Electrical Impedance Tomography and the Novikov-Veselov Equation

Samuli Siltanen

Department of Mathematics and Statistics
University of Helsinki, Finland
samuli.siltanen@helsinki.fi
www.siltanen-research.net

Classical and Quantum Integrability
Dijon, France
September 5, 2019
Finnish Centre of Excellence in Inverse Modelling and Imaging

2018-2025

Finland

• Universidad de Oulu
• Yliopisto Oulu
• University of Eastern Finland
• University of Jyväskylä
• Tampere University of Technology
• Finnish Meteorological Institute
• Aalto University

Academy of Finland
Outline

Electrical impedance tomography and the D-bar method

Interlude: Virtual Hybrid Edge Detection

Zero-energy exceptional points

Inverse scattering method for the Novikov-Veselov equation
This part is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA

Kim Knudsen, Technical University of Denmark

Matti Lassas, University of Helsinki, Finland

Jon Newell, Rensselaer Polytechnic Institute, USA

Jennifer Mueller, Colorado State University, USA
Electrical impedance tomography is useful for medical chest imaging

Lungs filled with air are resistive, shown as blue.

Blood in the heart appears red, as it is more conductive than air.
The inverse conductivity problem introduced by Alberto Calderón is a mathematical model for EIT

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\left\{ \begin{array}{l} \nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \\ u \big|_{\partial \Omega} = f. \end{array} \right.$$ 

The Dirichlet-to-Neumann map is a model for boundary measurements

$$\Lambda_{\sigma} : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}} \big|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_{\sigma}$.

This is a nonlinear task and an ill-posed inverse problem.
There exists a nonlinear Fourier transform adapted to electrical impedance tomography.
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

\[ \Lambda_\sigma \xrightarrow{\text{BIE}} \text{Ideal measurement} \xrightarrow{\text{Nonlinear IFFT}} \text{[Nachman 1996]} \]
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies.
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
The D-bar method is a regularization strategy for reconstructing the full conductivity distribution.
This is a brief history of the two-dimensional regularized D-bar method for EIT

1966 Faddeev: Complex geometric optics (CGO) solutions
1987 Sylvester and Uhlmann: CGO solutions for inverse boundary-value problems; uniqueness for 3D EIT with smooth conductivities and infinite-precision data
1988 R. G. Novikov: Core ideas of the D-bar method
1988 Nachman: D-bar method for 3D EIT
1996 Nachman: Uniqueness and reconstruction for 2D EIT with $C^2$ conductivities and infinite-precision data
2000 S, Mueller and Isaacson: Numerical implementation of Nachman’s proof using a Born approximation
2006 Isaacson, Mueller, Newell and S: Application of the D-bar method to EIT data measured from a human subject
2009 Knudsen, Lassas, Mueller and S: Regularization proof
Nachman (1996) transforms to the Schrödinger equation and uses CGO solutions

Define a potential \( q \) by setting \( q(z) \equiv 0 \) for \( z \) outside \( \Omega \) and

\[
q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for} \quad z \in \Omega.
\]

Then \( q \in C_0(\Omega) \). We look for solutions of the Schrödinger equation

\[
(-\Delta + q)\psi(\cdot, k) = 0 \quad \text{in} \quad \mathbb{R}^2
\]

parametrized by \( k \in \mathbb{C} \setminus 0 \) and satisfying the asymptotic condition

\[
e^{-ikz}\psi(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2),
\]

where \( \tilde{p} > 2 \) and \( ikz = i(k_1 + ik_2)(x + iy) \).
The CGO solutions are constructed using a generalized Lippmann-Schwinger equation

Define $\mu(z, k) = e^{-ikz}\psi(z, k)$. Then $(-\Delta + q)\psi = 0$ implies

$$(-\Delta - 4ik\overline{\partial}_z + q)\mu(\cdot, k) = 0,$$

where the D-bar operator is defined by $\overline{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

A solution of (1) satisfying $\mu(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2)$ can be constructed using the Lippmann-Schwinger type equation

$$\mu = 1 - g_k * (q\mu),$$

where $g_k$ satisfies $(-\Delta - 4ik\overline{\partial}_z)g_k = \delta$ and is defined by

$$g_k(z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)}\,d\xi_1\,d\xi_2.$$
One of the breakthroughs in Nachman’s 1996 article is showing uniqueness of $\mu$

A solution of $(-\Delta - 4i k \bar{\partial}_z + q)\mu(\cdot, k) = 0$ satisfying $\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ can be constructed using the formula

$$\mu - 1 = [I + g_k \ast (q \cdot)]^{-1}(g_k \ast q),$$

provided that the inverse operator exists.

Now $q \in L^p(\mathbb{R}^2)$ with $1 < p < 2$ and $1/\tilde{p} = 1/p - 1/2$, and

$$q \cdot : W^{1,\tilde{p}}(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \text{ is bounded,}$$
$$g_k \ast : L^p(\mathbb{R}^2) \to W^{1,\tilde{p}}(\mathbb{R}^2) \text{ is compact.}$$

Thus $I + g_k \ast (q \cdot): W^{1,\tilde{p}}(\mathbb{R}^2) \to W^{1,\tilde{p}}(\mathbb{R}^2)$ is Fredholm of index zero, and Nachman proved injectivity for all $k \neq 0$.

Progress on invertibility of $I + g_k \ast (q \cdot)$: Music, Perry & S (2013), Music (2014), Lakshtanov & Vainberg (2017)
The non-physical scattering transform \( t(k) \) is a nonlinear Fourier transform

We denote \( z = x + iy \in \mathbb{C} \) and \( z = (x, y) \in \mathbb{R}^2 \). The scattering transform \( t : \mathbb{C} \to \mathbb{C} \) is defined by

\[
t(k) := \int_{\mathbb{R}^2} e^{i\bar{k}z} q(z) \psi(z, k) \, dx \, dy.
\]

Sometimes \( t(k) \) is called the nonlinear Fourier transform of \( q \). This is because asymptotically \( \psi(z, k) \sim e^{ikz} \) as \( |z| \to \infty \), and substituting \( e^{ikz} \) in place of \( \psi(z, k) \) above gives

\[
\int_{\mathbb{R}^2} e^{i(kz + \bar{k}z)} q(z) \, dx \, dy = \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx \, dy
\]

\[
= \hat{q}(-2k_1, 2k_2).
\]
### Infinite-precision data:

Solve boundary integral equation

\[
\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda \sigma - \Lambda_1)\psi
\]

for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:

\[
t(k) = \int_{\partial \Omega} e^{ik\bar{z}}(\Lambda \sigma - \Lambda_1)\psi(\cdot, k) \, ds.
\]

Fix \( z \in \Omega \). Solve D-bar equation

\[
\frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu(z, k)
\]

with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

### Practical data:

Solve boundary integral equation

\[
\psi^\delta(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda^\delta \sigma - \Lambda_1)\psi^\delta
\]

for all \( 0 < |k| < R = -\frac{1}{10} \log \delta \).

For \( |k| \geq R \) set \( t^\delta_R(k) = 0 \). For \( |k| < R \)

\[
t^\delta_R(k) = \int_{\partial \Omega} e^{ik\bar{z}}(\Lambda^\delta \sigma - \Lambda_1)\psi^\delta(\cdot, k) \, ds.
\]

Fix \( z \in \Omega \). Solve D-bar equation

\[
\frac{\partial}{\partial k} \mu^\delta_R(z, k) = \frac{t^\delta_R(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu^\delta_R(z, k)
\]

with \( \mu^\delta_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma_{1/R^\delta}(\Lambda^\delta) := (\mu^\delta_R(z, 0))^2 \).
Computational solution of the D-bar equation

2000 S, Mueller and Isaacson: Nyström method
2004 Knudsen, Mueller and S: Vainikko’s method
2004 Eirola, Huhtanen and von Pfaler: real-linear GMRES
2017 Klein and McLaughlin: spectral method
Theorem (Knudsen, Lassas, Mueller & S 2009)

Fix a conductivity $\sigma \in \mathcal{D}(F)$. Assume given noisy data $\Lambda_\delta^{\sigma}$ satisfying

$$\|\Lambda_\sigma^{\delta} - \Lambda_\sigma\|_Y \leq \delta.$$ 

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\delta)}(\Lambda_\sigma^{\delta}) - \sigma\|_{L^\infty(\Omega)} \leq C(- \log \delta)^{-1/14}.$$
Regularized reconstructions from simulated data with noise amplitude \( \| \delta \| = \| \Lambda^\delta_{\sigma} - \Lambda_{\sigma} \|_Y \)

\[
\| \delta \| \approx 10^{-6} \quad \| \delta \| \approx 10^{-5} \quad \| \delta \| \approx 10^{-4} \quad \| \delta \| \approx 10^{-3} \quad \| \delta \| \approx 10^{-2}
\]

The percentages are the relative square norm errors in the reconstructions.
Medical application of EIT and the D-bar method: quantifying air-trapping in cystic fibrosis patients

All results on this slide are from Jennifer Mueller’s group at Colorado State University.

Images: ventilation-perfusion index maps, computed from three subjects at Children’s Hospital Colorado using EIT and the D-bar method.

Dark blue regions are well-perfused but poorly ventilated.

Radiologist’s report for Subject B: extensive regions of air trapping, regional to the lung areas affected by plugging, approximately 50% of both lungs.

Healthy control
Average index 0.46

CF Subject A
Average index 0.34

CF Subject B
Average index 0.10
Outline

Electrical impedance tomography and the D-bar method

Interlude: Virtual Hybrid Edge Detection

Zero-energy exceptional points

Inverse scattering method for the Novikov-Veselov equation
The results in this part are a joint work with

Allan Greenleaf, University of Rochester, NY, USA
Matti Lassas, University of Helsinki, Finland
Matteo Santacesaria, University of Genoa, Italy
Gunther Uhlmann, University of Washington, USA
Let us analyse Complex Geometric Optics solutions using complex two-vectors

Consider complex two-vectors $\eta \in \mathbb{C}^2$ of the form

$$\eta = (\theta_1 + i\theta_2, -\theta_2 + i\theta_1),$$

where $\theta = \theta_1 + i\theta_2 \in \mathbb{C}$ is a unitary complex number: $|\theta| = 1$.

Denote a planar point by $x = (x_1, x_2) \in \mathbb{R}^2$. Because $\eta \cdot \eta = 0$ we see that $e^{i\tau \eta \cdot x} = 0$ is harmonic in $x$.

We next analyse solutions of the conductivity equation $\nabla \cdot \sigma \nabla u = 0$ of the form

$$u(x) = e^{i\tau \eta \cdot x} w(x, \tau).$$

(They are connected to the previously discussed CGO solutions via $k = \tau \theta$ and $z = x_1 + ix_2$, since then $e^{ikz} = e^{i\tau \theta z} = e^{i\tau \eta \cdot x}$.)
Since \( u(x) = e^{i \tau \eta \cdot x} w(x, \tau) \) satisfies the conductivity equation,

\[
0 = \frac{1}{\sigma(x)} \nabla \cdot (\sigma(x) \nabla u(x)) \\
= (\Delta + \frac{1}{\sigma}(\nabla \sigma) \cdot \nabla)(e^{i \tau \eta \cdot x} w(x, \tau)) \\
= \left( \Delta w(x, \tau) + 2i \tau \eta \cdot \nabla w(x, \tau) + (\frac{1}{\sigma} \nabla \sigma) \cdot (\nabla + i \tau \eta) w(x, \tau) \right) e^{i \tau \eta \cdot x}.
\]

Hence, we have

\[
\Delta w(x, \tau) + 2i \tau \eta \cdot \nabla w(x, \tau) + (\frac{1}{\sigma} \nabla \sigma) \cdot (\nabla + i \tau \eta) w(x, \tau) = 0.
\]
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

Let \( \hat{w}(x, t) \) be the Fourier transform of \( w(x, \tau) \) in the \( \tau \) variable:

\[
\hat{w}(x, t) = \mathcal{F}w(x, t) = \int_{-\infty}^{\infty} e^{-it\tau} w(x, \tau) d\tau.
\]

We call \( t \) the pseudo-time corresponding to complex frequency \( \tau \). Then equation

\[
\Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot \left( \nabla + i\tau \eta \right) w(x, \tau) = 0
\]

yields

\[
\Delta \hat{w}(x, t) + 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot \left( \nabla + \eta \frac{\partial}{\partial t} \right) \hat{w}(x, t) = 0.
\]
Complex principal type operator leads to singularities propagating along leaves

Denote $\eta = \eta_R + i\eta_I$. The principal part of the equation

$$\Delta \hat{w}(x, t) + 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \frac{1}{\sigma}(\nabla \sigma) \cdot (\nabla + \eta \frac{\partial}{\partial t}) \hat{w}(x, t) = 0$$

is given by the complex principal type operator

$$\Delta + 2\eta \frac{\partial}{\partial t} \cdot \nabla = \left( \Delta + 2\eta_R \frac{\partial}{\partial t} \cdot \nabla \right) + i \left( 2\eta_I \frac{\partial}{\partial t} \cdot \nabla \right),$$

in the sense of Duistermaat and Hörmander (1972).

For a real principal type operator the characteristic singularities propagate along one-dimensional rays. For instance, for the wave equation the light-like singularities propagate along light rays.

For a complex principal type operator the characteristic singularities propagate along two-dimensional surfaces called leaves.
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
We use the Beltrami-type complex geometric optics (CGO) solutions

Set $\mu := (1 - \sigma)/(1 + \sigma)$. Write $f = u + iv$ and note that

\[ \overline{\partial}_z f_\mu = \mu \overline{\partial}_z f_\mu \quad \Leftrightarrow \quad \nabla \cdot \sigma \nabla u = 0 \quad \text{and} \quad \nabla \cdot \sigma^{-1} \nabla v = 0. \]

The CGO solutions of [Astala-Päivärinta 2006] have the form

\[ f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)), \]
\[ f_{-\mu}(z, k) = e^{ikz}(1 + \omega^-(z, k)), \]

with the asymptotic condition

\[ \omega^\pm(z, k) = \mathcal{O}\left(\frac{1}{|z|}\right) \text{ as } |z| \to \infty. \]

Here $ikz = i(k_1 + ik_2)(x + iy)$ and $\overline{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$. 
This is a brief history of computational solution methods for the Beltrami CGO solutions

1987 Sylvester and Uhlmann: Introduction of CGO solutions
2000 S, Mueller and Isaacson: Numerical CGOs
2006 Astala and Päivärinta: Original Beltrami-type construction

2010 Astala, Mueller, Päivärinta and S: First numerical solution method

2011 Astala, Mueller, Päivärinta, Perämäki and S: Novel EIT reconstruction method

2012 Huhtanen and Perämäki: Preconditioned Krylov subspace method for real-linear systems

2014 Astala, Päivärinta, Reyes and S: Computational high-frequency experiments

2018 Greenleaf, Lassas, Santacesaria, S and Uhlmann: Virtual Hybrid Edge Detection based on 1D Fourier technique
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

\[ f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)), \]
write the complex parameter in the form \( k = \tau e^{i\varphi} \) with \( \tau \in \mathbb{R} \). Denote \( \omega^+(z, \tau, e^{i\varphi}) = \omega^+(z, k) \).

Fourier transform \( \omega^+(z, \tau, e^{i\varphi}) \) in the \( \tau \) variable:

\[ \hat{\omega}^+(z, t, e^{i\varphi}) = \mathcal{F}_{\tau \rightarrow t}(\omega^+(z, \tau, e^{i\varphi})) = \int_{-\infty}^{\infty} e^{-it\tau} \omega^+(z, \tau, e^{i\varphi}) \, d\tau. \]

We call \( t \) the pseudo-time.
We can define an averaging operator

**Definition** (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018)
Define operator $T^\pm$ by complex contour integral:

$$T^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial \Omega} \hat{\omega}^\pm(z, t, e^{i\varphi}) dz.$$ 

In the case of $\Omega$ being the unit disc, we get

$$T^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\omega}^\pm(e^{i\gamma}, t, e^{i\varphi}) e^{i\gamma} d\gamma.$$
\( \hat{\omega}^+(1, 2t, 1) \)

\( [\hat{\omega}^+ - \hat{\omega}^-](1, 2t, 1) \)

\( [T^+ - T^-] \mu(2t, 1) \)

Computations by Rashmi Murthy
Cauchy and Beurling transforms

Define the Cauchy and Beurling transforms by

\[ Pf(z) = \bar{\partial}^{-1} f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\lambda)}{\lambda - z} d\lambda_1 d\lambda_2, \]

\[ Sg(z) = \bar{\partial}^{-1} g(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\lambda - z| > \varepsilon} \frac{g(\lambda)}{(\lambda - z)^2} d\lambda_1 d\lambda_2. \]

Also, set

\[ e_k(z) = \exp(i(kz + \bar{k}\bar{z})), \]

\[ \alpha(z, k) = -i \bar{k} e_{-k}(z) \mu(z), \]

\[ \nu(z, k) = e_{-k}(z) \mu(z), \]

and define the operator \( A \) by

\[ A := ( -\bar{\nu} S - \overline{\alpha} P ). \]
We introduce a new scattering series

Huhtanen and Perämäki (2012) modified the original construction of Astala and Päivärinta (2006) for computational purposes. We use the 2012 technique for the construction of a novel scattering series

\[ \omega = \sum_{n=1}^{\infty} \omega_n, \]

where \( A := (-\bar{\nu}S - \bar{\alpha}P) \) and

\[ \omega_n = -\bar{\partial}_z^{-1}u_n, \quad u_n = -Au_{n-1}, \quad u_1 = -\bar{\alpha}. \]

The single scattering term \( \omega_1 = \bar{\partial}_z^{-1}\alpha \) determines singularities of \( \mu \).

The terms \( \omega_n \) with \( n > 1 \) arise from multiple scattering.
Recovery by “filtered back-projection”

**Theorem.** (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018)
Define averaged operators $T_j^\pm$ for $j = 1, 2, 3, \ldots$ by the complex contour integral:

$$T_j^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial\Omega} \hat{\omega}_j^\pm(z, t, e^{i\varphi}) dz,$$

Then we have a filtered back-projection formula

$$(-\Delta)^{-1/2}(T_1^\pm)^* T_1^\pm \mu = \mu.$$
New result: inverse scattering methods can transform EIT into “X-ray tomography”

https://www.youtube.com/watch?v=37yOCfBfRJk
[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]
Conductivity Filtered back-projection “Λ-tomography”
Outline

Electrical impedance tomography and the D-bar method

Interlude: Virtual Hybrid Edge Detection

Zero-energy exceptional points

Inverse scattering method for the Novikov-Veselov equation
This part is a joint work with

Michael Music, University of Michigan, USA

Peter Perry, University of Kentucky, USA
Define a family $q_\lambda$ of central potentials

Assume that $q_0 \in C_0^\infty(\mathbb{R}^2)$ is
- of conductivity type: there exists a smooth, strictly positive function $\psi$ with $\lim_{|z| \to \infty} \psi(z) = 1$ so that $q = \psi^{-1}(\Delta \psi)$, and
- rotationally symmetric: $q_0(z) = q_0(|z|)$.

Let $w \in C_0^\infty(\mathbb{R}^2)$ be a non-negative rotationally symmetric test function, and set

$$q_\lambda = q_0 + \lambda w.$$  

Then the scattering transform $t_\lambda : \mathbb{C} \to \mathbb{C}$ corresponding to $q_\lambda$ is real-valued and rotationally symmetric: $t_\lambda(k) = t_\lambda(|k|)$.

For any $\lambda < 0$, the potential $q_\lambda$ is not of conductivity type [Murata 1987]. It is supercritical.
Consider the Dirichlet problem

\[ (-\Delta + q\lambda) u = 0 \text{ in } B_1 \]
\[ u|_{S^1} = f. \] 

(2)

Assume that zero is not a Dirichlet eigenvalue of \((-\Delta + q\lambda)\) in \(B_1\).

If \(u\) denotes the unique solution to (2), we set

\[ \Lambda_{q\lambda} f = \frac{\partial u}{\partial \nu}|_{S^1}. \]

Rotational symmetry implies \(\Lambda_{q\lambda} \varphi_n = \mu_n(q\lambda) \varphi_n\) for \(\varphi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}\).

We denote \(\mu(q\lambda) := \mu_0(q\lambda)\).
Simple example potential

Let’s take $q_\lambda = \lambda w$ with this radial, nonnegative test function $w$: 
Diagram of exceptional points for $q_\lambda$

$|k|$

Nachman 1996

Open problem

MIMPPSS 2012

Music 2014

$\lambda$
Theorem (Music, Perry & S 2012)

(1) For \( \lambda > 0 \) sufficiently small there are no exceptional points, and the scattering transform \( t_\lambda \) is \( C^\infty \) away from \( k = 0 \).

(2) For \( \lambda < 0 \) sufficiently small and a unique \( r(\lambda) > 0 \), the exceptional set \( \mathcal{E} \) is a circle \( C_\lambda \) of radius \( r(\lambda) \) about the origin, and the function \( t_\lambda \) is \( C^\infty \) on \( \mathbb{R}^2 \setminus [C_\lambda \cup \{0\}] \), while
\[
\lim_{|k| \to r(\lambda)} |t_\lambda(k)| = \infty.
\]

The radius \( r(\lambda) \) obeys the formula
\[
r(\lambda) \underset{\lambda \uparrow 0}{\sim} \exp \left[ -2\pi \left( h - \frac{(1 + O(\lambda))}{2\pi \mu(\lambda)} \right) \right]
\]
where \( h = -\gamma/(2\pi) \) with Euler’s constant \( \gamma \), and \( \mu(\lambda) \) is the eigenvalue of the DN map \( \Lambda_{q,\lambda} \) corresponding to constant functions.
The asymptotic behaviour predicted by theory matches the numerical results remarkably well.

\[ \exp\left(-\gamma + \frac{1}{\mu(\lambda)}\right) \]
For $\lambda > 0$ there are no exceptional points

**Theorem.** (Music 2014) Let $q_\lambda = q_0 + \lambda w$ be as above. For $\lambda > 0$, which is the case of *subcritical potentials*, there are no exceptional points.
Outline

Electrical impedance tomography and the D-bar method

Interlude: Virtual Hybrid Edge Detection

Zero-energy exceptional points

Inverse scattering method for the Novikov-Veselov equation
This part is a joint work with

**Ryan Croke**, University of Colorado—Boulder, USA

**Matti Lassas**, University of Helsinki, Finland

**Jennifer Mueller**, Colorado State University, USA

**Michael Music**, University of Michigan, USA

**Peter Perry**, University of Kentucky, USA

**Andreas Stahel**, BFH-TI Biel, Switzerland
Korteweg and de Vries formulated in 1895 an equation for waves in shallow water

\[ u_{\tau} + u_{xxx} + 6uu_x = 0, \quad x \in \mathbb{R}, \quad \tau \geq 0 \]

Assumptions: wave height is small compared to the depth, which in turn is small compared to the length of the wave.

The KdV equation is a nonlinear, dispersive wave equation.

It allows solitary wave solutions observed by Russell (1845), and was studied by Boussinesq (1871) and lord Rayleigh (1876).
Gardner, Greene, Kruskal and Miura (1967) found a striking connection between the KdV equation and Schrödinger scattering

\[(\lambda_n, c_n, R(k)) \rightarrow (\lambda_n, c_n e^{4k_n^3 \tau}, R(k) e^{8ik^3 \tau})\]

\[\uparrow\]
\[q_0(x) \quad \text{KdV} \quad \downarrow\]
\[\downarrow\]
\[q_\tau(x)\]

The inverse scattering step is due to

1946 Borg
1949 Levinson
1951 Gelfand-Levitan
1952 Marchenko
1953 Krein
Novikov-Veselov equation is the most natural 2D generalization of the KdV equation

Korteqev-de Vries equation, dimension (1+1):

\[ \dot{q} + \frac{\partial^3 q}{\partial x^3} + 6q \frac{\partial q}{\partial x} = 0. \]

Kadomtsev-Petviashvili equation, dimension (2+1):

\[ \frac{\partial}{\partial x} \left( \dot{q} + \frac{\partial^3 q}{\partial x^3} + 6q \frac{\partial q}{\partial x} \right) = \pm \frac{\partial^2 q}{\partial y^2}. \]

Novikov-Veselov equation, dimension (2+1):

\[ \dot{q} + \partial_z^3 q + \overline{\partial_z}^3 q - 3\partial_z(qv) - 3\overline{\partial_z}(q\overline{v}) = 0, \quad \overline{\partial_z} q = \partial_z v. \]

Here \( z = x + iy \) and \( \overline{\partial_z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \)
The inverse scattering method is one way to solve the Novikov-Veselov equation:

\[ t_0(k) \xrightarrow{\text{exp}(i\tau(k^3 + \bar{k}^3))} t_\tau(k) \]

\[ q_0(z) \xrightarrow{\text{nonlinear NV evolution}} q_{\tau}^{\text{NV}}(z), \]

\[ q_\tau^{\text{IS}}(z) \xrightarrow{\text{nonlinear NV evolution}} q_{\tau}^{\text{NV}}(z), \]
The direct and inverse nonlinear Fourier transforms $\mathcal{T}$ and $\mathcal{Q}$ are defined as follows:

The direct transform $q_{\tau} \mapsto \mathcal{T} q_{\tau}$ is given by

$$(\mathcal{T} q_{\tau})(k) = \int_{\mathbb{R}^2} e^{ik\bar{z}} q_{\tau}(z) \psi_{\tau}(z, k) dz,$$

where $(-\Delta + q_{\tau}) \psi_{\tau}(\cdot, k) = 0$ and $\psi_{\tau}(z, k) \sim e^{ikz}$ as $|z| \to \infty$.

The inverse transform $t_{\tau} \mapsto \mathcal{Q} t_{\tau}$ is given by

$$(\mathcal{Q} t_{\tau})(z) = \frac{i}{\pi^2} \overline{\partial_z} \int_{\mathbb{C}} \frac{t_{\tau}(k)}{k} e^{-ikz} \overline{\psi_{\tau}(z, k)} dk,$$

where $\psi_{\tau}(z, k) = e^{ikz} \mu_{\tau}(z, k)$ and $\mu_{\tau}$ satisfies the D-bar equation

$$\frac{\partial}{\partial k} \mu_{\tau}(z, k) = \frac{t_{\tau}(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \overline{\mu_{\tau}(z, k)}, \quad \mu_{\tau}(z, \cdot) \sim 1.$$
Zero-energy inverse scattering & NV equation

1984 Novikov & Veselov: Periodic case.
1993 Tsai: Formal analysis assuming no exceptional points.
1996 Nachman: Conductivity-type \( q_0 \) have no exceptional points.
2007 Lassas, Mueller & S: Inverse scattering evolution \( q^{IS}_\tau \) well-defined for conductivity-type initial data \( q_0 \).
2012 Lassas, Mueller, S & Stahel: Evolution \( q^{IS}_\tau \) preserves conductivity-type, numerical evidence for \( q^{IS}_\tau = q^{NV}_\tau \).
2012 Perry: \( q^{IS}_\tau = q^{NV}_\tau \) holds for conductivity-type \( q_0 \).
2013 Music, Perry & S: Supercritical exceptional points exist.
2014 Music: Subcritical \( q_0 \) have no exceptional points.
2015 Music & Perry: Global existence for critical and subcritical \( q_0 \).
2016 Angelopoulos: Local well-posedness of NV equation.
Let’s look at an example. Here is a smooth and rotationally symmetric conductivity function $\sigma(z)$
This is the initial potential $q_0(z) = \sigma^{-1/2}(z)\Delta\sigma^{1/2}(z)$.
This is the initial scattering transform $t_0(k)$
This is the Novikov-Veselov evolution
Dynamics of NV solutions for $\lambda > 0$ (A. Stahel); no apparent singularities

This is in accordance with [Music and Perry 2015]
Dynamics of NV solutions for $\lambda < 0$ (A. Stahel), seemingly with blowup in finite time
Negative-energy NV by Kazeykina and Klein 2017

Figure 16. Solution of the NV equation (1a) with $E = -10$ for the initial data (41) and $\beta = -10$; on the left the solution for $t = 0.2$, on the right the $L^\infty$ norm of the solution in dependence of time.

Initial data $v(x, y, 0) = \beta \exp(-x^2 - y^2)$
Computational CGO methods developed in my team can be used for EIT imaging and for solving the integrable Novikov-Veselov equation.

EIT is the zero-energy CGO case; positive and negative energies have applications as well.

With small modifications, the developed codes can be used for solving the KP II equation.
Samun tiedekanava

1,277 subscribers

Uploas

PLAY ALL

Sam's Science Splash: Fish Removal
3.5K views • 2 years ago
CC

Samun tiedepläijäys: Nopanheitto ja...
1.9K views • 2 years ago
CC

Tervetuloa Samun tiedekanavalle!
1.8K views • 2 years ago
CC

Sam's Science Splash: Tomography
1.8K views • 1 year ago
CC

Samun tiedepläijäys: Aivokuvaus antaineella
1.6K views • 1 year ago
CC

Matematiikan käänteis
Matematiikan [8:4]
Matematiikan [4:3]
Matematiikan
Matemaattinen
Selfie
Thank you for your attention!
EIT is the zero-energy CGO case; positive and negative energies have applications as well.

1988 Nachman
2013 R. G. Novikov and Santacesaria
2013 Santacesaria
2016 Kazeykina and Munõz

Computational studies:

2016 Tamminen, Tarvainen and S: Negative-energy D-bar method for diffuse optical tomography

The computational tools apply to the KPII equation with little modification

\[(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0\]

\[G(x, y; k) = \frac{\text{sgn}(-y)}{2\pi} \int_{-\infty}^{\infty} \theta(y(\xi^2 + 2k_R\xi)) \exp\{i\xi x - \xi(\xi + 2k)y\} \, d\xi\]

\[= \left\{ \theta(k_R) \left[ -\theta(-y) \int_{-2k_R}^{0} d\xi + \theta(y) \left( \int_{0}^{\infty} d\xi + \int_{-\infty}^{2k_R} d\xi \right) \right] + \theta(-k_R) \left[ -\theta(-y) \int_{0}^{2k_R} d\xi + \theta(y) \left( \int_{-\infty}^{0} d\xi + \int_{-2k_R}^{\infty} d\xi \right) \right] \right\} \]

\[\times \frac{1}{2\pi} \exp\{i\xi x - \xi(\xi + 2k)y\}. \quad (5.3.11)\]

Formulas from [Ablowitz & Clarkson 1991].

Anyone interested in collaboration?
Links to open computational resources

Open EIT datasets:

- Finnish Inverse Problems Society (FIPS) dataset page

Reconstruction algorithms: FIPS Computational Blog

- The D-bar Method for EIT—Simulated Data
- The D-bar Method for EIT—Experimental Data

For these slides see: http://www.siltanen-research.net/talks.html
Generalization to nonsymmetric potentials

Lakshtanov and Vainberg 2017 show that even when $q_0$ is not rotationally symmetric, the potential $q_\lambda = q_0 + \lambda w$ has

- one curve of exceptional points for $\lambda < 0$ close to zero,
- no exceptional points for $\lambda > 0$ close to zero.
References about foundations of Electrical Impedance Tomography

Classical review article by the legendary EIT group of RPI: Margaret Cheney, David Isaacson and Jon Newell (1999), *Electrical Impedance Tomography*. SIAM Review 41.


More recent review of EIT: Jennifer Mueller and S (2012)
Regularized reconstructions from simulated data with noise amplitude \( \| \delta \| = \| \Lambda_\sigma^\delta - \Lambda_\sigma \| \_Y \)

\[
\| \delta \| \approx 10^{-6} \quad \| \delta \| \approx 10^{-5} \quad \| \delta \| \approx 10^{-4} \quad \| \delta \| \approx 10^{-3} \quad \| \delta \| \approx 10^{-2}
\]

The percentages are the relative square norm errors in the reconstructions.
In Bohm’s approach, the motion of a quantum particle is equivalent to the motion of the fluid

Let \( n(x) = |\psi(x)|^2 \) denote particle density. Fluid velocity field is \( \mathbf{v} \). Then the equations for “quantum fluid” are

\[
\begin{align*}
\frac{dn}{dt} &= -n \nabla \cdot \mathbf{v} \\
\frac{m}{dt} \mathbf{v} &= \nabla (Q - U)
\end{align*}
\]

Here \(-\nabla U\) is the classical force and \( \nabla Q \) is quantum force. Furthermore, \( Q \) is the Bohm potential

\[
Q = \frac{\hbar}{2} \frac{\Delta \sqrt{n}}{\sqrt{n}}
\]

Slide adapted from Omar Morandi’s talk at ECMI 2018.
Here is a reconstruction of the conductivity, computed using a nonlinear Fourier transform.

Saline and agar phantom

D-bar reconstruction

Cut-off frequency $R = 4$

[Isaacson, Mueller, Newell & S 2004]
[Montoya 2012]