Noise-robust detection of inclusions and obstacles from partial data

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http://wiki.helsinki.fi/display/inverse/Home
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Outline

Electrical impedance tomography (EIT)

Regularization

The enclosure method of Ikehata

Generalizations of the enclosure method
Electrical impedance tomography (EIT) is an emerging medical imaging technique.
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón.

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases}
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
u|_{\partial \Omega} &= f.
\end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \mathbf{n}}|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_\sigma$. It is a nonlinear and ill-posed inverse problem.
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Generalizations of the enclosure method
The forward map $F : X \supset \mathcal{D}(F) \rightarrow Y$ of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.
A regularization strategy needs to be constructed so that the assumptions below are satisfied.

A family $\Gamma_\alpha : Y \to X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a regularization strategy for $F$ if

$$\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\sigma) - \sigma \|_X = 0$$

for each fixed $\sigma \in \mathcal{D}(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called admissible if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $\sigma \in \mathcal{D}(F)$ the following holds:

$$\sup_{\Lambda_\delta} \left\{ \| \Gamma_{\alpha(\delta)}(\Lambda_\delta) - \sigma \|_X : \| \Lambda_\delta - \Lambda_\sigma \|_Y \leq \delta \right\} \to 0 \text{ as } \delta \to 0.$$
There is currently only one regularized method for reconstructing the full conductivity distribution:

BIE $\rightarrow$ Lowpass $\rightarrow$ Nonlinear IFFT

[Practical measurement]

[Nachman 1996]
[S, Mueller & Isaacson 2000]
[Knudsen, Lassas, Mueller & S 2009]
However, the D-bar method gives blurred reconstructions and is not ideal for inclusions.

There are several computational methods for recovering inclusions robustly from localized EIT data. Here is an incomplete overview:

1997 Kang, Seo, Sheen
1998 Probe method Ikehata, Daido, Nakamura
1998 Factorization method Kirsch, Brühl, Hähner, Hakula, Hanke, Harrach, Hyvönen, Lechleiter, Schappel, Seo
1998 Point source method Potthast, Erhard
1998 Size estimation Alessandrini, Bilotta, Morassi, Rosset, Turco, Seo
1999 Linearization Isaacson, Mueller, Newell
2000 Enclosure method and its variants Ikehata and others
2004 Polarization tensors Ammari, Capdeboscq, Griesmaier, Hanke, Kang, Kwon, Seo, Vogelius, Woo, Yoon
2010 Monotonicity Harrach, Seo
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Generalizations of the enclosure method
We consider the restricted EIT problem of locating inclusions in background conductivity

Let $\sigma = 1 + \chi_D h : \Omega \rightarrow \mathbb{R}^+$. Here $D \subset \Omega$ and $h \in L^\infty(D)$ is such that $\sigma$ is bounded away from zero and has a jump along the inclusion boundary $\partial D$. 

\[\sigma \equiv 1\]

\[\sigma = 1 + h\]

\[\Omega\]
Enclosure method recovers the support function of an inclusion from the DN map.

The support function of $D$ with respect to $\omega \in S^1$ is

$$h_D(\omega) := \sup_{x \in D} x \cdot \omega.$$ 

Take $\omega^\perp$ so that $\omega \cdot \omega^\perp = 0$. For $x \in \partial \Omega$ and $\tau > 0$ set

$$f_\omega(x, \tau) := e^{\tau x \cdot \omega + i \tau x \cdot \omega^\perp}.$$ 

The indicator function is

$$l_\omega(\tau) = \langle (\Lambda_\sigma - \Lambda_1) \overline{f_\omega(\cdot, \tau)}, f_\omega(\cdot, \tau) \rangle.$$
The support function for many directions yields an approximation of the convex hull of the inclusion. According to Ikehata [1999] and Ikehata & S [2000] we get

\[ h_D(\omega) = \lim_{\tau \to \infty} \frac{\log |I_\omega(\tau)|}{2\tau}. \]
Ikehata [2002] shows that for all small enough $\delta > 0$ it holds that

$$\sup_{0 < \|\Lambda_\sigma - \Lambda_\sigma^\delta\|_{\mathcal{V}} \leq \delta} \left| h_D(\omega) - \frac{\log |I_\omega^\delta(\tau)|}{\tau} \right| = \mathcal{O} \left( \frac{|\log |\log \delta||}{|\log \delta|} \right)$$

as $\delta \to 0$. The parameter $\tau$ is a function of noise level $\delta$:

$$\tau = \tau(\delta) = \frac{1}{4} |\log \delta|.$$
Regularization of the enclosure method: practice

\[ \frac{1}{2} \log |l_\omega(\tau)| \]
Regularization of the enclosure method: practice

\[ \frac{1}{2} \log |l_\omega^\delta(\tau)| \]

Noisy data
Regularization of the enclosure method: practice

\[ \frac{1}{2} \log |l(\tau)| \]
Regularization of the enclosure method: practice

\[ \frac{1}{2} \log |I_\omega^\delta(\tau)| \]

Noisy data
Regularization of the enclosure method: practice

\[ \frac{1}{2} \log |I_\omega(\tau)| \]
Regularization of the enclosure method: practice

\[ \frac{1}{2} \log | l^{\delta}_\omega(\tau) | \]

Noisy data
Reconstruction of inclusion from simulated EIT data with 10% white Gaussian noise added
Outline

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Regularization

The enclosure method of Ikehata

Generalizations of the enclosure method
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<th>Year</th>
<th>Contribution</th>
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<td>2000</td>
<td>Enclosure method for EIT</td>
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<td>Ikehata &amp; S</td>
<td>2000</td>
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<td>Brühl &amp; Hanke</td>
<td>2000</td>
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<td>Ide, Isozaki, Nakata, S &amp; Uhlmann</td>
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<td>Ikehata, Niemi &amp; S</td>
<td>2012</td>
<td>2D inverse obstacle scattering</td>
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Different probing strategies

Half-space

Cone

Spherical
We denote the inclusion set by $\Omega_1$.

The background conductivity is $\sigma_0 \equiv 1$. Assume that $\sigma(x) = \sigma_0(x)$ for $x \notin \Omega_1$ and $\sigma(x) > \sigma_0(x)$ on $\Omega_1$.

Take $x_0$ from the outside of the convex hull of $\Omega$, and let $R > 0$.

Then there exists a smooth solution $u_\tau(x)$ of $\nabla \cdot (\sigma_0(x) \nabla u_\tau(x)) = 0$ on $\Omega$ with the following special properties.
Main theorem of spherical probing, part 2

Hyperbolic transformation: \( y_1 = \frac{x_1^2 + x_2^2 - r^2}{(x_1 + r)^2 + x_2^2}, \quad y_2 = \frac{2x_2r}{(x_1 + r)^2 + x_2^2} \)

\((x_1, x_2)\)-plane: \( u_\tau(x) \)

\((y_1, y_2)\)-plane: \( \exp(-\tau y_1 + i\tau y_2) \)
Main theorem of spherical probing, part 3

Let $f_\tau = u_\tau|_{\partial \Omega}$. There is a $0 < \delta = \delta(\Omega_1, x_0, R)$ such that

$$\langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau, f_\tau \rangle < C e^{-\delta \tau}$$

and

$$\langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau, f_\tau \rangle > C e^{\delta \tau}$$
Practical scheme for detecting inclusions

Take $0 < \tau_1 < \tau_2$.

\[
\langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_{\tau}, f_{\tau}\rangle < Ce^{-\delta \tau}
\]

\[
\langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_{\tau_1}, f_{\tau_1}\rangle > \langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_{\tau_2}, f_{\tau_2}\rangle
\]
Probing with complex spherical waves in the ideal case of infinite-precision data
Spherical probing from ideal and noisy EIT data

<table>
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<th>Conductivity</th>
<th>Maximal region</th>
<th>Reconstruction (ideal data)</th>
<th>Reconstruction (noisy data)</th>
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Spherical probing from ideal and noisy EIT data in 3D (Ide, Isozaki, Nakata & S 2010)
Enclosure method for inverse obstacle scattering (Ikehata, Niemi & S 2012)

Reconstruction from far field pattern with no added noise

Reconstruction from far field pattern with 1% white Gaussian noise
Enclosure method for inverse obstacle scattering
(Ikehata, Niemi & S 2012)

Reconstruction from far field pattern with no added noise

Reconstruction from far field pattern with 1% white Gaussian noise
All Matlab codes freely available on a website!

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Part II: Nonlinear Inverse Problems
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Inverse Days, Dec 11–13, 2013, Inari, Finland

http://inverse-problems.org/id2013/

Organizers:
Maarten de Hoop
Matti Lassas
Markku Lehtinen
Lassi Roininen
S. S.
Gunther Uhlmann
In practice, efficient regularized algorithms are needed for linear and nonlinear inverse problems.

Assume given a forward map $F$ and noisy data $y^\delta$.

An efficient regularized inversion algorithm should compute a numerical approximation to $\Gamma^{\alpha(\delta)}(y^\delta)$ quickly and accurately, where $\Gamma^{\alpha}$ is a regularization strategy with an admissible choice of regularization parameter.

The Tikhonov approach provides efficient regularized inversion algorithms only for linear and almost linear forward maps $F$.

Electrical impedance tomography is the only strongly nonlinear inverse problem with efficient regularized inversion algorithms, based on the problem-specific approach.

Let us emphasize the difference between stability analysis and regularization strategies.

Conditional stability results have the form

$$\|x - x'\|_X \leq f(\|y - y'\|_Y),$$

where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying $f(0) = 0$. However, the above inequality is practically irrelevant: the noisy measurement $y^\delta$ is almost surely not in the range of $F$. 

![Diagram showing the mapping $F(D(F))$ from $X$ to $Y$ with $y^\delta$ outside the range of $F$.](image)
The observed radii are better (=larger) than those given by the theoretical formula $R(\delta) = -\frac{1}{10} \log \delta$. 

\[ R(\delta) = -\frac{1}{10} \log \delta \]
Why is Calderón’s problem nonlinear?

Define a quadratic form $P_\sigma$ for functions $f : \partial \Omega \rightarrow \mathbb{R}$ by

$$P_\sigma(f) = \int_\Omega \sigma |\nabla u|^2 \, dz,$$

where $u$ is the solution of the Dirichlet problem

$$\begin{cases} 
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
\quad u|_{\partial \Omega} &= f.
\end{cases}$$

Now the map $\sigma \mapsto P_\sigma$ is nonlinear because $u$ depends on $\sigma$ in (1). Physically, $P_\sigma(f)$ is the power needed for maintaining the voltage potential $f$ on the boundary $\partial \Omega$. Integrate by parts in (1):

$$P_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) \, ds = \int_{\partial \Omega} f \left( \Lambda_\sigma f \right) \, ds.$$

Thus the map $\sigma \mapsto \Lambda_\sigma$ cannot be linear in $\sigma$. 
We define spaces for our regularization strategy

Model space $X = L^\infty(\Omega)$

Data space $Y$

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \rightarrow \mathbb{R}$ with

- $\|\sigma\|_{C^2(\overline{\Omega})} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators $A : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ satisfying

- $A(1) = 0$,
- $\int_{\partial\Omega} A(f) \, ds = 0$. 
Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

*There exists a constant $0 < \delta_0 < 1$, depending only on $M$ and $\rho$, with the following properties. Let $\sigma \in D(F)$ be arbitrary and assume given noisy data $\Lambda_\delta^\sigma$ satisfying

$$\|\Lambda_\delta^\sigma - \Lambda_\sigma\|_Y \leq \delta < \delta_0.$$*

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_\alpha(\delta)(\Lambda_\delta^\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(- \log \delta)^{-1/14}.$$