Electrical impedance tomography imaging via the Radon transform

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Seminar on analysis and geometry
Aalto University, April 19, 2017
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Outline

Electrical impedance tomography (EIT) and stroke

The D-bar method for EIT

Complex geometric optics solutions and the Fourier trick

CGO-based inclusion detection
Motivation: imaging stroke with EIT

Ischemic stroke: lower conductivity.

Hemorrhagic stroke: higher conductivity.

Same symptoms!
MRI image from (Hellerhoff 2010)

Difficulties: resistive skull layer and unknown background. However, see
- Holder 1992
- Romsauerova, McEwan, Horesh, Yerworth, Bayford & Holder 2006
- Shi, You, Xu, Wang, Fu, Liu & Dong 2009
- Bayford & Tizzard 2012
- Malone, Jehl, Arridge, Betcke & Holder 2014
The idea would be to equip every ambulance with an EIT device for classifying strokes.
We consider three simulated 2D stroke phantoms: here healthy brain.
We consider three simulated 2D stroke phantoms: here ischemic stroke
We consider three simulated 2D stroke phantoms: here hemorrhagic stroke
New result: inverse scattering methods can transform EIT into “X-ray tomography”

Video:

https://www.youtube.com/watch?v=37yOcfBfRJk

[Greenleaf, Lassas, Santacesaria, Siltanen and Uhlmann 2016]
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases} 
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
\left. u \right|_{\partial \Omega} &= f.
\end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \left. \frac{\partial u}{\partial \vec{n}} \right|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_\sigma$. It is a nonlinear and ill-posed inverse problem.
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There exists a nonlinear Fourier transform adapted to electrical impedance tomography
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

\[ \Lambda_{\sigma} \xrightarrow{\text{BIE}} \text{Ideal measurement} \xrightarrow{\text{Nonlinear IFFT}} \text{Image} \]

[Nachman 1996]
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies.
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
The D-bar method is a regularization strategy for reconstructing the full conductivity distribution.

[S, Mueller & Isaacson 2000]
[Knudsen, Lassas, Mueller & S 2009]
**Infinite-precision data:**

<table>
<thead>
<tr>
<th>Solve boundary integral equation</th>
<th>Practical data:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\cdot, k)</td>
<td><em>{\partial\Omega} = e^{ikz} - S_k(\Lambda</em>\sigma - \Lambda_1)\psi$</td>
</tr>
<tr>
<td>for every complex number $k \in \mathbb{C} \setminus 0$.</td>
<td>$\psi^\delta(\cdot, k)</td>
</tr>
<tr>
<td>Evaluate the scattering transform:</td>
<td>for all $0 &lt;</td>
</tr>
<tr>
<td>$t(k) = \int_{\partial\Omega} e^{ikz}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) , ds.$</td>
<td>Fix $</td>
</tr>
<tr>
<td>Fix $z \in \Omega$. Solve D-bar equation</td>
<td>$t^\delta_R(k) = \int_{\partial\Omega} e^{ikz}(\Lambda_\delta - \Lambda_1)\psi^\delta(\cdot, k) , ds$.</td>
</tr>
<tr>
<td>$\frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \frac{1}{\mu(z, k)}$</td>
<td>Fix $z \in \Omega$. Solve D-bar equation</td>
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<td>with $\mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$.</td>
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</tr>
<tr>
<td>with $\mu^\delta_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$.</td>
<td>Set $\Gamma_{1/R(\delta)}(\Lambda^\delta_\sigma) := (\mu^\delta_R(z, 0))^2$.</td>
</tr>
<tr>
<td>Reconstruct: $\sigma(z) = (\mu(z, 0))^2$.</td>
<td></td>
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</table>
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.

Model space $X = L^\infty(\Omega)$

Data space $Y = L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$

Regularization must be based on combining the incomplete measurement data with a priori information about the conductivity.
Nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

Fix a conductivity $\sigma \in D(F)$. Assume given noisy data $\Lambda^\varepsilon_\sigma$ satisfying

$$\|\Lambda^\varepsilon_\sigma - \Lambda_\sigma\|_Y \leq \varepsilon.$$

Then $\Gamma_\alpha$ with the choice

$$R(\varepsilon) = -\frac{1}{10} \log \varepsilon, \quad \alpha(\varepsilon) = \frac{1}{R(\varepsilon)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\varepsilon)}(\Lambda^\varepsilon_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \varepsilon)^{-1/14}.$$
Regularized reconstructions from simulated data with noise amplitude $\varepsilon = \|\Lambda^{\varepsilon}_\sigma - \Lambda_\sigma\|_Y$

$\varepsilon \approx 10^{-6}$  $\varepsilon \approx 10^{-5}$  $\varepsilon \approx 10^{-4}$  $\varepsilon \approx 10^{-3}$  $\varepsilon \approx 10^{-2}$

The percentages are the relative square norm errors in the reconstructions.
This is a brief history of the two-dimensional regularized D-bar method for EIT

1966 Faddeev: Complex geometric optics (CGO) solutions

1987 Sylvester and Uhlmann: CGO solutions for inverse boundary-value problems; uniqueness for 3D EIT with smooth conductivities and infinite-precision data

1988 R. G. Novikov: Outline of the core ideas of the D-bar method

1996 Nachman: Uniqueness and reconstruction for 2D EIT with $C^2$ conductivities and infinite-precision data

2000 S, Mueller and Isaacson: Numerical implementation of Nachman’s proof using a Born approximation

2006 Isaacson, Mueller, Newell and S: Application of the D-bar method to EIT data measured from a human subject

2009 Knudsen, Lassas, Mueller and S: Regularization proof
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CGO-based inclusion detection
We consider exponentially behaving Complex Geometric Optics (CGO) solutions

Denote \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( k = i\tau \theta \) where

\[
\theta = \theta_1 + i\theta_2 \in \mathbb{C} \quad \text{with} \quad |\theta| = 1.
\]

Let \( z = x_1 + ix_2 \in \mathbb{C} \) and

\[
\eta = \eta_R + i\eta_I = (\theta_1 + i\theta_2, -\theta_2 + i\theta_1) \in \mathbb{C}^2,
\]

so that \( z\theta = x_1\theta_1 - x_2\theta_2 + i(x_1\theta_2 + x_2\theta_1) = x \cdot \eta. \) Note that \( \eta \cdot \eta = 0. \) We consider solutions of the conductivity equation

\[
\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \ \Omega,
\]

with a strictly positive conductivity \( \sigma \in L^\infty(\Omega), \) of the form

\[
u(x) = e^{i\tau \theta z} w(x, \tau) = e^{i\tau x \cdot \eta} w(x, \tau).\]
Since \( u(x) = e^{i\tau \eta \cdot x} w(x, \tau) \) satisfies the conductivity equation,

\[
0 = \frac{1}{\sigma(x)} \nabla \cdot (\sigma(x) \nabla u(x)) = \left( \Delta + \frac{1}{\sigma} (\nabla \sigma) \cdot \nabla \right) (e^{i\tau \eta \cdot x} w(x, \tau))
\]

\[
= \left( \Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta) w(x, \tau) \right) e^{i\tau \eta \cdot x}.
\]

Hence, we have

\[
\Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta) w(x, \tau) = 0.
\]
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

Let \( \hat{w}(x, t) = \mathcal{F}_{\tau \to t}(w(x, \tau)) \) be the Fourier transform of \( w(x, \tau) \) in the \( \tau \) variable, that is,

\[
\hat{w}(x, t) = \mathcal{F}w(x, t) = \int_{\mathbb{R}} e^{-it\tau} w(x, \tau) \, d\tau.
\]

We say that \( t \) is the pseudo-time corresponding to the complex frequency \( \tau \). Then equation

\[
\Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta) w(x, \tau) = 0.
\]

yields

\[
\Delta \hat{w}(x, t) + 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + \eta \frac{\partial}{\partial t}) \hat{w}(x, t) = 0.
\]
Complex principal type operator leads to singularities propagating along leaves

In the equation

\[ \Delta \hat{w}(x, t) - 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \frac{1}{\sigma} (\nabla \sigma) \cdot (\nabla - \eta \frac{\partial}{\partial t}) \hat{w}(x, t) = 0 \]

the principal part is

\[ \Delta - 2\eta \frac{\partial}{\partial t} \cdot \nabla, \]

which is a complex principal type operator in the sense of Duistermaat and Hörmander.

For a real principal type operator the characteristic singularities propagate along one-dimensional rays. For instance, for the wave equation the light-like singularities propagate along light rays.

For a complex principal type operator the characteristic singularities propagate along two dimensional surfaces, called leaves.
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
This project started in 2000
Original conductivity

|F(ψ(1,k)−e^{ik})(t)|

Profile of reconstruction

1 .9
−3 −1 1 −1 −0.75 0.75 1
1 .9
−3 −1 1 −1 0.25 0.75 1
Matteo Santacesaria joined the project in 2014
We use the Beltrami-type complex geometric optics (CGO) solutions

Set $\mu := (1 - \sigma)/(1 + \sigma)$. Write $f = u + iv$ and note that

$$\overline{\partial}_z f_\mu = \mu \overline{\partial}_z f_\mu \Leftrightarrow \nabla \cdot \sigma \nabla u = 0 \text{ and } \nabla \cdot \sigma^{-1} \nabla v = 0.$$

The CGO solutions of [Astala-Päivärinta 2006] have the form

$$f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)),$$

$$f_{-\mu}(z, k) = e^{ikz}(1 + \omega^-(z, k)),$$

with the asymptotic condition

$$\omega^\pm(z, k) = \mathcal{O}\left(\frac{1}{|z|}\right) \text{ as } |z| \to \infty.$$

Here $ikz = i(k_1 + ik_2)(x + iy)$ and $\overline{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$. 
This is a brief history of computational solution methods for the Beltrami CGO solutions

2006 Astala and Päivärinta: Construction

2010 Astala, Mueller, Päivärinta and S: First numerical solution method

2011 Astala, Mueller, Päivärinta, Perämäki and S: Novel EIT reconstruction method

2012 Huhtanen and Perämäki: Preconditioned Krylov subspace method for real-linear systems

2014 Astala, Päivärinta, Reyes and S: Computational high-frequency experiments
Comparison between the transport matrix approach and the shortcut method

[R = 50]

[Astala, Päivärinta, Reyes & S 2014]
Convergence of the shortcut method: nonlinear Gibbs phenomenon

[Astala, Päivärinta, Reyes & S 2014]
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

\[ f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)), \]

write parameter \( k \) in polar form:

\[ k = \tau e^{i\varphi} \]

with \( \tau \in \mathbb{R} \). Denote \( \omega^+(x, t, e^{i\varphi}) = \omega^+(z, k) \).

Fourier transform \( \omega^+(x, \tau, e^{i\varphi}) \) in the \( \tau \) variable:

\[ \hat{\omega}^+(x, t, e^{i\varphi}) = \mathcal{F}_{\tau \rightarrow t}(\omega^+(x, t, e^{i\varphi})) = \int_{-\infty}^{\infty} e^{-it \tau} \omega^+(x, \tau, e^{i\varphi}) d\tau. \]

We say that \( t \) is the “pseudo-time” corresponding to the complex frequency \( \tau \).
Singularities in \( \hat{\omega}^+(1, t/2, 1) \)
Singularities in $\hat{\omega}^+(1, t/2, 1) - \hat{\omega}^-(1, t/2, 1)$
We can define an averaging operator

**Definition.** (Greenleaf, Lassas, Santacesaria, S and Uhlmann) Define operator $T^{a,\pm}$ by complex contour integral:

$$
T^{a,\pm}(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial \Omega} \hat{\omega}^{\pm}(z, t, e^{i\varphi}) dz.
$$

In the case of $\Omega$ being the unit disc, we get

$$
T^{a,\pm}(t, e^{i\varphi}) = \frac{1}{2\pi} \int_{0}^{2\pi} \hat{\omega}^{\pm}(z, t, e^{i\varphi}) e^{i\varphi} d\varphi.
$$
Singularities in $T^{a,+} \mu(t/2, 1) - T^{a,-} \mu(t/2, 1)$
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Inclusion detection algorithms for EIT: review


**Convex source support.** Hanke-Hyvönen-Reusswig 2008.


Recall the Beltrami equation

\[ \overline{\partial}_z f_\mu = \mu \overline{\partial}_z f_\mu, \quad f_\mu(z, k) = e^{ikz}(1 + \omega(z, k)), \]

where \( \mu := (1 - \sigma)/(1 + \sigma) \) and

\[ \omega(z, k) = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty. \]

[Huhtanen-Perämäki 2012]: note that \( \omega \) satisfies the equation

\[ \overline{\partial}\omega - \nu \overline{\partial}\omega - \alpha \omega - \alpha = 0, \]

where the derivatives are taken with respect to \( z \).
Modified construction of CGO solutions

Note that $\bar{\partial} \omega = \nu \bar{\partial} \omega + \alpha \bar{\omega} + \alpha$ shows that $\bar{\partial} \omega$ is supported in $\Omega$.

Define $u \in L^p(\Omega)$ by $\bar{u} = -\bar{\partial} \omega$. Then $\omega = -P \bar{u}$ and $\partial \omega = -S \bar{u}$. Substituting $u$ leads to $-\bar{u} - \nu(-S \bar{u}) - \alpha(-P \bar{u}) = \alpha$ and further to

$$u + (-\nu S - \alpha P) \bar{u} = -\bar{\alpha}.$$ 

Denote complex conjugation by $\bar{f} = \rho(f)$. Then

$$(I + A \rho) u = -\bar{\alpha},$$

where $A := (-\nu S - \alpha P)$. By [Huhtanen-Perämäki 2012] we know that $I + A \rho$ is invertible in $L^p(\Omega)$. 
The key to singularity propagation is this Neumann series

We can write a series for \( u \):

\[
u = \sum_{n=0}^{\infty} u_n, \quad u_0 = -\alpha, \quad u_{n+1} = -A\overline{u_n}.
\]

For \( \omega \) we then get

\[
\omega = -\overline{\partial_z^{-1} u} = \sum_{n=0}^{\infty} \omega_n, \quad \omega_n = -\overline{\partial_z^{-1} u_n}.
\]

Next we consider the term corresponding to single scattering,

\[
u_0 = -\alpha, \quad \text{and} \quad \omega_0 = \overline{\partial_z^{-1} \alpha},
\]

The term \( \omega_0(z)|_{z \in \partial\Omega} \) determines singularities of \( \mu \).

The terms \( \omega_n(z)|_{z \in \partial\Omega}, n \geq 1 \) contribute \text{“multiple scattering,”} which explain artifacts in numerics.
Recovery by “filtered back-projection”

**Theorem.** (Greenleaf, Lassas, Santacesaria, S and Uhlmann) Define averaged operators $T_{j}^{a,\pm}$ for $j = 0, 1, 2, 3, \ldots$ by the complex contour integral:

\[
T_{j}^{a,\pm} \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial \Omega} \hat{\omega}_{j}^{\pm}(z, t, e^{i\varphi}) dz,
\]

Then we have

\[
(-\Delta)^{-1/2} (T_{0}^{a,\pm})^* T_{0}^{a,\pm} \mu = \mu.
\]
Let us consider a rotationally symmetric conductivity with jump along the circle $|z| = 0.2$. 

$z = 1 \quad k = \tau e^{i\varphi} \quad \varphi = 0$
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
Here are the two lowest-order terms in the scattering series (after subtracting artefacts)
$T_{0, \pm}^a \mu(t, 1)$

$T_{2, \pm}^a \mu(t, 1)$
Reconstruction algorithm

**Step 1.** Given the measurement $\Lambda_\sigma$, follow [Astala, Mueller, Päivärinta, Perämäki & S 2011] to compute both $\omega^+(x,k)$ and $\omega^-(x,k)$ for $x \in \partial \Omega$ by solving the boundary integral equation derived in [Astala & Päivärinta 2006].

*Note:* In practice this can only be done in a disc $|k| < R$ with $R$ depending on measurement noise amplitude.

**Step 2.** Write $k = \tau e^{i\phi}$ and compute the partial Fourier transform to get $\hat{\omega}^\pm(z,t,e^{i\phi})$.

*Note:* In practice the Fourier transform needs to be windowed.

**Step 3.** Reconstruct $\sigma = (\mu - 1)/(\mu + 1)$ approximately as $(\tilde{\mu} - 1)/(\tilde{\mu} + 1)$ using formula $\tilde{\mu} = (\tilde{\mu}^+ - \tilde{\mu}^-)/2$ with

$$\tilde{\mu}^\pm := \Delta^{-1/2}(T_{0,a}^{a,\pm})^* T^{a,\pm}_a \mu.$$
Brain hemorrhage example: ideal data
Brain hemorrhage example: more realistic data
Conclusion

We propose a novel method for detecting inclusions in EIT. Main benefits of the approach:

▶ Can see inclusions within inclusions.
▶ Works with unknown non-homogeneous background.

Real-data phantom experiments to follow, with open datasets! (Hauptmann, Kolehmainen, Mach, Savolainen & Seppänen)
Thank you for your attention!