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Erratum

An implementation of the reconstruction algorithm of A Nachman for the 2D inverse conductivity problem

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In theorem 3.1 of this paper we presented an estimate for $\mathbf{t}(k)$ and $\mu(x, k)$ for k near zero. The statement of theorem 3.1 holds, but the proof contains two errors.

- We incorrectly stated that $S_0 = R_1$ for a general C^2 domain Ω. This led to an erroneous formula (24). We prove below that the identity $S_0 = \frac{1}{2}R_1$ holds when Ω is the unit disc. The proof of theorem 3.1 can then be corrected by reducing it to that case.
- We proved the (correct) estimate $\|\mathcal{H}_k\|_{L(H^{1/2}(\partial\Omega))} \leqslant C|k|$ for small $|k|$. However, in the original proof $\mathcal{H}_k$ is an operator from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$. We provide a new argument showing that the above estimate can be used.

Define the Dirichlet-to-Neumann map corresponding to γ by $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$,

$$\langle \Lambda_\gamma f, h\rangle = \int_\Omega \gamma \nabla u \cdot \nabla v,$$

where v is any $H^1(\Omega)$ function with trace h on the boundary and $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega.$$

Let $R_\gamma : \tilde{H}^{-1/2}(\partial\Omega) \to \tilde{H}^{1/2}(\partial\Omega)$ denote the Neumann-to-Dirichlet map of γ, where $\tilde{H}^s$ spaces consist of H^s functions with mean value zero. We have $R_\gamma g = u|_{\partial\Omega}$, where u is the unique $H^1(\Omega)$ solution of the Neumann problem

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad \gamma \frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega,$$

satisfying $\int_{\partial\Omega} u \mathrm{d}\sigma = 0$. We note two key equalities concerning Λ_γ and R_γ. Define a projection operator $P\phi := |\partial\Omega|^{-1}\int_{\partial\Omega}\phi$. Then for any $f \in H^{1/2}(\partial\Omega)$ we have $P\Lambda_\gamma f = |\partial\Omega|^{-1}\int_{\partial\Omega}\gamma\frac{\partial u}{\partial\nu} = \int_\Omega \nabla\cdot\gamma\nabla u = 0$, so actually $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to \tilde{H}^{-1/2}(\partial\Omega)$. From the definitions of Λ_γ and R_γ we now have

$$\Lambda_\gamma R_\gamma = I \qquad : \tilde{H}^{-1/2}(\partial\Omega) \to \tilde{H}^{-1/2}(\partial\Omega), \tag{1}$$

$$R_\gamma \Lambda_\gamma = I - P \qquad : H^{1/2}(\partial\Omega) \to \tilde{H}^{1/2}(\partial\Omega). \tag{2}$$

Definition. *Let $S_0 : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ denote the standard single layer operator*

$$(S_0\phi)(x) := \int_{\partial\Omega} \Phi(x, y)\phi(y)\mathrm{d}\sigma(y),$$

where $\Phi(x, y) = G_0(x - y) = -\frac{1}{2\pi}\log|x - y|$.

Lemma. *Let $\Omega \subset \mathbb{R}^2$ be the unit disc and $f \in \tilde{H}^{-1/2}(\partial\Omega)$. Then $S_0 f = \frac{1}{2}R_1 f$.*

Proof. By theorem 3.1 of [1] in dimension 2,

$$(S_0 f)(x) = u(x) + (D_0 u)(x), \qquad x \in \Omega, \tag{3}$$

where u satisfies

$$\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = f \text{ on } \partial\Omega,$$

and

$$(D_0\phi)(x) := \int_{\partial\Omega} \frac{\partial \Phi(x, y)}{\partial \nu(y)} \phi(y) \mathrm{d}\sigma(y)$$

is the double layer potential with $x \in \Omega$.

We let x approach $\partial\Omega$ in (3) using jump relations. First, by theorem 2.13 of [1] $D_0\phi$ can be continuously extended from Ω to $\overline{\Omega}$ with

$$\lim_{x\to\tilde{x}} (D_0\phi)(x) = \int_{\partial\Omega} \frac{\partial \Phi(\tilde{x}, y)}{\partial \nu(y)} \phi(y) \mathrm{d}\sigma(y) - \frac{1}{2}\phi(\tilde{x}), \qquad \tilde{x} \in \partial\Omega, \tag{4}$$

where $x \in \Omega$. Second, $S_0\phi$ is continuous in $\mathbb{R}^2$ by theorem 2.12 of [1].

When Ω is the unit disc, D_0 on the boundary takes the simple form

$$(D_0\phi)(\tilde{x}) = -\frac{1}{4\pi} \int_{\partial\Omega} \phi(y) \mathrm{d}\sigma(y), \qquad \tilde{x} \in \partial\Omega, \tag{5}$$

since

$$-2\pi \frac{\partial \Phi(\tilde{x}, y)}{\partial \nu(y)} = \frac{y - \tilde{x}}{|y - \tilde{x}|^2} \cdot y = \frac{1 - \tilde{x} \cdot y}{(y - \tilde{x}) \cdot (y - \tilde{x})} = \frac{1 - \tilde{x} \cdot y}{2 - 2\tilde{x} \cdot y} = \frac{1}{2}.$$

The lemma follows now from (3), (4) and (5):

$$\begin{aligned}(S_0 f)|_{\partial\Omega} &= u|_{\partial\Omega} + \int_{\partial\Omega} \frac{\partial \Phi(\tilde{x}, y)}{\partial \nu(y)} u(y) \mathrm{d}\sigma(y) - \frac{1}{2} u|_{\partial\Omega} \\ &= \frac{1}{2}\left(u|_{\partial\Omega} - \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u(y) \mathrm{d}\sigma(y)\right) = \frac{1}{2} R_1 f.\end{aligned}$$

□

We are now ready to re-prove theorem 3.1 of the original paper.

Proof. With no loss of generality we can assume that Ω is the unit disc $D(0, 1) \subset \mathbb{R}^2$. Namely, by theorem 3.3 of the original paper, replacing γ by $\tilde{\gamma}(x) := \gamma(\lambda x)$ with large enough $\lambda > 0$ yields $\operatorname{supp}(\tilde{\gamma} - 1) \subset D(0, 1)$ and $\tilde{\boldsymbol{t}}(k) = \boldsymbol{t}(k/\lambda)$. Thus the estimate $|\boldsymbol{t}(k)| \leqslant C|k|^2$ holds for $|k| \leqslant \varepsilon$ if and only if $|\tilde{\boldsymbol{t}}(k)| \leqslant \tilde{C}|k|^2$ holds for $|k| \leqslant \lambda\varepsilon$. Furthermore, by chapter 6 of [2] it is equivalent to work on $\partial D(0, 1)$ or on $\partial\Omega'$ for any $\Omega' \subset D(0, 1)$.

Write using the above lemma and equation (2)

$$A_\gamma := I + S_0(\Lambda_\gamma - \Lambda_1) = \frac{1}{2}(I + R_1\Lambda_\gamma + P). \tag{6}$$

The operator $\Lambda_\gamma - \Lambda_1$ is infinitely smoothing (and thus compact from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$) since $\gamma \equiv 1$ in a neighbourhood of $\partial\Omega$. By the Fredholm alternative, injectivity of $I + R_1\Lambda_\gamma + P$ implies invertibility of A_γ. Assume

$$(I + R_1\Lambda_\gamma + P)\phi = 0 \tag{7}$$

for some $\phi \in H^{1/2}(\partial\Omega)$. Integrating (7) over $\partial\Omega$ gives $P\phi = 0$. This yields $\phi = -R_1\Lambda_\gamma\phi$, and using equation (1) we get

$$\Lambda_1\phi = -\Lambda_1 R_1 \Lambda_\gamma \phi = -\Lambda_\gamma \phi.$$

Let $u, \tilde{u} \in H^1(\Omega)$ be weak solutions of $\Delta u = 0$ and $\nabla \cdot \gamma \nabla \tilde{u} = 0$ in Ω with trace ϕ. Then

$$0 \leqslant \int_\Omega |\nabla u|^2 = \langle \Lambda_1 \phi, \phi \rangle = -\langle \Lambda_\gamma \phi, \phi \rangle = -\int_\Omega \gamma |\nabla \tilde{u}|^2 \leqslant 0.$$

Clearly u must be constant and since $P(u|_{\partial\Omega}) = P\phi = 0$, this constant is zero.

By (6) we have $A_\gamma 1 = 1$ and thus $A_\gamma^{-1} 1 = 1$.

Since $\Lambda_\gamma - \Lambda_1$ is infinitely smoothing we can treat it as an element of $L(H^{1/2}(\partial\Omega))$ and the estimate $\|\mathcal{H}_k\|_{L(H^{1/2}(\partial\Omega))} \leqslant C|k|$ can again be used as in the original paper to prove the theorem. Note that the last displayed formula before formula (25) in the original paper should be

$$\begin{aligned}\psi|_{\partial\Omega} &= [I + S_k(\Lambda_\gamma - \Lambda_1)]^{-1} \mathrm{e}^{\mathrm{i}kx} \\ &= 1 + A_\gamma^{-1}(\mathrm{e}^{\mathrm{i}kx} - 1) \\ &\quad - A_\gamma^{-1} \mathcal{H}_k(\Lambda_\gamma - \Lambda_1)[I + A_\gamma^{-1} \mathcal{H}_k(\Lambda_\gamma - \Lambda_1)]^{-1} \mathrm{e}^{\mathrm{i}kx}.\end{aligned}$$

□

References

[1] Colton D and Kress R 1983 *Integral Equation Methods in Scattering Theory* (New York: Wiley)

[2] Nachman A I 1996 Global uniqueness for a two-dimensional inverse boundary value problem *Ann. Math.* **143** 71–96