3.11}$$

with $M_2 < \infty$ independent of x and k.

There are positive constants C_1, C_2 and R depending only on the Schwartz seminorms of ϕ such that $\|\hat{\phi}\|_1 = C_1 < \infty$, and $|\hat{\phi}(\xi)| \leq C_2|\xi|^{-2}$ for $|\xi| \geq R$. Here $\hat{\phi}$ denotes the Fourier transform of ϕ. Note that g can be written in the form

$$g(x,k) = \frac{1}{4\pi}\left(\frac{1}{\pi k'} * (\phi(k')e_{-x}(k'))\right)(k).$$

Fourier transforming g with respect to k yields

$$\hat{g}(x,\xi) = \frac{1}{4\pi^2\xi}(\phi e_{-x})\hat{\ }(\xi) = \frac{1}{4\pi^2\xi}(\hat{\phi}(\xi) * \widehat{e_{-x}}(\xi)) = \frac{\hat{\phi}(\xi+2\bar{x})}{4\pi^2\xi}.$$

We want to estimate the L^1 norm of $\hat{g}(x,\cdot)$ for large $|x|$. Fix $x \in \mathbb{R}^2$ with $|x| > R$.

$$\begin{aligned}
4\pi^2\|\hat{g}(x,\cdot)\|_1 &= \int_{|\xi|\geq|x|} \frac{|\hat{\phi}(\xi+2\bar{x})|}{|\xi|}d\xi + \int_{|\xi|\leq|x|} \frac{|\hat{\phi}(\xi+2\bar{x})|}{|\xi|}d\xi \\
&\leq \frac{1}{|x|}\int_{\mathbb{R}^2} |\hat{\phi}(\xi+2\bar{x})|d\xi + \int_{|\zeta-2\bar{x}|\leq|x|} \frac{|\hat{\phi}(\zeta)|}{|\zeta-2\bar{x}|}d\zeta \\
&\leq \frac{C_1}{|x|} + C_2\int_{|\zeta-2\bar{x}|\leq|x|} \frac{1}{|\zeta|^2|\zeta-2\bar{x}|}d\zeta \\
&\leq \frac{C_1}{|x|} + \frac{2\pi C_2}{|x|^2}\int_0^{|x|} \frac{1}{r}rdr \\
&= (C_1 + 2\pi C_2)|x|^{-1}.
\end{aligned}$$

Since the inverse Fourier transform is continuous from L^1 to L^∞ we have

$$\|g(x,\cdot)\|_\infty \leq M_3|x|^{-1} \quad \text{for } |x| > R, \tag{3.12}$$

where $M_3 < \infty$ does not depend on x.

A combination of (3.10), (3.11), and (3.12) now yields the estimate (3.9) for the case $\alpha_1 = \alpha_2 = 0$.

Next we show the continuity of the map $x \mapsto g(x,\cdot)$. Write

$$\begin{aligned}
g(x,k) - g(y,k) &= \frac{1}{4\pi^2}\int_{\mathbb{C}} \frac{\phi(k')}{k-k'}(e_{-x}(k') - e_{-y}(k'))dk' \\
&= \frac{1}{4\pi^2}\int_{\mathbb{C}} \frac{\phi(k')}{k-k'}e_{-x}(k')(e_{x-y}(k') - 1)dk'.
\end{aligned}$$

Clearly $\lim_{y\to x}\|\phi(k)\big(e_{x-y}(k)-1\big)\|_{L^\lambda}=0$ for any $1\leq\lambda\leq\infty$ since ϕ is rapidly decaying. Thus replacing $g(x,k)$ by $g(x,k)-g(y,k)$ and ϕ by $\phi(e_{x-y}(\cdot)-1)$ in estimates (3.10) and (3.11) shows that

$$\lim_{y\to x}\|g(x,\cdot)-g(y,\cdot)\|_r=0,$$

implying continuity of $x\mapsto g(x,\cdot)$ from $\mathbb{R}^2$ to L^r.

Let now $\alpha_1,\alpha_2\geq 0$. Then by repeated applications of Lebesgue's dominated convergence theorem we can write

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}g(x,k)=\frac{1}{4\pi^2}\int_{\mathbb{C}}\frac{\phi(k')}{k-k'}(-i2k_1')^{\alpha_1}(i2k_2')^{\alpha_2}e_{-x}(k')dk'.$$

Estimate (3.9) follows from repeating the above proof with $\phi(k)\in\mathcal{S}$ replaced by $k_1^{\alpha_1}k_2^{\alpha_2}\phi(k)\in\mathcal{S}$.

Finally, continuity of the map $x\mapsto(\partial/\partial x_j)\,g(x,\cdot)$ follows analogously by repeating the above proof with ϕ replaced by $k_j\phi(k)$. □

Now Lemma 3.2 and formula (3.7) yield the following estimate:

Lemma 3.3. *Let $\mathbf{t}:\mathbb{C}\to\mathbb{C}$ satisfy $\mathbf{t}(k)/\bar{k}\in\mathcal{S}(\mathbb{C})$ and take $2<r<\infty$. Let μ be the unique solution of (3.5) satisfying $\mu(x,\cdot)-1\in L^r\cap L^\infty$. Then the map $x\mapsto(\mu(x,\cdot)-1)$ is continuous from $\mathbb{R}^2$ to L^r and*

$$\|\mu(x,\cdot)-1\|_r\leq C\langle x\rangle^{-1}. \tag{3.13}$$

Proof. By Lemma 3.2 we know that $\mathscr{C}T_x1\in L^r$ depends continuously on x.

Let $f\in L^r$ with $\|f\|_r=1$ and write for any $y\in\mathbb{R}^2$

$$\mathscr{C}T_xf-\mathscr{C}T_yf=\mathscr{C}\left(\frac{\mathbf{t}(k)}{4\pi\bar{k}}(e_{x-y}(k)-1)\,e_{-x}(k)f(k)\right).$$

By the proof of Nachman (1993, Lemma 4.2) we know that

$$\|\mathscr{C}(af)\|_r\leq c_0\|a\|_2\|f\|_r=c_0\|a\|_2. \tag{3.14}$$

Thus we see that the map $x\mapsto\mathscr{C}T_x$ is continuous from $\mathbb{R}^2$ to $L(L^r(\mathbb{C}))$:

$$\lim_{y\to x}\|\mathscr{C}T_xf-\mathscr{C}T_yf\|_r\leq\frac{c_0}{4\pi}\lim_{y\to x}\left\|\frac{\mathbf{t}(k)}{\bar{k}}\big(e_{x-y}(k)-1\big)\right\|_2=0.$$

Since the operator $I-\mathscr{C}T_x$ is invertible for all $x\in\mathbb{R}^2$, the map $x\mapsto[I-\mathscr{C}T_x]^{-1}$ is continuous from $\mathbb{R}^2$ to $L(L^r(\mathbb{C}))$ as well. Hence the right hand side of (3.7) depends continuously on x.

It remains to prove estimate (3.13). Lemma 3.2 shows that

$$\|\mathscr{C}T_x1\|_r\leq M\left(\langle x\rangle^{-r}\int_{\mathbb{C}}\langle k\rangle^{-r}dk\right)^{1/r}\leq C\langle x\rangle^{-1}. \tag{3.15}$$

Further, by Lemma 2.2.1 of Liu (1997) we have

$$\|[I - \mathscr{C}T_x]^{-1}\|_{L(L^r)} \leq C, \tag{3.16}$$

where C does not depend on x. Now (3.13) follows from (3.7), (3.15) and (3.16). □

Now we are ready to prove the crucial estimate on the x-derivatives of μ.

Lemma 3.4. *Let $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ satisfy $\mathbf{t}(k)/\bar{k} \in \mathscr{S}(\mathbb{C})$ and take $2 < r < \infty$. Let μ be the unique solution of (3.5) satisfying $\mu(x, \cdot) - 1 \in L^r \cap L^\infty$. Then the map $x \mapsto \partial\mu(x, \cdot)/\partial x_j$ is continuous from $\mathbb{R}^2$ to L^r and*

$$\left\| \frac{\partial}{\partial x_j} \mu(x, \cdot) \right\|_r \leq C\langle x\rangle^{-1} \quad \text{for } j = 1, 2. \tag{3.17}$$

Proof. Let us write formally

$$\frac{\partial \mu(x, k)}{\partial x_j} = [I + \mathscr{C}T_x]^{-1}\left(\frac{\partial}{\partial x_j}\mathscr{C}T_x\right)[I + \mathscr{C}T_x]^{-1}\mathscr{C}T_x 1 + [I + \mathscr{C}T_x]^{-1}\frac{\partial}{\partial x_j}(\mathscr{C}T_x 1). \tag{3.18}$$

By Lemma 3.2 we know that the maps $x \mapsto \frac{\partial}{\partial x_j}(\mathscr{C}T_x 1)$ with $j = 1, 2$ are continuous from $\mathbb{R}^2$ to $L^r(\mathbb{C})$. We need to prove that

(i) The operator $\mathscr{C}T_x$ is differentiable with respect to x in the strong operator topology of $L(L^r)$, and
(ii) The maps $x \mapsto \frac{\partial}{\partial x_j}\mathscr{C}T_x$ with $j = 1, 2$ are continuous from $\mathbb{R}^2$ to $L(L^r(\mathbb{C}))$.

Then (i), (ii) and (3.18) together yield the continuity of the map $x \mapsto \partial\mu(x, \cdot)/\partial x_j$.

Denote

$$(\mathscr{C}T_x)_1 f := -\frac{i}{2\pi^2}\int_{\mathbb{C}} \frac{k_1'\mathbf{t}(k')}{(k - k')\overline{k'}} e_{-x}(k')f(k')dk'. \tag{3.19}$$

We will show that

$$\lim_{h\to 0}\left(\frac{\mathscr{C}T_{x+h}f - \mathscr{C}T_x f}{h}\right) = (\mathscr{C}T_x)_1 f \tag{3.20}$$

in the L^r topology uniformly in f. Let $\varepsilon > 0$ and take $f \in L^r$ with $\|f\|_r = 1$. With c_0 as in (3.14), choose such a $h_0 > 0$ that

$$\left\|(e_{-h'}(k) - 1)\frac{k_1\mathbf{t}(k)}{\bar{k}}\right\|_{L^2(\mathbb{C})} < \frac{2\pi\varepsilon}{c_0}$$

for all $0 < h' < h_0$. Compute using the mean value theorem and estimate (3.14)

$$\begin{aligned}\left\|\frac{\mathscr{C}T_{x+h}f - \mathscr{C}T_x f}{h} - (\mathscr{C}T_x)_1 f\right\|_r &= \left\|\mathscr{C}\left(\left(\frac{e_{-h}(k) - 1}{h} + 2ik_1\right)e_{-x}(k)\frac{\mathbf{t}(k)}{4\pi\bar{k}}f(k)\right)\right\|_r \\ &= \frac{1}{4\pi}\left\|\mathscr{C}\left((-2ik_1 e_{-h'}(k) + 2ik_1)e_{-x}(k)\frac{\mathbf{t}(k)}{\bar{k}}f(k)\right)\right\|_r\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\left\|\mathscr{C}\left((e_{-h'}(k)-1)e_{-x}(k)\frac{k_1\mathbf{t}(k)}{\bar{k}}f(k)\right)\right\|_r \\
&\leq \frac{c_0}{2\pi}\left\|(e_{-h'}(k)-1)\frac{k_1\mathbf{t}(k)}{\bar{k}}\right\|_2 \\
&< \varepsilon,
\end{aligned}
$$

where $0 < h' < h < h_0$ and $h' = h'(k)$. This proves (3.20), which in turn implies (i) for $j = 1$. The proof for $j = 2$ is analogous, and claim (ii) can be proved using a similar argument.

Finally, to prove estimate (3.17) take L^r norm of both sides of equation (3.18). Using Lemma 3.2 we see similarly to (3.15) that

$$
\left\|\frac{\partial}{\partial x_j}(\mathscr{C}T_x 1)\right\|_r \leq C\langle x\rangle^{-1}. \tag{3.21}
$$

Using (3.19) and (3.14) we get the following uniform bound:

$$
\left\|\frac{\partial}{\partial x_j}\mathscr{C}T_x\right\|_{L(L^r)} \leq C. \tag{3.22}
$$

Now (3.15), (3.16), (3.21), and (3.22) yield the estimate (3.17). □

4. Proof of Theorem 1.1

Let q be an infinitely smooth, compactly supported potential of conductivity type. By Theorems 2.1 and 2.2 we see that $\mathbf{t} := \mathscr{T}q$ is well-defined and

$$
\frac{\mathbf{t}(k)}{\bar{k}} \in \mathscr{S}(\mathbb{C}), \quad \frac{\mathbf{t}(k)}{k} \in \mathscr{S}(\mathbb{C}). \tag{4.1}
$$

It is shown in Nachman (1996) that for any $k \in \mathbb{C}\backslash 0$ the equation

$$
\bar{\partial}_x(\partial_x + ik)\mu(x,k) = \frac{q}{4}\mu(x,k) \tag{4.2}
$$

has a unique solution satisfying $\mu(\cdot,k) - 1 \in L^{\tilde{p}} \cap L^\infty(\mathbb{R}^2)$ for any $2 < \tilde{p} < \infty$. Furthermore, it is shown in Nachman (1996) that the functions μ are also the unique solutions of the $\bar{\partial}$ equation (3.1) with asymptotic condition (3.2) with any $2 < r < \infty$.

Define $f := \bar{\partial}_x(\partial_x + ik)\mu$. Use (4.2) and (3.7) to write

$$
0 = f - \frac{q}{4}(\mu(x,k)-1) - \frac{q}{4} = f - \frac{q}{4}[I - \mathscr{C}T_x]^{-1}\mathscr{C}T_x 1 - \frac{q}{4}.
$$

Because q is real-valued and does not depend on k, we can write

$$
f - [I - \mathscr{C}T_x]^{-1}\mathscr{C}T_x\frac{q}{4} - \frac{q}{4} = 0. \tag{4.3}
$$

Applying $[I - \mathscr{C}T_x]$ to both sides of (4.3) yields

$$
0 = f - \frac{q}{4} - \mathscr{C}T_x f + \mathscr{C}T_x\frac{q}{4} - \mathscr{C}T_x\frac{q}{4} = f - \frac{q}{4} - \mathscr{C}T_x f. \tag{4.4}
$$

Compute the commutator of T_x with the operator $\bar{\partial}_x(\partial_x + ik)$:

$$
\begin{aligned}
&[\bar{\partial}_x(\partial_x + ik), T_x]\varphi(x,k) \\
&\quad = \bar{\partial}_x(\partial_x + ik)\frac{\mathbf{t}(k)}{4\pi\bar{k}}e_{-k}(x)\overline{\varphi} - \frac{\mathbf{t}(k)}{4\pi\bar{k}}e_{-k}(x)\overline{\bar{\partial}_x(\partial_x + ik)\varphi} \\
&\quad = \frac{\mathbf{t}(k)}{4\pi\bar{k}}\big(\bar{\partial}_x(e_{-k}(x)\partial_x\overline{\varphi}) - e_{-k}(x)\partial_x(\bar{\partial}_x - i\bar{k})\overline{\varphi}\big) \\
&\quad = \frac{\mathbf{t}(k)}{4\pi\bar{k}}\big(e_{-k}(x)(\bar{\partial}_x - i\bar{k})\partial_x\overline{\varphi} - e_{-k}(x)\partial_x(\bar{\partial}_x - i\bar{k})\overline{\varphi}\big) = 0.
\end{aligned}
\tag{4.5}
$$

Take $k \neq 0$ and compute similarly for the operator $\mathscr{C}$:

$$
\begin{aligned}
&[\bar{\partial}_x(\partial_x + ik), \mathscr{C}]\varphi \\
&\quad = \bar{\partial}_x(\partial_x + ik)\frac{1}{\pi}\int_{\mathbb{C}}\frac{\varphi(x,k')}{k-k'}dk' - \frac{1}{\pi}\int_{\mathbb{C}}\frac{\bar{\partial}_x(\partial_x + ik')\varphi(x,k')}{k-k'}dk' \\
&\quad = \frac{ik}{\pi}\bar{\partial}_x\int_{\mathbb{C}}\frac{\varphi(x,k')}{k-k'}dk' - \frac{i}{\pi}\bar{\partial}_x\int_{\mathbb{C}}\frac{k'\varphi(x,k')}{k-k'}dk' \\
&\quad = \frac{i}{\pi}\bar{\partial}_x\int_{\mathbb{C}}\varphi(x,k')dk'.
\end{aligned}
\tag{4.6}
$$

Differentiation outside the integral sign in (4.6) is justified for continuous $\varphi(x,\cdot) \in L^1(\mathbb{C})$.

Applying the commutator relations (4.5) and (4.6) to equation (4.4) gives

$$
\begin{aligned}
\bar{\partial}_x(\partial_x + ik)\mu &= \mathscr{C}T_x\bar{\partial}_x(\partial_x + ik)\mu + \frac{q}{4} \\
&= \mathscr{C}\bar{\partial}_x(\partial_x + ik)T_x\mu + \frac{q}{4} \\
&= \bar{\partial}_x(\partial_x + ik)\mathscr{C}T_x\mu + \frac{q}{4} - \frac{i}{\pi}\bar{\partial}_x\int_{\mathbb{C}}T_x\mu(x,k)dk.
\end{aligned}
\tag{4.7}
$$

Further, from (4.7) and (3.5)

$$
\frac{q}{4} - \frac{i}{\pi}\bar{\partial}_x\int_{\mathbb{C}}T_x\mu(x,k)dk = \bar{\partial}_x(\partial_x + ik)\big(1 + \mathscr{C}T_x\mu - \mathscr{C}T_x\mu\big) = 0.
$$

Thus,

$$
q = \frac{4i}{\pi}\bar{\partial}_x\int_{\mathbb{C}}\frac{\mathbf{t}(k)}{4\pi\bar{k}}e_{-x}(k)\overline{\mu(x,k)}dk = \frac{i}{\pi^2}\bar{\partial}_x\int_{\mathbb{C}}\frac{\mathbf{t}(k)}{\bar{k}}e_{-x}(k)\overline{\mu(x,k)}dk = \mathscr{Q}\mathbf{t},
$$

as required.

5. Proof of Theorem 1.2

By Lemma 3.4 the map $x \mapsto \mu(x, \cdot)$ is differentiable from $\mathbb{C} \to L^r(\mathbb{C})$ with the derivative defined in the norm topology. Thus

$$\lim_{h\to 0} \left\| \frac{\mu(x+he_j, \cdot) - \mu(x, \cdot)}{h} - \frac{\partial}{\partial x_j}\mu(x, \cdot) \right\|_{L^r} = 0$$

where $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let r' be the exponent conjugate to r: $\frac{1}{r} + \frac{1}{r'} = 1$. Then in the weighted $L^{r'}$-norm with weight $w(k) = (1 + |k|)^{-3}$ we have

$$\lim_{h\to 0} \left\| \left(\frac{e_{-x}(x+he_j, \cdot) - e_{-x}(x, \cdot)}{h} - \frac{\partial}{\partial x_j} e_{-x}(\cdot) \right) w(\cdot) \right\|_{L^{r'}} = 0.$$

Thus, $k \mapsto \mu(x, \cdot)e_{-x}(\cdot)w(\cdot)$ is an $L^1(\mathbb{C})$-valued continuously differentiable function, and as $\frac{\mathbf{t}(k)}{\bar{k}} w(k)^{-1} \in L^\infty(\mathbb{C})$, we see that

$$\begin{aligned}
&\left| \frac{\partial}{\partial x_j} \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e_{-x}(k)\overline{\mu(x, k)}dk - \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} \frac{\partial}{\partial x_j} (e_{-x}(k)\overline{\mu(x, k)})dk \right| \\
&= \left| \lim_{h\to 0} \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} \left(\frac{e_{-(x+he_j)}(k)\overline{\mu(x+he_j, k)} - e_{-x}(k)\overline{\mu(x, k)}}{h} \right.\right. \\
&\qquad \left.\left. - \frac{\partial}{\partial x_j}(e_{-x}(k)\overline{\mu(x, k)}) \right) dk \right| \\
&= 0.
\end{aligned}$$

Thus we can differentiate under the integral sign below to obtain

$$\begin{aligned}
\pi^2(\mathscr{Q}\mathbf{t})(x) &= i\bar{\partial}_x \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e_{-x}(k)\overline{\mu(x, k)}dk \\
&= i\bar{\partial}_x \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e_{-x}(k)(\overline{\mu(x, k) - 1})dk + i\bar{\partial}_x \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e_{-x}(k)dk \\
&= i \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} (\bar{\partial}_x e_{-x}(k))(\overline{\mu(x, k) - 1})dk \\
&\quad + i \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e_{-x}(k)\bar{\partial}_x\overline{\mu(x, k)}dk + i \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} (-i\bar{k})e_{-x}(k)dk \\
&= \int_{\mathbb{C}} \mathbf{t}(k)e_{-x}(k)(\overline{\mu(x, k) - 1})dk + i \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar{k}} e_{-x}(k)\bar{\partial}_x\overline{\mu(x, k)}dk \qquad (5.1) \\
&\quad + \int_{\mathbb{C}} \mathbf{t}(k)e_{-x}(k)dk. \qquad (5.2)
\end{aligned}$$

Now (5.2) is rapidly decaying since it is the Fourier transform of a Schwartz function. By Lemmas 3.3 and 3.4 the L^r norms of $\mu(x, k) - 1$ and $\partial_x\mu(x, k)$ are finite and depend continuously on x. Thus the fact that $\mathbf{t}(k)/\bar{k} \in \mathscr{S}$ together with Hölder's inequality shows that (5.1) is well-defined and continuous in x.

To prove estimate (1.6) for term (5.1) write the $\bar\partial$ equation (3.1) in the form

$$\bar\partial_k \mu(x,k) = \frac{\mathbf{t}(k)}{4\pi\bar k} e_{-x}(k)\left(\overline{\mu(x,k)-1}\right) + \frac{\mathbf{t}(k)}{4\pi\bar k} e_{-x}(k). \tag{5.3}$$

Now integration by parts yields for $x \neq 0$

$$\begin{aligned}
|x| & \left| \int_{\mathbb{C}} \mathbf{t}(k) e_{-x}(k)\left(\overline{\mu(x,k)-1}\right) dk + i \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar k} e_{-x}(k) \bar\partial_x \overline{\mu(x,k)} dk \right| \\
&= \left| -i(x_1 + i x_2)\left(\int_{\mathbb{C}} \mathbf{t}(k) e_{-x}(k)\left(\overline{\mu(x,k)-1}\right) dk \right.\right. \\
&\qquad \left.\left. + i \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar k} e_{-x}(k) \bar\partial_x \overline{\mu(x,k)} dk \right) \right| \\
&= \left| \int_{\mathbb{C}} \mathbf{t}(k) \frac{\partial e_{-x}(k)}{\partial k}\left(\overline{\mu(x,k)-1}\right) dk + i \int_{\mathbb{C}} \frac{\mathbf{t}(k)}{\bar k} \frac{\partial e_{-x}(k)}{\partial k} \bar\partial_x \overline{\mu(x,k)} dk \right| \\
&= \left| - \int_{\mathbb{C}} e_{-x}(k)\left[(\partial_k \mathbf{t}(k))\left(\overline{\mu(x,k)-1}\right) + \mathbf{t}(k) \overline{\bar\partial_k \mu(x,k)} \right] dk \right. \\
&\qquad \left. - i \int_{\mathbb{C}} e_{-x}(k) \partial_k \left(\frac{\mathbf{t}(k)}{\bar k} \bar\partial_x \overline{\mu(x,k)} \right) dk \right|.
\end{aligned} \tag{5.4}$$

Making use of equation (5.3) we have that (5.4) equals

$$\begin{aligned}
&\left| - \int_{\mathbb{C}} e_{-x}(k)\left[(\partial_k \mathbf{t}(k))\left(\overline{\mu(x,k)-1}\right) \right.\right. \\
&\qquad \left. + \mathbf{t}(k)\left(\frac{\overline{\mathbf{t}(k)}}{4\pi k} e_x(k)(\mu(x,k)-1) + \frac{\overline{\mathbf{t}(k)}}{4\pi k} e_x(k) \right) \right] dk \\
&- i \int_{\mathbb{C}} e_{-x}(k) \partial_k \left(\frac{\mathbf{t}(k)}{\bar k} \right) \bar\partial_x \overline{\mu(x,k)} \\
&\left. + \frac{\mathbf{t}(k)}{\bar k} e_{-x}(k) \bar\partial_x \left(\frac{\overline{\mathbf{t}(k)}}{4\pi k} e_x(k)(\mu(x,k)-1) + \frac{\overline{\mathbf{t}(k)}}{4\pi k} e_x(k) \right) dk \right|
\end{aligned}$$

After some simplification we have that (5.4) equals

$$\left| - \int_{\mathbb{C}} \left[e_{-x}(k)(\partial_k \mathbf{t}(k))\left(\overline{\mu(x,k)-1}\right) + \frac{|\mathbf{t}(k)|^2}{4\pi k}(\mu(x,k)-1) + \frac{|\mathbf{t}(k)|^2}{4\pi k} \right] dk \right. \tag{5.5}$$

$$- i \int_{\mathbb{C}} \left[e_{-x}(k) \frac{\partial}{\partial k}\left(\frac{\mathbf{t}(k)}{\bar k} \right) \bar\partial_x \overline{\mu(x,k)} + \frac{|\mathbf{t}(k)|^2}{4\pi |k|^2} e_{-x}(k) \bar\partial_x (e_x(k)(\mu(x,k)-1)) \right. \tag{5.6}$$

$$\left. \left. + \frac{|\mathbf{t}(k)|^2}{4\pi |k|^2} e_{-x}(k)(i\bar k e_x(k)) \right] dk \right| \tag{5.7}$$

The third term in (5.5) cancels with the term (5.7). Expanding the $\overline{\partial_x}$ derivative in (5.6) allows us to estimate (5.4) as follows:

$$\begin{aligned}
&\Bigg| -\int_{\mathbb{C}} \left[e_{-x}(k) \frac{\partial \mathbf{t}(k)}{\partial k} \overline{(\mu(x,k)-1)} + \frac{|\mathbf{t}(k)|^2}{4\pi k}(\mu(x,k)-1) \right] dk \\
&-i \int_{\mathbb{C}} \left[e_{-x}(k) \frac{\partial}{\partial k}\left(\frac{\mathbf{t}(k)}{\bar{k}} \right) \bar{\partial}_x \overline{\mu(x,k)} + \frac{|\mathbf{t}(k)|^2}{4\pi k} i(\mu(x,k)-1) \right. \\
&\qquad \left. + \frac{|\mathbf{t}(k)|^2}{4\pi |k|^2} \overline{\partial_x} \mu(x,k) \right] dk \Bigg| \\
&\leq \left\| \frac{\partial \mathbf{t}(k)}{\partial k} \right\|_{r'} \|\mu(x,k)-1\|_r + \frac{1}{2\pi} \left\| \frac{|\mathbf{t}(k)|^2}{k} \right\|_{r'} \|\mu(x,k)-1\|_r \\
&\quad + \left\| \frac{\partial}{\partial k}\left(\frac{\mathbf{t}(k)}{\bar{k}} \right) \right\|_{r'} \|\partial_x \mu(x,k)\|_r + \frac{1}{4\pi} \left\| \frac{|\mathbf{t}(k)|^2}{|k|^2} \right\|_{r'} \|\partial_x \mu(x,k)\|_r \\
&\leq C \left(\left\| \frac{\partial \mathbf{t}(k)}{\partial k} \right\|_{r'} + \left\| \frac{|\mathbf{t}(k)|^2}{k} \right\|_{r'} + \left\| \frac{\partial}{\partial k}\left(\frac{\mathbf{t}(k)}{\bar{k}} \right) \right\|_{r'} + \left\| \frac{|\mathbf{t}(k)|^2}{|k|^2} \right\|_{r'} \right) \langle x \rangle^{-1}.
\end{aligned}$$

where $C < \infty$. Thus we can conclude that (5.4) is bounded by $C'\langle x \rangle^{-1}$, and the proof is complete.

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References

Ablowitz, M. J., Nachman, A. I. (1986). Multidimensional nonlinear evolution equations and inverse scattering. *Physica D* 18:223–241.

Astala, K., Päivärinta, L. (2006). Calderón's inverse conductivity problem in the plane. *Annals of Mathematics* 163:265–299.

Barceló, J. A., Barceló, T., Ruiz, A. (2001). Stability of the inverse conductivity problem in the plane for less regular conductivities *J. Differ. Equations* 173:231–70.

Beals, R., Coifman, R. R. (1980). Scattering, transformations spectrales et equations d'evolution non lineaires I, II. Seminaire Goulaouic-Meyer-Schwartz 1980–1981, exp. 22, 1981–1982, exp. 21, Palaiseau: Ecole Polytechnique.

Beals, R., Coifman, R. R. (1985). Multidimensional inverse scattering and nonlinear partial differential equations. *AMS* 43:45–70.

Beals, R., Coifman, R. R. (1986). The d-bar approach to inverse scattering and nonlinear evolution equations. *Physica* 18:242–249.

Beals, R., Coifman, R. R. (1988). The spectral problem for the Davey-Stewartson and Ishimori hierarchies. *Nonlinear Evolution Equations: Integrability and Spectral Methods*. Manchester University Press, pp. 15–23.

Beals, R., Coifman, R. R. (1989). Linear spectral problems, non-linear equations and the $\bar{\partial}$-method. *Inverse Problems* 5:87–130.

Boiti, M., Leon, J. P., Manna, M., Pempinelli, F. (1987). On a spectral transform of a KdV-like equation related to the Schrödinger operator in the plane. *Inverse Problems* 3:25–36.

Brown, R. M., Uhlmann, G. (1997). Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. *Communications in Partial Differential Equations* 22(5):1009–1027.

Calderón, A. P. (1980). On an inverse boundary value problem. *Seminar on Numerical Analysis and Its Applications to Continuum Physics*. Soc. Brasileira de Matemàtica, pp. 65–73.

Cheney, M., Isaacson, D., Newell, J. C. (1999). Electrical impedance tomography. *SIAM Review* 41:85–101.

Faddeev, L. D. (1966). Increasing solutions of the Schrödinger equation. *Sov. Phys. Dokl.* 10:1033–1035.

Grinevich, P. G. (1986). Rational solutions of the Veselov–Novikov equations—Two-dimensional potentials that are reflectionless for a given energy. *Teoret. Mat. Fiz.* 69(2):307–310.

Grinevich, P. G. (2000). Scattering transformation at fixed non-zero energy for the two-dimensional Schrödinger operator with potential decaying at infinity. *Russian Math. Surveys* 55(6):1015–1083.

Grinevich, P. G., Novikov, R. G. (1985). Analogs of multisoliton potentials for the two-dimensional Schrödinger operator. *Functional Anal. Appl.* 19:276–285.

Grinevich, P. G., Manakov, S. V. (1986). Inverse scattering problem for the two-dimensional Schrödinger operator, the $\bar{\partial}$-method and nonlinear equations. *Functional Anal. Appl.* 20:94–103.

Grinevich, P. G., Novikov, R. G. (1995). Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials. *Communications in Mathematical Physics* 174:409–446.

Henkin, G. M., Novikov, R. G. (1988). A multidimensional inverse problem in quantum and acoustic scattering. *Inverse Problems* 4:103–121.

Hörmander, L. (1994). *The Analysis of Linear Partial Differential Operators III. Pseudo-Differential Operators*. Springer-Verlag.

Isaacson, D., Mueller, J. L., Newell, J. C., Siltanen, S. (2004). Reconstructions of chest phantoms by the D-Bar method for electrical impedance tomography. *IEEE Transactions on Medical Imaging* 23(7):821–828.

Knudsen, K. (2002). On the Inverse Conductivity Problem. Ph.D. thesis, University of Aalborg, Ph.D. report series 5/2002.

Knudsen, K. (2003). A new direct method for reconstructing isotropic conductivities in the plane. *Physiol. Meas.* 24:391–401.

Knudsen, K., Tamasan, A. (2004). Reconstruction of less regular conductivities in the plane. *Comm. Partial Differential Equations* 29:361–381.

Knudsen, K., Mueller, J. L., Siltanen, S. (2004). Numerical solution method for the dbar-equation in the plane. *J. Comp. Phys.* 198(2):500–517.

Lee, J., Uhlmann, G. (1989). Determining anisotropic real-analytic conductivities by boundary measurements. *Comm. Pure Appl. Math.* 42(8):1097–1112.

Liu, L. (1997). Stability Estimates for the Two-Dimensional Inverse Conductivity Problem. Ph.D. thesis, University of Rochester.

Mueller, J. L., Siltanen, S. (2003). Direct reconstructions of conductivities from boundary measurements. *SIAM Journal of Scientific Computation* 24(4):1232–1266.

Nachman, A. I. (1988). Reconstructions from boundary measurements. *Annals of Mathematics* 128:531–576.

Nachman, A. I. (1993). Global uniqueness for a two-dimensional inverse boundary value problem. University of Rochester, Dept. of Mathematics Preprint Series, No. 19.

Nachman, A. I. (1996). Global uniqueness for a two-dimensional inverse boundary value problem. *Annals of Mathematics* 143:71–96.

Nachman, A. I., Ablowitz, M. J. (1984). A multidimensional inverse-scattering method. *Studies in Applied Mathematics* 71:243–250.

Nizhnik, L. P. (1980). Integration of multidimensional nonlinear equations by the method of the inverse problem. *Sov. Phys. Dokl.* 25(9):706–708.

Novikov, R. G. (1988). Multidimensional inverse spectral problem for the equation $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. *Funkt. Anal. i Pril.* 22(4):11–22 (in Russian); *English transl.: Funct. Anal. and Appl.* 22:263–272.

Novikov, S. P., Veselov, A. P. (1986). Two-dimensional Shrödinger operator: inverse scattering transform and evolutional equations. *Physica D* 18:267–273.

Siltanen, S., Mueller, J., Isaacson, D. (2000). An implementation of the reconstruction algorithm of Nachman A. for the 2-D inverse conductivity problem. *Inverse Problems* 16:681–699. *Erratum Inverse problems* 17:1561–1563.

Sung, L.-Y. (1994). An inverse scattering transform for the Davey-Stewardson II equations, II. *J. Math. Anal. Appl.* 183:289–325.

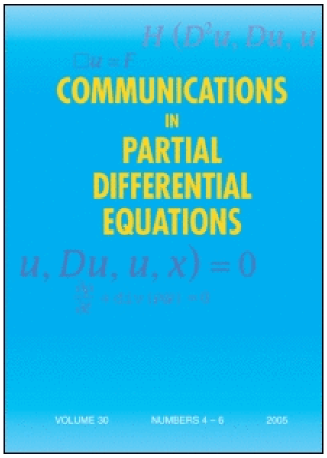
Sylvester, J., Uhlmann, G. (1988). Inverse boundary value problems at the boundary—continuous dependence. *Comm. Pure Appl. Math.* 41(2):197–219.

Tsai, T. (1993). The Schrödinger operator in the plane. *Inverse Problems* 9:763–787.

Tsai, T. (1994). The associated evolution equations of the Schrödinger operator in the plane. *Inverse Problems* 10:1419–1432.

Vekua, I. N. (1962). *Generalized Analytic Functions*. Pergamon Press.

Veselov, A. P., Novikov, S. P. (1984). Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formulas and evolution equations. *Sov. Math. Dokl.* 30:558–591.



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