Electrical impedance tomography using nonlinear Fourier transform

Samuli Siltanen

Department of Mathematics and Statistics
University of Helsinki, Finland
samuli.siltanen@helsinki.fi
http://www.siltanen-research.net

Mathematical Methods and Modeling of Biophysical Phenomena
Cabo Frio, Rio de Janeiro, Brazil
March 6, 2013
This is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA

Kim Knudsen, Technical University of Denmark

Matti Lassas, University of Helsinki, Finland

Jon Newell, Rensselaer Polytechnic Institute, USA

Jennifer Mueller, Colorado State University, USA
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT: overview

Nonlinear low-pass filtering for EIT: details

Computational example with simulated data

Computational examples with real data
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT: overview

Nonlinear low-pass filtering for EIT: details

Computational example with simulated data

Computational examples with real data
Electrical impedance tomography (EIT) is an emerging medical imaging technique.

Feed electric currents through electrodes. Measure the resulting voltages. Repeat the measurement for several current patterns.

Reconstruct distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient’s inner structure.

EIT is a harmless and painless imaging method suitable for long-term monitoring.
We recently started to study EIT for imaging changes in vocal folds due to excessive voice use.

Sao Paulo, February 27, 2013

Laukkanen
León
Lima
Liu
Moura
Seppänen
S
The most promising use of EIT is detection of breast cancer in combination with mammography ACT4 and mammography devices. Radiolucent electrodes.

Cancerous tissue is up to four times more conductive than healthy breast tissue [Jossinet 1998]. The above experiment by David Isaacson’s team measures 3D X-ray mammograms and EIT data at the same time.
Which of these three breasts have cancer?
Spectral EIT can detect cancerous tissue

[Kim, Isaacson, Xia, Kao, Newell & Saulnier 2007]
This talk concentrates on applications of EIT to chest imaging

Applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
This is a phantom experiment from 2004 for giving you an idea of how EIT works.

Saline and agar phantom

Reconstruction

[Isaacson, Mueller, Newell & S 2004]
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT: overview

Nonlinear low-pass filtering for EIT: details

Computational example with simulated data

Computational examples with real data
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases}
\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \; \Omega , \\
u|_{\partial \Omega} = f .
\end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega} .$$

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_\sigma$. It is a nonlinear and ill-posed inverse problem.
Why is Calderón’s problem nonlinear?

Define a quadratic form $\mathcal{P}_\sigma$ for functions $f : \partial \Omega \rightarrow \mathbb{R}$ by

$$\mathcal{P}_\sigma(f) = \int_{\Omega} \sigma |\nabla u|^2 \, dz,$$

(1)

where $u$ is the solution of the Dirichlet problem

\[
\begin{cases}
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
\quad u |_{\partial \Omega} = f.
\end{cases}
\]

Now the map $\sigma \mapsto \mathcal{P}_\sigma$ is nonlinear because $u$ depends on $\sigma$ in (1). Physically, $\mathcal{P}_\sigma(f)$ is the power needed for maintaining the voltage potential $f$ on the boundary $\partial \Omega$. Integrate by parts in (1):

$$\mathcal{P}_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) ds = \int_{\partial \Omega} f \left( \Lambda_\sigma f \right) ds.$$

Thus the map $\sigma \mapsto \Lambda_\sigma$ cannot be linear in $\sigma$. 
We illustrate the ill-posedness of Calderón’s problem using a simulated example.
We apply the voltage distribution \( f(\theta) = \cos \theta \) at the boundary of the two different phantoms \( \sigma_1 \) and \( \sigma_2 \).
The measurement is the distribution of current through the boundary

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \]

\[ \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
The current data are very similar, although the conductivities are quite different.
Let us apply the more oscillatory distribution $f(\theta) = \cos 2\theta$ of voltage at the boundary.
The measurement is again the distribution of current through the boundary.
The current distribution measurements are almost the same
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data

\[ \sigma_1 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \sigma_2 \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
The forward map $F : X ⊃ D(F) \to Y$ of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.
A **regularization strategy** needs to be constructed so that the assumptions below are satisfied

A family \( \Gamma_\alpha : Y \to X \) of continuous mappings parameterized by \( 0 < \alpha < \infty \) is a **regularization strategy** for \( F \) if

\[
\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\sigma) - \sigma \|_X = 0
\]

for each fixed \( \sigma \in \mathcal{D}(F) \).

Further, a regularization strategy with a choice \( \alpha = \alpha(\delta) \) of regularization parameter is called **admissible** if

\[
\alpha(\delta) \to 0 \text{ as } \delta \to 0,
\]

and for any fixed \( \sigma \in \mathcal{D}(F) \) the following holds:

\[
\sup_{\Lambda_\delta} \{ \| \Gamma_{\alpha(\delta)}(\Lambda_\delta) - \sigma \|_X : \| \Lambda_\delta - \Lambda_\sigma \|_Y \leq \delta \} \to 0 \text{ as } \delta \to 0.
\]
There are many EIT reconstruction methods:

**Linearization:** Barber, Bikowski, Brown, Calderón, Cheney, Isaacson, Mueller, Newell

**Iterative regularization:** Dobson, Gehre, Hua, Jin, Kaipio, Kindermann, Kluth, Leitão, Lechleiter, Lipponen, Maass, Neubauer, Rieder, Rondi, Santosa, Seppänen, Tompkins, Webster, Woo

**Bayesian inversion:** Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Seppänen, Somersalo, Vauhkonen, Voutilainen

**Resistor network methods:** Borcea, Druskin, Mamonov, Vasquez

**Layer stripping:** Cheney, Isaacson, Isaacson, Somersalo

**D-bar methods:** Astala, Bikowski, Bowerman, Delbary, Hamilton, Hansen, Herrera, Isaacson, Kao, Knudsen, Lassas, Montoya, Mueller, Murphy, Nachman, Newell, Päivärinta, Perämäki, Saulnier, S, Tamasan, Tamminen

**Teichmüller space methods:** Kolehmainen, Lassas, Ola, S

Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT: overview

Nonlinear low-pass filtering for EIT: details

Computational example with simulated data

Computational examples with real data
There exists a nonlinear Fourier transform specially adapted to EIT
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

$\Lambda_\sigma$ 

[BIE] 

Nonlinear IFFT

[Nachman 1996]
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies.
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
The resulting nonlinear EIT algorithm is regularized by the low-pass filter

[S, Mueller & Isaacson 2000]
History of CGO-based methods for real 2D EIT

<table>
<thead>
<tr>
<th>Infinite-precision data</th>
<th>Practical data</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980 Calderón</td>
<td>2008 Bikowski-Mueller</td>
</tr>
<tr>
<td>1987 Sylvester-Uhlmann ((d \geq 3))</td>
<td>2008 Boverman-Isaacson-Kao-Saulnier-Newell</td>
</tr>
<tr>
<td>1988 Nachman</td>
<td>2010 Bikowski-Knudsen-Mueller</td>
</tr>
<tr>
<td>1996 Nachman ((\sigma \in C^2(\Omega)))</td>
<td>2000 S-Mueller-Isaacson</td>
</tr>
<tr>
<td>1997 Liu</td>
<td>2003 Mueller-S</td>
</tr>
<tr>
<td>1997 Brown-Uhlmann ((\sigma \in C^1(\Omega)))</td>
<td>2004 Isaacson-Mueller-Newell-S</td>
</tr>
<tr>
<td>2000 Francini</td>
<td>2007 Murphy-Mueller</td>
</tr>
<tr>
<td>2001 Barceló-Barceló-Ruiz</td>
<td>2012 S-Tamminen</td>
</tr>
<tr>
<td>2003 Astala-Päivärinta ((\sigma \in L^\infty(\Omega)))</td>
<td>2009 Astala-Mueller-Päivärinta-S</td>
</tr>
<tr>
<td>2005 Astala-Lassas-Päivärinta</td>
<td>2011 Astala-Mueller-Päivärinta-Perämäki-S</td>
</tr>
<tr>
<td>2007 Barceló-Faraco-Ruiz</td>
<td></td>
</tr>
</tbody>
</table>
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT: overview

Nonlinear low-pass filtering for EIT: details

Computational example with simulated data

Computational examples with real data
Nachman’s 1996 uniqueness proof in 2D uses complex geometric optics (CGO) solutions

Assume the conductivity $\sigma \in C^2(\Omega)$ satisfies $\sigma(z) \equiv 1$ near $\partial \Omega$. Define a potential $q$ by setting $q(z) \equiv 0$ for $z$ outside $\Omega$ and

$$
q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for } z \in \Omega.
$$

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$
(\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2
$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$
e^{-ikz}\psi(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2),$$

where $\tilde{p} > 2$ and $ikz = i(k_1 + ik_2)(x + iy)$. 
The CGO solutions are constructed using a generalized Lippmann-Schwinger equation.

Define \( \mu(z, k) = e^{-ikz}\psi(z, k) \). Then \((-\Delta + q)\psi = 0\) implies

\[
(\Delta - 4ik\bar{\partial}_z + q)\mu(\cdot, k) = 0, \tag{2}
\]

where the D-bar operator is defined by \( \bar{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \).

A solution of (2) satisfying \( \mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2) \) can be constructed using the Lippmann-Schwinger type equation

\[
\mu = 1 - g_k * (q\mu),
\]

where \( g_k \) satisfies \((-\Delta - 4ik\bar{\partial}_z)g_k = \delta \) and is defined by

\[
g_k(z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} \, d\xi_1 d\xi_2.
\]
The Faddeev fundamental solution $g_1(z)$ has a logarithmic singularity at $z = 0$

It is enough to know $g_1(z)$ because of the relation $g_k(z) = g_1(kz)$. 
One of the breakthroughs in Nachman’s 1996 article is showing uniqueness of $\mu$

A solution of $(-\Delta - 4i k \bar{\partial}_z + q)\mu(\cdot, k) = 0$ satisfying $\mu(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2)$ can be constructed using the formula

$$\mu - 1 = [I + g_k * (q \cdot)]^{-1}(g_k * q),$$

provided that the inverse operator exists.

Now $q \in L^p(\mathbb{R}^2)$ with $1 < p < 2$ and $1/\tilde{p} = 1/p - 1/2$, and

$$q \cdot : W^{1, \tilde{p}}(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \text{ is compact,}$$

$$g_k * : L^p(\mathbb{R}^2) \to W^{1, \tilde{p}}(\mathbb{R}^2) \text{ is bounded.}$$

Thus $I + g_k * (q \cdot) : W^{1, \tilde{p}}(\mathbb{R}^2) \to W^{1, \tilde{p}}(\mathbb{R}^2)$ is Fredholm of index zero, and Nachman proved injectivity for all $k \neq 0$. 
The conductivity $\sigma$ can be recovered from the functions $\mu(z, k)$ at $k = 0$

Recall that

$(-\Delta - 4ik\overline{\partial}_z + q)\mu(\cdot, k) = 0$

with the asymptotics

$\mu(z, k) - 1 \in W^{1,p}(\mathbb{R}^2)$. \hspace{1cm} (1)

Substituting $k = 0$ gives

$(-\Delta + \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}})\mu(\cdot, 0) = 0$, \hspace{1cm} (3)

and setting $\mu(z, 0) = \sqrt{\sigma(z)}$ gives the unique solution of (3) satisfying $\mu(z, 0) - 1 \in W^{1,p}(\mathbb{R}^2)$. 

The crucial intermediate object in the proof is the non-physical scattering transform \( t(k) \)

We denote \( z = x + iy \in \mathbb{C} \) or \( z = (x, y) \in \mathbb{R}^2 \) whenever needed. The scattering transform \( t : \mathbb{C} \to \mathbb{C} \) is defined by

\[
t(k) := \int_{\mathbb{R}^2} e^{i\bar{k}z} q(z) \psi(z, k) \, dx \, dy.
\] (4)

Sometimes (4) is called the nonlinear Fourier transform of \( q \). This is because asymptotically \( \psi(z, k) \sim e^{ikz} \) as \( |z| \to \infty \), and substituting \( e^{ikz} \) in place of \( \psi(z, k) \) into (4) results in

\[
\int_{\mathbb{R}^2} e^{i(kz + \bar{k}z)} q(z) \, dx \, dy = \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx \, dy
\]

\[
= \hat{q}(-2k_1, 2k_2).
\]
Alessandrini’s equation gives a way to write $t$ in terms of $\Lambda_\sigma$ and traces of the CGO solutions

The following boundary integral equation is a Fredholm equation of the second kind and uniquely solvable in the space $H^{1/2}(\partial \Omega)$:

$$
\psi(\cdot, k)|_{\partial \Omega} = e^{ikz}|_{\partial \Omega} - S_k(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k).
$$

Here $S_k$ is the single-layer operator with Faddeev Green’s function:

$$(S_k \phi)(z) := \int_{\partial \Omega} G_k(z - \zeta)\phi(\zeta)\,ds(\zeta),$$

where $G_k(z) := e^{ikz}g_k(z)$ satisfies $-\Delta G_k = \delta$.

The scattering transform can be evaluated by

$$
t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k)\,ds.
$$
The difference between the usual Green’s function and Faddeev Green’s function is exponential

Usual: \( G_0(z) = -\frac{1}{2\pi} \log|z| \)

Faddeev: \( G_1(z) = e^{iz}g_1(z) \)
The functions $\mu$ can be recovered from the scattering transform $t$ using a D-bar equation

It is natural to ask whether $\mu(z, k)$ depends analytically on the parameter $k$. If it does, the D-bar operator

$$\frac{\partial}{\partial k} = \frac{1}{2} \left( \frac{\partial}{\partial k_1} + i \frac{\partial}{\partial k_2} \right)$$

will give zero when applied to $\mu(z, k)$.

It turns out that the $\bar{k}$-differential of $g_k*$ is a rank-one operator, and differentiating $\mu = 1 - g_k * (q\mu)$ yields

$$\frac{\partial}{\partial k} \mu(z, k) = \frac{1}{4\pi k} t(k) e^{-i(kz + \bar{k}\bar{z})} \mu(z, k).$$

Thus the dependence of $\mu(z, k)$ on $k$ is not analytic. The D-bar equation was discovered by Beals and Coifman in the 1980’s.
These are the steps of Nachman’s 1996 proof:

Solve boundary integral equation

$$\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi$$

for every complex number $k \in \mathbb{C} \setminus 0$.

---

Evaluate the scattering transform:

$$t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds.$$  

---

Fix $z \in \Omega$. Solve D-bar equation

$$\frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz+\bar{k}z)} \frac{\mu(z, k)}{\mu(z, k)}$$

with $\mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$.

---

Reconstruct: $\sigma(z) = (\mu(z, 0))^2$.
We define spaces for our regularization strategy

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ with

- $\|\sigma\|_{C^2(\overline{\Omega})} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators $A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying

- $A(1) = 0$,
- $\int_{\partial\Omega} A(f) \, ds = 0$. 
### Infinite-precision data:

Solve boundary integral equation

\[ \psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_{\sigma} - \Lambda_1)\psi \]

for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:

\[ t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial k}\mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)}\mu(z, k) \]

with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

### Practical data:

Solve boundary integral equation

\[ \psi^\delta(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_{\sigma}^\delta - \Lambda_1)\psi^\delta \]

for all \( 0 < |k| < R = -\frac{1}{10} \log \delta \).

For \( |k| \geq R \) set \( t_R^\delta(k) = 0 \). For \( |k| < R \)

\[ t_R^\delta(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_{\sigma}^\delta - \Lambda_1)\psi^\delta(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial k}\mu_R^\delta(z, k) = \frac{t_R^\delta(k)}{4\pi k} e^{-i(kz + \bar{k}z)}\mu_R^\delta(z, k) \]

with \( \mu_R^\delta(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma_{1/R(\delta)}(\Lambda_{\sigma}^\delta) := (\mu_R^\delta(z, 0))^2 \).
Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

There exists a constant $0 < \delta_0 < 1$, depending only on $M$ and $\rho$, with the following properties. Let $\sigma \in \mathcal{D}(F)$ be arbitrary and assume given noisy data $\Lambda^\delta_\sigma$ satisfying

$$\|\Lambda^\delta_\sigma - \Lambda_\sigma\|_Y \leq \delta < \delta_0.$$  

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\delta)}(\Lambda^\delta_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(- \log \delta)^{-1/14}.$$
We still need to define the regularization strategy on all of the data space $Y$, not only near $F(\mathcal{D}(F))$.

The previous results show the claim only for operators $\delta_0$-close to the range $F(\mathcal{D}(F)) \subset Y$.

The structure of the set $F(\mathcal{D}(F))$ is not understood at the moment.

However, the proof can be extended to the whole data space $Y$ using spectral-theoretic arguments.
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT: overview

Nonlinear low-pass filtering for EIT: details

Computational example with simulated data

Computational examples with real data
Let us analyze how the regularization works using a simulated heart-and-lungs phantom.
Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

$$\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}.$$ 

We invert the linear operator appearing in the equation

$$\psi^\delta(\cdot, k)|_{\partial\Omega} = [I + S_k(\Lambda^\delta - \Lambda_1)]^{-1} e^{ikz}|_{\partial\Omega}$$

as a matrix in \(\text{span}(\{\varphi_n\}_{n=-N}^N)\).

The single-layer operator

$$(S_k\phi)(z) = \int_{\partial\Omega} G_k(z-w)\phi(w) \, ds(w)$$

uses Faddeev’s Green’s function.
This is how the actual scattering transform looks like in the disc $|k| < 10$, computed by knowing $\sigma$.
Scattering transform in the disc $|k| < 10$, here computed from noisy measurement $\Lambda_\sigma^\delta$
Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

The generalization of Vainikko’s method for the D-bar equation is described in [Knudsen, Mueller & S 2004]. The D-bar equation

\[ \frac{\partial}{\partial k} \mu_R = \frac{1}{4\pi k} t_R(k) e^{-i(kz + k\bar{z})} (k) \mu_R, \]

together with the asymptotics \( \mu_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \), can be combined in a generalized Lippmann-Schwinger equation:

\[ \mu_R(z, k) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{t_R(k')}{(k - k')k'} e^{-i(kz + k\bar{z})} (k') \mu_R(z, k') \, dk'_1 dk'_2 \]

\[ = 1 - PT_R \left( \mu_R(z, \cdot) \right) . \]

Here \( P = \overline{\partial_k}^{-1} \) is the solid Cauchy transform.
The trick is to solve a periodic equation whose solution coincides with $\mu_R^\delta(z, k)$ in the disc $|k| < R$.

Take $\epsilon > 0$ and $s = 2R + 3\epsilon$. Let $\eta \in C_0^\infty(\mathbb{R}^2)$ satisfy

$$\eta(k) = \begin{cases} 1 & \text{for } |k| < 2R + \epsilon, \\ 0 & \text{for } |k| \geq 2R + 2\epsilon. \end{cases}$$

Define a $2s$-periodic function $\widetilde{\beta}$:

$$\widetilde{\beta}(k + n2s + im2s) = \frac{\eta(k)}{\pi k},$$

and a periodic Cauchy transform:

$$\widetilde{P}f(k) = \int \widetilde{\beta}(k-k')f(k') \, dk_1' \, dk_2'. $$

Now we consider the equation

$$\phi = 1 - \widetilde{P} \mathcal{T}_R \overline{\phi}.$$
The D-bar equation is not complex-linear, so real and imaginary parts must be written separately.

The grid points are numbered with one index as shown.

Any function $\phi : [-s, s)^2 \to \mathbb{C}$ is represented by a vector of values at the grid points:

$$
\begin{bmatrix}
\text{Re}\phi(z_1) \\
\text{Re}\phi(z_2) \\
\vdots \\
\text{Re}\phi(z_{64}) \\
\text{Im}\phi(z_1) \\
\text{Im}\phi(z_2) \\
\vdots \\
\text{Im}\phi(z_{64})
\end{bmatrix} \in \mathbb{R}^{128}
$$
This is the real-linear operation given to GMRES

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\Rightarrow
\text{FFT} \rightarrow
\begin{bmatrix}
T_{R \phi} & T_{R \phi} & T_{R \phi} & T_{R \phi} \\
T_{R \phi} & T_{R \phi} & T_{R \phi} & T_{R \phi} \\
T_{R \phi} & T_{R \phi} & T_{R \phi} & T_{R \phi} \\
T_{R \phi} & T_{R \phi} & T_{R \phi} & T_{R \phi}
\end{bmatrix}
\Rightarrow
\text{Element-wise multiplication}
\Rightarrow
\text{IFFT}
\Rightarrow
\begin{bmatrix}
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi
\end{bmatrix}
\Rightarrow
\phi - \tilde{P} T_{R \phi}
Regularized reconstructions from simulated data with noise amplitude \( \delta = \| \Lambda^\delta - \Lambda^\sigma \|_Y \)

\( \delta \approx 10^{-6} \) \quad \( \delta \approx 10^{-5} \) \quad \( \delta \approx 10^{-4} \) \quad \( \delta \approx 10^{-3} \) \quad \( \delta \approx 10^{-2} \)

\( R = 6.7 \) \quad 5.9 \quad 4.3 \quad 3.5 \quad 2.5

12\% \quad 12\% \quad 14\% \quad 19\% \quad 52\%

The percentages are the relative square norm errors in the reconstructions.
The observed radii are better (=larger) than those given by the theoretical formula $R(\delta) = -\frac{1}{10} \log \delta$.
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT: overview

Nonlinear low-pass filtering for EIT: details

Computational example with simulated data

Computational examples with real data
The method works for real data as well, including laboratory phantoms and *in vivo* human data.

Saline and agar phantom  
Reconstruction \((R = 4)\)

[Isaacson, Mueller, Newell & S 2004]  
[Montoya 2012]
D-bar reconstruction of *in vivo* chest data

[Isaacson, Mueller, Newell & S 2006]
Unknown boundary shape can be estimated from EIT data using Teichmüller space methods

[Kolehmainen, Lassas, Ola & S 2013]
1. **Tikhonov regularization:** write a penalty functional

\[ \Phi(x) = \| F(x) - y^\delta \|^2_Y + \alpha \| x \|^2_X, \]

and \( \Gamma_\alpha(y^\delta) \) is defined by

\[ \Phi(\Gamma_\alpha(y^\delta)) = \min_{x \in X} \{ \Phi(x) \}. \]

**Pro:** The same code applies to many problems.

**Con:** Repeated solution of direct problem needed.

**Con:** Prone to get stuck in local minima.

See [Bissantz, Burger, Engl, Hanke, Hofmann, Hohage, Justen, Kaltenbacher, Kindermann, Lechleiter, Lu, Mathé, Morozov, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Ramm, Resmerita, Rieder, Scherzer, Seidman, Teschke, Vogel, Yagola]

2. **Problem-specific regularization**

**Pro:** Can deal efficiently with a specific nonlinearity.

**Con:** Each code applies to only one problem.

In practice, efficient regularized algorithms are needed for linear and nonlinear inverse problems.

Assume given a forward map $F$ and noisy data $y^\delta$.

An efficient regularized inversion algorithm should compute a numerical approximation to $\Gamma_{\alpha(\delta)}(y^\delta)$ quickly and accurately, where $\Gamma_{\alpha}$ is a regularization strategy with an admissible choice of regularization parameter.

The Tikhonov approach provides efficient regularized inversion algorithms only for linear and almost linear forward maps $F$.

Electrical impedance tomography is the only strongly nonlinear inverse problem with efficient regularized inversion algorithms, based on the problem-specific approach.

Let us emphasize the difference between stability analysis and regularization strategies.

Conditional stability results have the form
\[ \|x - x'\|_X \leq f(\|y - y'\|_Y), \]
where \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function satisfying \( f(0) = 0 \).
However, the above inequality is practically irrelevant: the noisy measurement \( y^\delta \) is almost surely not in the range of \( F \).