Solution of inverse boundary value problems using nonlinear Fourier transform

Samuli Siltanen

Department of Mathematics and Statistics
University of Helsinki, Finland
samuli.siltanen@helsinki.fi
http://www.siltanen-research.net

Geo-Mathematical Imaging Group
Project Review and Advisory Board Meeting
Chicago, Illinois, USA
April 19, 2013
Finnish Centre of Excellence in Inverse Problems Research

University of Jyväskylä

University of Eastern Finland

Tampere University of Technology

University of Helsinki

http://wiki.helsinki.fi/display/inverse/Home
The first part (EIT) is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA
Kim Knudsen, Technical University of Denmark
Matti Lassas, University of Helsinki, Finland
Jon Newell, Rensselaer Polytechnic Institute, USA
Jennifer Mueller, Colorado State University, USA
The seismic imaging part is a joint work with

Sarah Hamilton, University of Helsinki, Finland
Maarten de Hoop, Purdue University, USA
Janne Tamminen, University of Helsinki, Finland
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT

Computational example with simulated data

Seismic imaging
Electrical impedance tomography (EIT) is an emerging medical imaging technique.

**Feed** electric currents through electrodes. **Measure** the resulting voltages. Repeat the measurement for several current patterns.

**Reconstruct** distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient’s inner structure.

EIT is a harmless and painless imaging method suitable for long-term monitoring.
We recently started to study EIT for imaging changes in vocal folds due to excessive voice use.
This talk concentrates on applications of EIT to chest imaging

Applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
This is a phantom experiment from 2004 for giving you an idea of how EIT works.

Saline and agar phantom

Reconstruction

[Isaacson, Mueller, Newell & S 2004]
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT

Computational example with simulated data

Seismic imaging
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let \( \Omega \subset \mathbb{R}^2 \) be the unit disc and let conductivity \( \sigma : \Omega \to \mathbb{R} \) satisfy

\[
0 < M^{-1} \leq \sigma(z) \leq M.
\]

Applying voltage \( f \) at the boundary \( \partial \Omega \) leads to the elliptic PDE

\[
\begin{aligned}
\nabla \cdot (\sigma \nabla u) &= 0 \text{ in } \Omega, \\
u|_{\partial \Omega} &= f.
\end{aligned}
\]

Boundary measurements are modelled by the Dirichlet-to-Neumann map

\[
\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega}.
\]

Calderón’s problem is to recover \( \sigma \) from the knowledge of \( \Lambda_\sigma \). It is a nonlinear and ill-posed inverse problem.
We illustrate the ill-posedness of Calderón’s problem using a simulated example
We apply the voltage distribution $f(\theta) = \cos \theta$ at the boundary of the two different phantoms.
The measurement is the distribution of current through the boundary.
The current data are very similar, although the conductivities are quite different

\[ \sigma_1 \quad \sigma_2 \]

\[ \frac{\partial u_1}{\partial \vec{n}} \quad \frac{\partial u_2}{\partial \vec{n}} \]
Let us apply the more oscillatory distribution $f(\theta) = \cos 2\theta$ of voltage at the boundary.
The measurement is again the distribution of current through the boundary

\[ \sigma_1 u_1 \quad \sigma_2 u_2 \]

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \quad \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
The current distribution measurements are almost the same
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data

$\sigma_1 \quad \cos \theta \quad \cos 4\theta$

$\sigma_2 \quad \cos 2\theta \quad \cos 5\theta$

$\cos 3\theta \quad \cos 6\theta$
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data.

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
The forward map \( F : X \supset D(F) \rightarrow Y \) of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.
A **regularization strategy** needs to be constructed so that the assumptions below are satisfied.

A family $\Gamma_\alpha : Y \to X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a *regularization strategy* for $F$ if

$$\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\sigma) - \sigma \|_X = 0$$

for each fixed $\sigma \in \mathcal{D}(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called *admissible* if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $\sigma \in \mathcal{D}(F)$ the following holds:

$$\sup_{\Lambda_\delta} \left\{ \| \Gamma_\alpha(\delta) (\Lambda_\sigma) - \sigma \|_X : \| \Lambda_\delta - \Lambda_\sigma \|_Y \leq \delta \right\} \to 0 \text{ as } \delta \to 0.$$
1. Tikhonov regularization: write a penalty functional

\[ \Phi(x) = \| F(x) - y^\delta \|_Y^2 + \alpha \| x \|_X^2, \]

and \( \Gamma_\alpha(y^\delta) \) is defined by \( \Phi(\Gamma_\alpha(y^\delta)) = \min_{x \in X} \{ \Phi(x) \} \).

**Pro:** The same code applies to many problems.

**Con:** Repeated solution of direct problem needed.

**Con:** Prone to get stuck in local minima.

See [Bissantz, Burger, Engl, Hanke, Hofmann, Hohage, Justen, Kaltenbacher, Kindermann, Lechleiter, Lu, Mathé, Morozov, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Ramm, Resmerita, Rieder, Scherzer, Seidman, Teschke, Vogel, Yagola]

2. Problem-specific regularization

**Pro:** Can deal efficiently with a specific nonlinearity.

**Con:** Each code applies to only one problem.

Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT

Computational example with simulated data

Seismic imaging
There exists a nonlinear Fourier transform adapted to electrical impedance tomography.
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

\[ \Lambda \sigma \quad \text{BIE} \quad \text{Ideal measurement} \quad \text{Nonlinear IFFT} \]

[Nachman 1996]
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies.
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
The resulting nonlinear EIT algorithm is regularized by the low-pass filter

[S, Mueller & Isaacson 2000]
## History of CGO-based methods for real 2D EIT

<table>
<thead>
<tr>
<th>Infinite-precision data</th>
<th>Practical data</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1980</strong> Calderón</td>
<td><strong>2008</strong> Bikowski-Mueller</td>
</tr>
<tr>
<td><strong>1987</strong> Sylvester-Uhlmann ($d \geq 3$)</td>
<td><strong>2008</strong> Boverman-Isaacson-Kao-Saulnier-Newell</td>
</tr>
<tr>
<td>1988 Nachman</td>
<td><strong>2010</strong> Bikowski-Knudsen-Mueller</td>
</tr>
<tr>
<td><strong>1996</strong> Nachman ($\sigma \in C^2(\Omega)$)</td>
<td><strong>2000</strong> S-Mueller-Isaacson</td>
</tr>
<tr>
<td>1997 Liu</td>
<td><strong>2003</strong> Mueller-S</td>
</tr>
<tr>
<td></td>
<td><strong>2004</strong> Isaacson-Mueller-Newell-S</td>
</tr>
<tr>
<td></td>
<td><strong>2006</strong> Isaacson-Mueller-Newell-S</td>
</tr>
<tr>
<td></td>
<td><strong>2007</strong> Murphy-Mueller</td>
</tr>
<tr>
<td></td>
<td><strong>2008</strong> Knudsen-Lassas-Mueller-S</td>
</tr>
<tr>
<td></td>
<td><strong>2009</strong> Knudsen-Lassas-Mueller-S</td>
</tr>
<tr>
<td></td>
<td><strong>2012</strong> S-Tamminen</td>
</tr>
<tr>
<td><strong>1997</strong> Brown-Uhlmann ($\sigma \in C^1(\Omega)$)</td>
<td><strong>2001</strong> Knudsen-Tamasan</td>
</tr>
<tr>
<td><strong>2001</strong> Barceló-Barceló-Ruiz</td>
<td><strong>2003</strong> Knudsen</td>
</tr>
<tr>
<td><strong>2000</strong> Francini</td>
<td><strong>2012</strong> Hamilton-Herrera-Mueller-Herrmann</td>
</tr>
<tr>
<td><strong>2010</strong> Beretta-Francini</td>
<td></td>
</tr>
<tr>
<td><strong>2003</strong> Astala-Päivärinta ($\sigma \in L^\infty(\Omega)$)</td>
<td><strong>2009</strong> Astala-Mueller-Päivärinta-S</td>
</tr>
<tr>
<td><strong>2005</strong> Astala-Lassas-Päivärinta</td>
<td><strong>2011</strong> Astala-Mueller-Päivärinta-Perämäki-S</td>
</tr>
<tr>
<td><strong>2007</strong> Barceló-Faraco-Ruiz</td>
<td></td>
</tr>
<tr>
<td><strong>2008</strong> Clop-Faraco-Ruiz</td>
<td></td>
</tr>
</tbody>
</table>
Nachman’s 1996 uniqueness proof in 2D uses complex geometric optics (CGO) solutions.

Assume the conductivity $\sigma \in C^2(\Omega)$ satisfies $\sigma(z) \equiv 1$ near $\partial \Omega$. Define a potential $q$ by setting $q(z) \equiv 0$ for $z$ outside $\Omega$ and

$$q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}}$$

for $z \in \Omega$.

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(-\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz}\psi(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2),$$

where $\tilde{p} > 2$ and $ikz = i(k_1 + ik_2)(x + iy)$. 
The conductivity $\sigma$ can be recovered from the functions $\mu(z, k)$ at $k = 0$

Define $\mu(z, k) = e^{-ikz}\psi(z, k)$. Then $(-\Delta + q)\psi = 0$ implies

$$(-\Delta - 4ik\partial_z + q)\mu(\cdot, k) = 0,$$

where the D-bar operator is defined by $\overline{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

We require the asymptotics

$$\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2).$$

Substituting $k = 0$ gives

$$(-\Delta + \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}})\mu(\cdot, 0) = 0,$$

and setting $\mu(z, 0) = \sqrt{\sigma(z)}$ gives the unique solution of (2) satisfying $\mu(z, 0) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$. 
The crucial intermediate object in the proof is the non-physical scattering transform $t(k)$

We denote $z = x + iy \in \mathbb{C}$ or $z = (x, y) \in \mathbb{R}^2$ whenever needed. The scattering transform $t : \mathbb{C} \to \mathbb{C}$ is defined by

$$t(k) := \int_{\mathbb{R}^2} e^{i \bar{k} z} q(z) \psi(z, k) \, dx \, dy. \quad (3)$$

Sometimes (3) is called the nonlinear Fourier transform of $q$. This is because asymptotically $\psi(z, k) \sim e^{ikz}$ as $|z| \to \infty$, and substituting $e^{ikz}$ in place of $\psi(z, k)$ into (3) results in

$$\int_{\mathbb{R}^2} e^{i(kz + \bar{k}z)} q(z) \, dx \, dy = \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx \, dy$$

$$= \hat{q}(-2k_1, 2k_2).$$
These are the steps of Nachman’s 1996 proof:

Solve boundary integral equation
\[ \psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]
for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:
\[ t(k) = \int_{\partial\Omega} e^{i\bar{z}k} (\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{z}k)} \mu(z, k) \]
with \( \mu(z, \cdot) - 1 \in L^r \cap L^{\infty}(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

Fredholm equation of 2nd kind, ill-posedness shows up here.

Simple integration.

Well-posed problem, can be analyzed by scattering theory.

Trivial step.
### Infinite-precision data:

Solve boundary integral equation
\[ \psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]
for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:
\[ t(k) = \int_{\partial\Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)}\overline{\mu(z, k)} \]
with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

### Practical data:

Solve boundary integral equation
\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta \]
for all \( 0 < |k| < R = -\frac{1}{10} \log \delta \).

For \( |k| \geq R \) set \( t^\delta_R(k) = 0 \). For \( |k| < R \)
\[ t^\delta_R(k) = \int_{\partial\Omega} e^{i\bar{k}z}(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial k} \mu^\delta_R(z, k) = \frac{t^\delta_R(k)}{4\pi k} e^{-i(kz + \bar{k}z)}\overline{\mu^\delta_R(z, k)} \]
with \( \mu^\delta_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma_{1/R(\delta)}(\Lambda_\sigma^\delta) := (\mu^\delta_R(z, 0))^2 \).
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT

Computational example with simulated data

Seismic imaging
Let us analyze how the regularization works using a simulated heart-and-lungs phantom.
Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

\[ \varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}. \]

We invert the linear operator appearing in the equation

\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = [I + S_k(\Lambda^\delta - \Lambda_1)]^{-1} e^{ikz}|_{\partial\Omega} \]

as a matrix in \( \text{span}(\{\varphi_n\}_{n=-N}^{N}) \).

The single-layer operator

\[ (S_k\phi)(z) = \int_{\partial\Omega} G_k(z-w)\phi(w) \, ds(w) \]

uses Faddeev’s Green’s function.
This is how the actual scattering transform looks like in the disc $|k| < 10$, computed by knowing $\sigma$. 

Real part of $t(k)$

Imaginary part

![Graph showing real and imaginary parts of $t(k)$](image)
Scattering transform in the disc $|k| < 10$, here computed from noisy measurement $\Lambda_\delta^\sigma$.
Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

The generalization of Vainikko’s method for the D-bar equation is described in [Knudsen, Mueller & S 2004]. The D-bar equation

$$\frac{\partial}{\partial k} \mu_R^\delta = \frac{1}{4\pi k} t_R^\delta(k) e^{-i(kz + k\bar{z})} (k) \mu_R^\delta,$$

together with the asymptotics $\mu_R^\delta(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$, can be combined in a generalized Lippmann-Schwinger equation:

$$\mu_R^\delta(z, k) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{t_R^\delta(k')}{(k - k')} e^{-i(kz + k\bar{z})} (k') \mu_R^\delta(z, k') dk'_1 dk'_2$$

$$= 1 - PT_R \left( \mu_R^\delta(z, \cdot) \right).$$

Here $P = \overline{\partial_k}^{-1}$ is the solid Cauchy transform.
The trick is to solve a periodic equation whose solution coincides with \( \mu^\delta_R(z, k) \) in the disc \( |k| < R \).

Take \( \epsilon > 0 \) and \( s = 2R + 3\epsilon \). Let \( \eta \in C_0^\infty(\mathbb{R}^2) \) satisfy

\[
\eta(k) = \begin{cases} 
1 & \text{for } |k| < 2R + \epsilon, \\
0 & \text{for } |k| \geq 2R + 2\epsilon.
\end{cases}
\]

Define a \( 2s \)-periodic function \( \widetilde{\beta} \):

\[
\widetilde{\beta}(k + n2s + im2s) = \frac{\eta(k)}{\pi k},
\]

and a periodic Cauchy transform:

\[
\widetilde{P}f(k) = \int_{[-s, s)^2} \widetilde{\beta}(k-k')f(k') \, dk_1 \, dk_2.
\]

Now we consider the equation

\[
\phi = 1 - \widetilde{P} T_R \bar{\phi}.
\]
The D-bar equation is not complex-linear, so real and imaginary parts must be written separately.

The grid points are numbered with one index as shown.

Any function $\phi : [-s, s)^2 \rightarrow \mathbb{C}$ is represented by a vector of values at the grid points:

$$\begin{bmatrix}
\text{Re}\phi(z_1) \\
\text{Re}\phi(z_2) \\
\vdots \\
\text{Re}\phi(z_{64}) \\
\text{Im}\phi(z_1) \\
\text{Im}\phi(z_2) \\
\vdots \\
\text{Im}\phi(z_{64})
\end{bmatrix} \in \mathbb{R}^{128}$$
This is the real-linear operation given to GMRES

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi} \\
T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi} \\
T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi} \\
T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi} & T_R\bar{\phi}
\end{bmatrix}
\]

Element-wise multiplication

IFFT

\[
\begin{bmatrix}
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi
\end{bmatrix}
\]

\[
\phi - \tilde{P} T_R\bar{\phi}
\]
Regularized reconstructions from simulated data with noise amplitude $\delta = \|\Lambda^\delta_\sigma - \Lambda_\sigma\|_{\mathcal{Y}}$

\[\delta \approx 10^{-6}\]
\[\delta \approx 10^{-5}\]
\[\delta \approx 10^{-4}\]
\[\delta \approx 10^{-3}\]
\[\delta \approx 10^{-2}\]

The percentages are the relative square norm errors in the reconstructions.
Recall these phantoms. Can we distinguish between them using the D-bar method?

\[ \sigma_1 \]

\[ \sigma_2 \]
Here are the D-bar reconstructions from simulated EIT data using frequency cutoff $R = 4$.

\[ \sigma_1 \]

\[ \sigma_2 \]
The difference image shows clearly where the two patients are not the same.

\[ \sigma_1 \]

\[ \sigma_2 \]
The D-bar method works for real EIT data, such as laboratory phantoms and *in vivo* human data.

Saline and agar phantom

Reconstruction \((R = 4)\)

[Isaacson, Mueller, Newell & S 2004]
[Montoya 2012]
Outline

Electrical impedance tomography (EIT) and its applications

EIT is an ill-posed inverse problem

Nonlinear low-pass filtering for EIT

Computational example with simulated data

Seismic imaging
How does the D-bar method need to be modified for application to seismic imaging?

1. We can only measure on a small part of the boundary.
2. The physics is different, requiring complex geometric optics solutions for the Helmholtz equation (instead of Schrödinger)
3. The D-bar equation has an extra term
4. All of the above needs to be taken to 3D
This is the first D-bar reconstruction using locally measured data

True conductivity

Reconstruction (full boundary data)

Reconstruction (partial boundary data)

[Hamilton & S, submitted]
The local reconstructions are based on localized solution of the boundary integral equation

\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi^\delta \]

Real part of solution (k=-4i) Imaginary part of solution

[Hamilton & S, submitted]
Assuming time-harmonic solutions, we arrive at a Helmholtz equation

Substituting \( f(z, t) = p(z)e^{i\omega t} \) into the wave equation

\[-\kappa(z) \frac{\partial^2}{\partial t^2} f(z, t) = \nabla \cdot \left( \frac{1}{\varrho(z)} \nabla f(z, t) \right)\]

leads to the equation

\[\kappa(z) \omega^2 p(z) = \nabla \cdot \left( \frac{1}{\varrho(z)} \nabla p(z) \right).\]

Furthermore, the change of variables \( p = \varrho^{1/2} w \) gives

\[-\Delta w + qw = 0,\]

where \( q = \varrho^{1/2} \Delta \varrho^{-1/2} - \omega^2 \kappa \varrho \). Now \( q \) is continued outside \( \Omega \) continuously as the negative constant \(-\omega^2 \kappa \varrho = -E\). We denote \( q_0 = q + E \). The number \( E > 0 \) is called energy.
### EIT D-bar:

\[ (-\Delta + q)\psi = 0 \]
\[ \psi(z, \zeta) = e^{i\zeta \cdot z} \mu(z, \zeta) \]
\[ q = \sigma^{-1/2} \Delta \sigma^{1/2} \]

\[ \zeta = \begin{bmatrix} k \\ \pm ik \end{bmatrix} \in \mathbb{C}^2, \quad k \in \mathbb{C} \]
\[ \Delta e^{i\zeta \cdot z} = 0 \quad \zeta \cdot \zeta = 0 \]
\[ (-\Delta - 2i\zeta \cdot \nabla + q)\mu = 0 \]

\[ t(k) = \int_{\mathbb{C}} e_k(z)q(z)\mu(z, k)dz \]
\[ e_k(z) = \exp \left( i(kz + \overline{k}z) \right) \]

### Seismic D-bar:

\[ (-\Delta + q)\psi = 0 \]
\[ \psi(z, \zeta) = e^{i\zeta \cdot z} \mu(z, \zeta) \]
\[ q = \varrho^{1/2} \Delta \varrho^{-1/2} - \omega^2 \kappa \varrho \]

\[ \zeta = \begin{bmatrix} (\lambda + \frac{1}{\lambda}) \frac{\sqrt{E}}{2} \\ (\frac{1}{\lambda} - \lambda) \frac{i\sqrt{E}}{2} \end{bmatrix} \in \mathbb{C}^2, \quad \lambda \in \mathbb{C} \]
\[ (\Delta + E)e^{i\zeta \cdot z} = 0 \quad \zeta \cdot \zeta = E \]
\[ (-\Delta - 2i\zeta \cdot \nabla + \zeta \cdot \zeta + q)\mu = 0 \]

\[ t(\lambda) = \int_{\mathbb{C}} e_\lambda(z)q_0(z)\mu(z, \lambda)dz \]
\[ e_\lambda(z) = \exp \left( \frac{i}{2} (\lambda \overline{z} + \overline{\lambda}z + \frac{z}{\lambda} + \frac{\overline{z}}{\lambda}) \right) \]
The zero-energy (EIT) and positive-energy D-bar equations are different

Zero-energy D-bar equation for electrical impedance tomography, with \( e_{-z}(k) = \exp(-i(kz + \bar{k}\bar{z})) \) and \( k \neq 0 \):

\[
\frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e_{-z}(k) \overline{\mu(z, k)}.
\]

Positive-energy D-bar equation for seismic imaging, with \( e_{-z}(\lambda) = \exp(-i\sqrt{E}(\lambda\bar{z} + \bar{\lambda}z + z/\lambda + \bar{z}/\bar{\lambda})/2) \) and \( |\lambda| \neq 1 \):

\[
\frac{\partial}{\partial \lambda} \mu(z, \lambda) = \text{sgn}(|\lambda|^2 - 1) \frac{t(\lambda)}{4\pi \lambda} e_{-z}(\lambda) \overline{\mu(z, \lambda)}.
\]

In both cases there is the additional requirement that \( \mu(z, \cdot) \sim 1 \) asymptotically when \( |k| \to \infty \) or \( |\lambda| \to \infty \), respectively.
The zero-energy and positive-energy integral equations differ by a contour integral term

Zero-energy D-bar equation for EIT:

\[ \mu(z, k) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{t(k')}{k'(k - k')} e^{-z(k')} \mu(z, k') \, dk' \]

Positive-energy D-bar equation for seismic imaging:

\[ \mu(z, \lambda) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{\text{sgn}(|\lambda'|^2 - 1)t(\lambda')}{\lambda'(\lambda - \lambda')} e^{-\lambda'(z)} \mu(z, \lambda') \, d\lambda'_1 d\lambda'_2 \]

\[ - \frac{1}{2\pi i} \int_{|\lambda''|=1} \frac{d\lambda''}{\lambda'' - \lambda} \int_{|\lambda'|=1} R(z, \lambda', \lambda'') \mu_-(z, \lambda') \, d\lambda' \]

### D-bar method for EIT:

Solve boundary integral equation

\[ \psi|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]

for all \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:

\[ t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi \, ds \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e_{-z}(k) \bar{\mu}(z, k) \]

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

---

### Seismic D-bar method:

Solve boundary integral equation

\[ \psi|_{\partial \Omega} = e^{i\sqrt{E}(\lambda \bar{z} + \frac{z}{\lambda})/2} - S_\lambda(\Lambda_q - \Lambda_E)\psi \]

for all \( |\lambda| \neq 1 \).

Evaluate the scattering transform:

\[ t(\lambda) = \int_{\partial \Omega} e^{i\sqrt{E}(\bar{\lambda}z + \frac{z}{\lambda})/2}(\Lambda_q - \Lambda_E)\psi \, ds \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial \lambda} \mu(z, \lambda) = \text{sgn}(|\lambda|^2 - 1) \frac{t(\lambda)}{4\pi \lambda} e_{-z}(\lambda) \bar{\mu}(z, \lambda) \]

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).
Inverse Days, Dec 11–13, 2013, Inari, Finland

http://inverse-problems.org/id2013/

Organizers:
Maarten de Hoop
Matti Lassas
Markku Lehtinen
Lassi Roininen
S. S.
Gunther Uhlmann
All Matlab codes freely available on a website!

Part I: Linear Inverse Problems
1 Introduction
2 Naïve reconstructions and inverse crimes
3 Ill-Posedness in Inverse Problems
4 Truncated singular value decomposition
5 Tikhonov regularization
6 Total variation regularization
7 Besov space regularization using wavelets
8 Discretization-invariance
9 Practical X-ray tomography with limited data
10 Projects

Part II: Nonlinear Inverse Problems
11 Nonlinear inversion
12 Electrical impedance tomography
13 Simulation of noisy EIT data
14 Complex geometrical optics solutions
15 A regularized D-bar method for direct EIT
16 Other direct solution methods for EIT
17 Projects
In practice, efficient regularized algorithms are needed for linear and nonlinear inverse problems

Assume given a forward map $F$ and noisy data $y^\delta$.

An efficient regularized inversion algorithm should compute a numerical approximation to $\Gamma_{\alpha(\delta)}(y^\delta)$ quickly and accurately, where $\Gamma_{\alpha}$ is a regularization strategy with an admissible choice of regularization parameter.

The Tikhonov approach provides efficient regularized inversion algorithms only for linear and almost linear forward maps $F$.

Electrical impedance tomography is the only strongly nonlinear inverse problem with efficient regularized inversion algorithms, based on the problem-specific approach.
Let us emphasize the difference between stability analysis and regularization strategies

Conditional stability results have the form

$$\|x - x'\|_X \leq f(\|y - y'\|_Y),$$

where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying $f(0) = 0$. However, the above inequality is practically irrelevant: the noisy measurement $y^\delta$ is almost surely not in the range of $F$. 
The observed radii are better (=larger) than those given by the theoretical formula $R(\delta) = -\frac{1}{10} \log \delta$.
Why is Calderón’s problem nonlinear?

Define a quadratic form $\mathcal{P}_\sigma$ for functions $f : \partial \Omega \to \mathbb{R}$ by

$$\mathcal{P}_\sigma(f) = \int_\Omega \sigma |\nabla u|^2 \, dz,$$  \hspace{1cm} (4)

where $u$ is the solution of the Dirichlet problem

$$\begin{cases} 
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
 u|_{\partial \Omega} &= f.
\end{cases}$$

Now the map $\sigma \mapsto \mathcal{P}_\sigma$ is nonlinear because $u$ depends on $\sigma$ in (4). Physically, $\mathcal{P}_\sigma(f)$ is the power needed for maintaining the voltage potential $f$ on the boundary $\partial \Omega$. Integrate by parts in (4):

$$\mathcal{P}_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) \, ds = \int_{\partial \Omega} f \left( \Lambda_\sigma f \right) \, ds.$$

Thus the map $\sigma \mapsto \Lambda_\sigma$ cannot be linear in $\sigma$. 
We define spaces for our regularization strategy

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ with

- $\|\sigma\|_{C^2(\Omega)} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators $A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying

- $A(1) = 0$,
- $\int_{\partial\Omega} A(f) \, ds = 0$. 

Model space $X = L^\infty(\Omega)$

Data space $Y$
Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

There exists a constant $0 < \delta_0 < 1$, depending only on $M$ and $\rho$, with the following properties. Let $\sigma \in D(F)$ be arbitrary and assume given noisy data $\Lambda^\delta_\sigma$ satisfying

$$\|\Lambda^\delta_\sigma - \Lambda_\sigma\|_Y \leq \delta < \delta_0.$$ 

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\delta)}(\Lambda^\delta_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(- \log \delta)^{-1/14}.$$