Electrical impedance imaging using nonlinear Fourier transform

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Outline

Electrical impedance tomography

Regularization of nonlinear inverse problems

D-bar method for infinite-precision data

Regularization using non-linear low-pass filtering

Practical implementation
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Practical implementation
Electrical impedance tomography (EIT) is an emerging medical imaging technique.

Feed electric currents through electrodes. Measure the resulting voltages. Repeat the measurement for several current patterns.

Reconstruct distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient’s inner structure.

EIT is a harmless and painless imaging method suitable for long-term monitoring.
This talk concentrates on applications of EIT to chest imaging

Applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases}
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = f.
\end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_\sigma$. It is a nonlinear and ill-posed inverse problem.
Why is Calderón’s problem nonlinear?

The weak formulation of the Dirichlet-to-Neumann map \( \Lambda_\sigma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \) is given by Green’s formula

\[
\langle \Lambda_\sigma f, g \rangle = \int_{\partial \Omega} g \Lambda_\sigma f \, ds = \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dz,
\]

where \( v \) is any \( H^1(\Omega) \) function with trace \( g \), and \( u \) is the solution of the Dirichlet problem

\[
\begin{align*}
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
u|_{\partial \Omega} &= f.
\end{align*}
\]

Now the map \( \sigma \mapsto \Lambda_\sigma \) is nonlinear because \( u \) depends on \( \sigma \).
We illustrate the ill-posedness of Calderón’s problem using a simulated example.
We apply the voltage distribution $f(\theta) = \cos \theta$ at the boundary of the two different phantoms.
The measurement is the distribution of current through the boundary

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \]

\[ \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
The measurements are very similar, although the conductivities are quite different.

\( \sigma_1 \) \( \frac{\partial u_1}{\partial \vec{n}} \) \( \sigma_2 \) \( \frac{\partial u_2}{\partial \vec{n}} \)
Let us apply the more oscillatory distribution $f(\theta) = \cos 2\theta$ of voltage at the boundary.
The measurement is again the distribution of current through the boundary

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \]

\[ \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
The current distribution measurements are again very similar

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \quad \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data

$$\sigma_1$$

$$\sigma_2$$

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
Many different types of reconstruction methods have been suggested for EIT in the literature

- **Linearization**: Barber, Bikowski, Brown, Calderón, Cheney, Isaacson, Mueller, Newell
- **Iterative regularization**: Dobson, Gehre, Hua, Jin, Kaipio, Kindermann, Kluth, Leitão, Lechleiter, Lipponen, Maass, Neubauer, Rieder, Rondi, Santosa, Seppänen, Tompkins, Webster, Woo
- **Bayesian inversion**: Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen
- **Resistor network methods**: Borcea, Druskin, Mamonov, Vasquez
- **Layer stripping**: Cheney, Isaacson, Isaacson, Somersalo
- **D-bar methods**: Astala, Bikowski, Bowerman, Delbary, Hansen, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Perämäki, Saulnier, S, Tamasan
- **Teichmüller space methods**: Kolehmainen, Lassas, Ola, S
### History of CGO-based methods for real 2D EIT

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Outline

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Practical implementation
The forward map $F : \mathbb{X} \supset \mathcal{D}(F) \rightarrow \mathbb{Y}$ of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \to X$ that inverts $F$ approximately.
A regularization strategy needs to be constructed so that the assumptions below are satisfied

A family $\Gamma_\alpha : Y \to X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a regularization strategy for $F$ if

$$\lim_{\alpha \to 0} \| \Gamma_\alpha(y) - x \|_X = 0$$

for each fixed $x \in \mathcal{D}(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called admissible if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $x \in \mathcal{D}(F)$ the following holds:

$$\sup_{y^\delta} \left\{ \| \Gamma_{\alpha(\delta)}(y^\delta) - x \|_X : \| y^\delta - y \|_Y \leq \delta \right\} \to 0 \text{ as } \delta \to 0.$$
Let us emphasize the difference between stability analysis and regularization strategies

Conditional stability results have the form

$$\|x - x'\|_X \leq f(\|y - y'\|_Y),$$

where \(f : \mathbb{R}^+ \to \mathbb{R}^+\) is a continuous function satisfying \(f(0) = 0\). However, the above inequality is practically irrelevant: the noisy measurement \(y^\delta\) is almost surely not in the range of \(F\).
Two main methods for nonlinear regularization:

In the Tikhonov approach, one writes a penalty functional such as

$$\Phi(x) = \|F(x) - y^\delta\|_Y^2 + \alpha\|x\|_X^2,$$

and finds $$\tilde{x}$$ such that $$\Phi(\tilde{x}) = \min\{\Phi(x) \mid x \in D(F)\}$$.

**Pro:** The same code applies to many problems.

**Con:** Prone to get stuck in local minima.

[Bissantz, Burger, Engl, Hanke, Hofmann, Hohage, Justen, Kaltenbacher, Kindermann, Lechleiter, Lu, Mathé, Morozov, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Ramm, Resmerita, Rieder, Scherzer, Seidman, Teschke, Vogel, Yagola]

In the problem-specific approach, one constructs a regularization strategy tailored for a given inverse problem.

**Pro:** Can deal efficiently with a specific nonlinearity.

**Con:** Each code applies to only one problem.
In practice, efficient regularized algorithms are needed for linear and nonlinear inverse problems.

Assume given a forward map $F$ and noisy data $y^\delta$.

An efficient regularized inversion algorithm should compute a numerical approximation to $\Gamma_{\alpha}(\delta)(y^\delta)$ quickly and accurately, where $\Gamma_{\alpha}$ is a regularization strategy with an admissible choice of regularization parameter.

The Tikhonov approach provides efficient regularized inversion algorithms only for linear and almost linear forward maps $F$.

Electrical impedance tomography is the only strongly nonlinear inverse problem with efficient regularized inversion algorithms, based on the problem-specific approach.

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Nachman’s 1996 uniqueness proof in 2D relies on complex geometric optics solutions

Define a potential $q$ by setting $q(z) \equiv 0$ for $z$ outside $\Omega$ and

$$q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for } z \in \Omega.$$

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(-\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz} \psi(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2),$$

where $\tilde{p} > 2$ and $ikz = i(k_1 + ik_2)(x + iy)$. 
The CGO solutions are constructed using a generalized Lippmann-Schwinger equation

Define $\mu(z, k) = e^{-ikz}\psi(z, k)$. Then $(-\Delta + q)\psi = 0$ implies

$$(-\Delta - 4ik\overline{\partial}_z + q)\mu(\cdot, k) = 0,$$

where the D-bar operator is defined by $\overline{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

A solution of (1) satisfying $\mu(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2)$ can be constructed using the Lippmann-Schwinger type equation

$$\mu = 1 - g_k \ast (q\mu),$$

where $g_k$ satisfies $(-\Delta - 4ik\overline{\partial}_z)g_k = \delta$ and is defined by

$$g_k(z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} \, d\xi_1 \, d\xi_2.$$
The Faddeev fundamental solution $g_1(z)$ has a logarithmic singularity at $z = 0$

It is enough to know $g_1(z)$ because of the relation $g_k(z) = g_1(kz)$. 
One of the breakthroughs in Nachman’s 1996 article is showing uniqueness of \( \mu \)

A solution of \((-\Delta - 4ik\overline{\partial}_z + q)\mu(\cdot, k) = 0\) satisfying \(\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)\) can be constructed using the formula

\[
\mu - 1 = [I + g_k \ast (q \cdot)]^{-1}(g_k \ast q),
\]

provided that the inverse operator exists.

Now \(q \in L^p(\mathbb{R}^2)\) with \(1 < p < 2\) and \(1/\tilde{p} = 1/p - 1/2\), and

\[
q \cdot : W^{1,\tilde{p}}(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \text{ is bounded,}
\]

\[
g_k \ast : L^p(\mathbb{R}^2) \to W^{1,\tilde{p}}(\mathbb{R}^2) \text{ is compact.}
\]

Thus \(I + g_k \ast (q \cdot) : W^{1,\tilde{p}}(\mathbb{R}^2) \to W^{1,\tilde{p}}(\mathbb{R}^2)\) is Fredholm of index zero, and Nachman proved injectivity for all \(k \neq 0\).
The conductivity $\sigma$ can be recovered from the functions $\mu(z, k)$ at $k = 0$

Recall that

$$(-\Delta - 4ik\bar{\partial}z + q)\mu(\cdot, k) = 0$$

with the asymptotics

$$\mu(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2).$$

Substituting $k = 0$ gives

$$(-\Delta + \frac{\Delta \sqrt{\sigma}}{\sqrt{\sigma}})\mu(\cdot, 0) = 0,$$

and setting $\mu(z, 0) = \sqrt{\sigma(z)}$ gives the unique solution of (2) satisfying $\mu(z, 0) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2)$. 
The crucial intermediate object in the proof is the non-physical scattering transform \( t(k) \).

We denote \( z = x + iy \in \mathbb{C} \) or \( z = (x, y) \in \mathbb{R}^2 \) whenever needed. The scattering transform \( t : \mathbb{C} \to \mathbb{C} \) is defined by

\[
t(k) := \int_{\mathbb{R}^2} e^{i\bar{k}z} q(z) \psi(z, k) \, dx \, dy.
\]

Sometimes (3) is called the nonlinear Fourier transform of \( q \). This is because asymptotically \( \psi(z, k) \sim e^{ikz} \) as \( |z| \to \infty \), and substituting \( e^{ikz} \) in place of \( \psi(z, k) \) into (3) results in

\[
\int_{\mathbb{R}^2} e^{i(kz + \bar{k}z)} q(z) \, dx \, dy = \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx \, dy
\]

\[
= \hat{q}(-2k_1, 2k_2).
\]
Alessandrini’s equation gives a way to write \( t \) in terms of \( \Lambda_\sigma \) and traces of the CGO solutions

The following boundary integral equation is a Fredholm equation of the second kind and uniquely solvable in the space \( H^{1/2}(\partial \Omega) \):

\[
\psi(\cdot, k)|_{\partial \Omega} = e^{ikz}|_{\partial \Omega} - S_k(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k).
\]

Here \( S_k \) is the single-layer operator with Faddeev Green’s function:

\[
(S_k \phi)(z) := \int_{\partial \Omega} G_k(z - \zeta)\phi(\zeta) \, ds(\zeta),
\]

where \( G_k(z) := e^{ikz}g_k(z) \) satisfies \(-\Delta G_k = \delta\).  

The scattering transform can be evaluated by

\[
t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds.
\]
The difference between the usual Green’s function and Faddeev Green’s function is exponential.

Usual: $G_0(z) = -\frac{1}{2\pi} \log |z|$

Faddeev: $G_1(z) = e^{iz} g_1(z)$
The functions $\mu$ can be recovered from the scattering transform $t$ using a D-bar equation

It is natural to ask whether $\mu(z, k)$ depends analytically on the parameter $k$. If it does, the D-bar operator

$$\frac{\partial}{\partial k} = \frac{1}{2} \left( \frac{\partial}{\partial k_1} + i \frac{\partial}{\partial k_2} \right)$$

will give zero when applied to $\mu(z, k)$.

It turns out that the $\overline{k}$-differential of $g_k^*$ is a rank-one operator, and differentiating $\mu = 1 - g_k^* (q\mu)$ yields

$$\frac{\partial}{\partial k} \mu(z, k) = \frac{1}{4\pi k} t(k) e_{-k}(z) \mu(z, k).$$

Thus the dependence of $\mu(z, k)$ on $k$ is not analytic. The D-bar equation was discovered by Beals and Coifman in the 1980’s.
These are the steps of Nachman’s 1996 proof:

Solve boundary integral equation:

\[ \psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]

for every complex number \( k \in \mathbb{C} \).

Evaluate the scattering transform:

\[ t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation:

\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu(z, k) \]

with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

Fredholm equation of 2nd kind, ill-posedness shows up here.

Simple integration.

Well-posed problem, can be analyzed by scattering theory.

Trivial step.
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### Infinite-precision data:

Solve boundary integral equation

\[ \psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]

for every complex number \( k \in \mathbb{C} \).

Evaluate the scattering transform:

\[ t(k) = \int_{\partial\Omega} e^{ik\bar{z}}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \frac{1}{\mu(z, k)} \]

with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

### Practical data:

Solve boundary integral equation

\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta \]

for all \( |k| < R = R(\delta) \).

For \( |k| \geq R \) set \( t^\delta_R(k) = 0 \). For \( |k| < R \)

\[ t^\delta_R(k) = \int_{\partial\Omega} e^{ik\bar{z}}(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial k} \mu_R(z, k) = \frac{t_R^\delta(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \frac{1}{\mu_R(z, k)} \]

with \( \mu_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma^\alpha(\delta)(\Lambda_\sigma^\delta) := (\mu_R^\delta(z, 0))^2 \).
We define spaces for our regularization strategy

Consider $F : X \supset D(F) \to Y$ with $X = L^\infty(\Omega)$. Let $M > 0$ and $0 < \rho < 1$. Now $D(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ satisfying

$$\|\sigma\|_{C^2(\overline{\Omega})} \leq M, \quad \sigma(z) \geq M^{-1}, \quad \text{and} \quad \sigma(z) \equiv 1 \text{ for } \rho < |z| < 1.$$

$Y$ comprises bounded linear operators $A : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$ satisfying $A(1) = 0$ and $\int_{\partial \Omega} A(f) \, ds = 0$. 
Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

There exists a constant $0 < \delta_0 < 1$, depending only on $M$ and $\rho$, with the following properties. Let $\sigma \in \mathcal{D}(F)$ be arbitrary and assume given noisy data $\Lambda^\delta_\sigma$ satisfying

$$\|\Lambda^\delta_\sigma - \Lambda_\sigma\|_Y \leq \delta < \delta_0.$$ 

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_\alpha(\delta)(\Lambda^\delta_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \delta)^{-1/14}.$$
The proof of the main theorem is divided into several steps. First a D-bar estimate:

**Lemma 1.** Let $4/3 < r_0 < 2$ and suppose that $\phi_1, \phi_2 \in L^r(\mathbb{R}^2)$ for all $r \geq r_0$. Let $\mu_1, \mu_2$ be the solutions of

$$
\mu_j(z, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\phi_j(k')}{(k - k')} \mu_j(z, k') \, dk_1' \, dk_2',
$$

$j = 1, 2$. Then for fixed $z \in \overline{\Omega}$ we have

$$
\|\mu_1(z, \cdot) - \mu_2(z, \cdot)\|_{C^\beta(\mathbb{R}^2)} \leq C \|\phi_1 - \phi_2\|_{L^{r_0} \cap L^{r_0'}(\mathbb{R}^2)},
$$

where $\beta < 2/r_0 - 1$ and $1/r_0' = 1 - 1/r_0$.

**Proof.** Combination of well-known results.
These results follow from careful analysis of Faddeev Green’s function

Lemma 2. Let $\phi_0 \in H^{-1/2}(\partial \Omega)$ with $\int_{\partial \Omega} \phi_0 ds = 0$. Then

$$\|S_k \phi_0\|_{H^{1/2}(\partial \Omega)} \leq C e^{2|k|} (1 + |k|) \|\phi_0\|_{H^{-1/2}(\partial \Omega)}.$$ 

Lemma 3. For $k \in \mathbb{C}$ we have the estimate

$$\|\left[ I + S_k (\Lambda_\sigma - \Lambda_1) \right]^{-1} \|_{L(H^{1/2}(\partial \Omega))} \leq C e^{2|k|} (1 + |k|),$$

where $C$ depends only on $M$ and $\rho$. 
Combining previous results, a perturbation argument, and delicate $L^p$ analysis shows

**Lemma 4.** There exists $\delta_0 > 0$, depending only on $M$ and $\rho$, such that the equation

$$
\psi^{\delta}(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda^{\delta}_\sigma - \Lambda_1)\psi^{\delta}(\cdot, k)|_{\partial\Omega}
$$

is solvable in $H^{1/2}(\partial\Omega)$ for all $0 < \delta < \delta_0$ and $|k| < R$ with

$$
R = R(\delta) = -\frac{1}{10} \log \delta.
$$

Furthermore, for $p > 1$ we have the estimate

$$
\left\| \frac{t(k) - t^{\delta}_{R}(k)}{k} \right\|_{L^p(|k|<R)} \leq C \delta^{1/10} \left(-\frac{1}{10} \log \delta \right)^{2/p},
$$

where $C$ is independent of $p$ and $R$ and $\delta$. 
Sketch of the proof that nonlinear low-pass filtering gives a regularization strategy for EIT

We need to show the following:

(i) \( \lim_{\alpha \to 0} \| \Gamma_{\alpha}(\Lambda_{\sigma}) - \sigma \|_X = 0 \),

(ii) \( \alpha(\delta) \to 0 \) as \( \delta \to 0 \),

(iii) \( \sup_{\Lambda_{\sigma}^\delta} \{ \| \Gamma_{\alpha(\delta)}(\Lambda_{\sigma}^\delta) - \sigma \|_X : \| \Lambda_{\sigma}^\delta - \Lambda_{\sigma} \|_Y \leq \delta \} \to 0 \) as \( \delta \to 0 \).

Claim (i) follows from Lemma 1 and careful choices of Lebesgue exponents in Nachman's original proof. Claim (ii) holds by the definition

\[ \alpha(\delta) = \frac{1}{R(\delta)} = -10(\log \delta)^{-1}. \]
Proof of claim (iii)

To prove that the worst-case reconstruction error

$$\sup_{\Lambda_{\delta}} \left\{ \| \Gamma_{\alpha(\delta)}(\Lambda_{\delta}) - \sigma \|_X : \| \Lambda_{\delta} - \Lambda_{\sigma} \|_Y \leq \delta \right\}$$

tends to zero as $\delta \to 0$ we combine Nachman’s results with
Lemmas 1 and 4 to estimate

$$\| \mu(z, \cdot) - \mu_R(z, \cdot) \|_{C^\beta(\mathbb{R}^2)} \leq C \left\| \frac{t(k) - t^\delta_R(k)}{k} \right\|_{L^p \cap L^{p'}(\mathbb{C})} \leq C \left\| \frac{t(k) - t^\delta_R(k)}{k} \right\|_{L^p \cap L^{p'}(|k|<R)} + C \left\| \frac{t(k)}{k} \right\|_{L^p(|k|>R)} \leq C \left( -\log \delta \right)^{10/7} \delta^{1/10} + R(\delta)^{-1/7} + R(\delta)^{-1/14} \right) \leq C \left( -\log \delta \right)^{10/7} \delta^{1/10} + (- \log \delta)^{-1/7} + (- \log \delta)^{-1/14} \right).$$
We still need to define the regularization strategy on all of the data space $Y$, not only near $F(D(F))$.

The previous results show the claim only for operators $\delta_0$-close to the range $F(D(F)) \subset Y$.

The structure of the set $F(D(F))$ is not understood at the moment.

However, the proof can be extended to the whole data space $Y$ using spectral-theoretic arguments.
Outline

Electrical impedance tomography

Regularization of nonlinear inverse problems

D-bar method for infinite-precision data

Regularization using non-linear low-pass filtering

Practical implementation
Let us analyze how the regularization works using a simulated heart-and-lungs phantom.
Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

\[ \varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}. \]

We invert the linear operator appearing in the equation

\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = \left[ I + S_k(\Lambda_\delta - \Lambda_1) \right]^{-1} e^{ikz}|_{\partial\Omega} \]

as a matrix in \( \text{span}(\{\varphi_n\}_{n=-N}^N) \).

The single-layer operator

\[ (S_k\phi)(z) = \int_{\partial\Omega} G_k(z-w)\phi(w) \, ds(w) \]

uses Faddeev’s Green’s function.
This is how the actual scattering transform looks like in the disc $|k| < 10$, computed by knowing $\sigma$.
Scattering transform in the disc $|k| < 10$, here computed from noisy measurement $\Lambda^\delta_\sigma$
Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

The generalization of Vainikko’s method for the D-bar equation is described in [Knudsen, Mueller & S 2004].

The D-bar equation

\[
\frac{\partial}{\partial k} \mu_R^\delta = \frac{1}{4\pi k} t_R^\delta(k) e^{-z(k)} \mu_R^\delta
\]

together with the asymptotics

\[
\mu_R^\delta(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})
\]

can be combined in a generalized Lippmann-Schwinger equation:

\[
\mu_R^\delta(z, k) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{t_R^\delta(k')}{(k - k')k'} e^{-z(k')} \mu_R^\delta(z, k') \, dk_1' \, dk_2'.
\]
This is the real-linear operation given to GMRES

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\]

Element-wise multiplication

\[
\begin{bmatrix}
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi
\end{bmatrix}
\]

\[
\phi - \frac{1}{\pi k} \ast (T_R \overline{\phi})
\]
Regularized reconstructions from simulated data with noise amplitude $\delta = \|\Lambda_\sigma^\delta - \Lambda_\sigma\|_Y$

\[
\delta \approx 10^{-6} \\
\delta \approx 10^{-5} \\
\delta \approx 10^{-4} \\
\delta \approx 10^{-3} \\
\delta \approx 10^{-2}
\]

The percentages are the relative square norm errors in the reconstructions.
The observed radii are better (=larger) than those given by the theoretical formula $R(\delta) = -\frac{1}{10} \log \delta$
The method works for real data as well, including laboratory phantoms and *in vivo* human data

[Isaacson, Mueller, Newell & S 2006]
Unknown boundary shape can be estimated from EIT data using Teichmüller space methods

[Kolehmainen, Lassas, Ola & S 2012]
Conclusion

We have constructed the first direct (non-iterative) regularization strategy for a global nonlinear PDE coefficient recovery problem.

Efficient implementation available, based on Vainikko’s method.

The nonlinear low-pass filter regularization approach has an explicit speed of convergence in a Banach space setting.
The Astala-Päivärinta approach can recover discontinuous conductivities quite accurately.

Conductivity $\sigma_1$  

Reconstruction from ideal data, $R = 6$  

Reconstruction with 0.01% noise, $R = 5.5$

D. Isaacson, J.L. Mueller, J.C. Newell, and S. S.
Imaging cardiac activity by the d-bar method for electrical impedance tomography.

K. Knudsen, M. Lassas, J.L. Mueller, and S. S.
Regularized d-bar method for the inverse conductivity problem.

A. I. Nachman.
Global uniqueness for a two-dimensional inverse boundary value problem.

An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem.