The magic of math: three-dimensional X-ray vision

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Finland

UNIVERSITY OF JYVÄSKYLÄ
TAMPERE UNIVERSITY OF TECHNOLOGY
FINNISH METEOROLOGICAL INSTITUTE
Aalto University

UNIVERSITY OF HELSINKI
LUT
Lappeenranta University of Technology

UNIVERSITY OF EASTERN FINLAND

ACADEMY OF FINLAND
This is my industrial-academic background

2009: Professor, University of Helsinki, Finland

2006: Professor, Tampere University of Technology, Finland

2005: R&D scientist at Palodex Group

2004: R&D scientist at GE Healthcare

2002: Postdoc at Gunma University, Japan

2000: R&D scientist at Instrumentarium Imaging

1999: PhD, Helsinki University of Technology, Finland
Kumpula Science Campus of University of Helsinki
Outline

What is an X-ray image?

Slice imaging: X-ray tomography

Are you a natural tomographer?

Filtered back-projection (FBP)

X-ray vision with small number of X-rays

Industrial case study: low-dose 3D dental X-ray imaging

X-ray vision without X-rays
Wilhelm Conrad Röntgen invented X-rays and was awarded the first Nobel Prize in Physics in 1901.
X-ray intensity attenuates inside matter, here shown with a homogeneous block

https://www.youtube.com/watch?v=lfXo2S1xXCQ
We can see through a box of candy!

https://www.dropbox.com/s/e7i3exqc4sdpr1s/Sisu2.mp4?dl=0
X-ray images are very useful for doctors. For example, they can see fractures.
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X-ray vision without X-rays
Here is a 2D slice through a human head
Now the attenuation process is more complicated because there are different tissues

https://youtu.be/lvUAOeS1sv8
After calibration we are observing how much attenuating matter the X-ray encounters in total

https://youtu.be/RFArLtWEfsQ
This sweeping movement is the data collection mode of first-generation CT scanners

https://youtu.be/JHUz5oyeZb0
This sweeping movement is the data collection mode of first-generation CT scanners

https://youtu.be/JHUz5oyeZb0
Godfrey Hounsfield and Allan McLeod Cormack developed X-ray tomography

Hounsfield (top) and Cormack received Nobel prizes in 1979.
Reconstruction of a function from its line integrals was first invented by Johann Radon in 1917.

\[ f(P) = -\frac{1}{\pi} \int_0^\infty \frac{d\overline{F}_p(q)}{q} \]

Johann Radon (1887-1956)
Modern CT scanners look like this
Modern scanners rotate at high speed

https://commons.wikimedia.org/wiki/File:CT-Rotation.ogv
This is the inverse problem of tomography: we only know the data

https://youtu.be/pr8bXB0oAqI
This is an illustration of the standard reconstruction by filtered back-projection

https://youtu.be/tRD58lO1FKw
Diagnosing stroke with X-ray tomography

Ischemic stroke

CT image from Jansen 2008

Hemorrhagic stroke

CT image from Nakano et al. 2001
Unusual variant of the Nutcracker Fracture of the calcaneus and tarsal navicular

[Gajendran, Yoo & Hunter, Radiology Case Reports 3 (2008)]
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X-ray vision without X-rays
Let’s warm up before the tests.
Here is tomographic data of a simple object:
Can you guess the shape of the object from the tomographic data?
Test: can you guess the image?

https://youtu.be/NishyJWhXDk
Alternatives
Solution

https://youtu.be/MkAQoF3YOwg
Test: can you guess the image?

https://youtu.be/jP4AC7l8guo
Alternatives
Solution

https://youtu.be/epHR4x3up8I
Test: can you guess the image?

https://youtu.be/ZJaek4nkcRA
Alternatives

S  B  W
Solution

https://youtu.be/YHpG5HqDmZk
Test: can you guess the image?

Alternatives
Solution

https://youtu.be/k0ArBgCx0n0
Test: can you guess the image?

https://youtu.be/goddXsubZO8
Solution

https://youtu.be/RfKA3R2-pjk
Test: can you guess the image?

https://youtu.be/8ZrRazVdRjM
Alternatives
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X-ray vision without X-rays
The inverse problem of tomography is to recover the unknown target from the measured X-ray data.

https://youtu.be/YhClb0MaB70
Since we know the projection directions, we can back-project the data into the image.
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Back-projection becomes more useful by summing up the images
Summing all the back-projections results in a blurred reconstruction
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Summing all the back-projections results in a blurred reconstruction.
Here we use more directions, so the reconstruction quality is higher

https://youtu.be/5DUGTXd26nA
Final reconstruction involves filtering on top of the back-projection

\[ \hat{f}(\xi) \]

Multiplication with “ice-cream cone”

\[ |\xi| \hat{f}(\xi) \]

FFT \quad IFFT
This is an illustration of the standard reconstruction by filtered back-projection

https://youtu.be/tRD58lO1FKw
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X-ray vision without X-rays
We collected X-ray projection data of a walnut from 1200 directions.

Data collection: thanks to Keijo Hämäläinen and Aki Kallonen, University of Helsinki.

The data is openly available at http://fips.fi/dataset.php, thanks to Esa Niemi and Antti Kujanpää.
Reconstructions of a 2D slice through the walnut using filtered back-projection (FBP)

FBP with comprehensive data (1200 projections)  FBP with sparse data (20 projections)
Sparse-data reconstruction of the walnut using non-negative total variation regularization

Filtered back-projection

Constrained TV regularization

$$\arg \min_{f \in \mathbb{R}^n_+} \left\{ \| Af - m \|_2^2 + \alpha \| \nabla f \|_1 \right\}$$
Consider a simple example of a 2D square patient, whose internal structures consist of small squares $8 = 2 + 6$. 

```
 2 6
2 7
```
Two horizontal X-rays give us two numbers: row sums of the $2 \times 2$ array of attenuations.

\[
\begin{array}{cc}
2 & 6 \\
2 & 7 \\
\end{array}
\]

- Row sum of the first row: $8 \ (= 2 + 6)$
- Row sum of the second row: $9 \ (= 2 + 7)$
Tomographic imaging requires collecting X-ray data along another direction as well.

```
\begin{array}{cc}
2 & 6 \\
2 & 7 \\
\end{array}
\end{array}
```
“Direct problem” in this example is to compute row and column sums of a known interior

<table>
<thead>
<tr>
<th>2</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

Row sums: 8, 9
Column sums: 4, 13
“Inverse problem” in this example is to recover the interior numbers from the measurements.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

4 13

8

9
With such a limited amount of data, the inverse problem has multiple solutions!
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With such a limited amount of data, the inverse problem has multiple solutions!
So-called “ghosts,” or targets with zero data, are the source of multiple solutions.
Adding a ghost does not change the data!

\[
\begin{array}{c|c}
2 & 6 \\
\hline
2 & 7 \\
\end{array}
\quad + 
\begin{array}{c|c}
2 & -2 \\
\hline
-2 & 2 \\
\end{array}
\quad = 
\begin{array}{c|c}
4 & 4 \\
\hline
0 & 9 \\
\end{array}
\]
How can a reconstruction method pick out the correct image among all that match the data?
Consider these three candidates for reconstruction

True target

```
2  6
2  7
```

Wrong data, good “tissue type”

```
3  3
3  3
```

Right data, bad “tissue type”

```
4  4
0  9
```
Penalty calculation for candidate 1 (true target). First the penalty from (mis)matching X-ray data.

\[
\begin{array}{cc}
2 & 6 \\
2 & 7 \\
\end{array}
\]

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).
Penalty calculation for candidate 1.
Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).
Prior penalty: \(|2 - 6|\)
Penalty calculation for candidate 1.
Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).
Prior penalty: \(|2 - 6| + |2 - 7|\)
Penalty calculation for candidate 1.
Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).
Prior penalty: \(|2 - 6| + |2 - 7| + |2 - 2|\)
Penalty calculation for candidate 1.
Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).
Prior penalty: \(|2 - 6| + |2 - 7| + |2 - 2| + |6 - 7| = 4 + 5 + 0 + 1 = 10\).
Penalty calculation for candidate 1. Total penalty is the sum of data and prior penalties

\[
\begin{array}{c c}
2 & 6 \\
2 & 7
\end{array}
\]

\[
\begin{align*}
\text{data penalty} & \quad 0 \\
+ \quad \text{prior penalty} & \quad 10 \\
\hline
= \quad \text{total penalty} & \quad €10
\end{align*}
\]
Penalty calculation for candidate 1. Total penalty is the sum of data and prior penalties.

\[
\begin{align*}
\text{data penalty} & \quad 0 \\
+ \text{prior penalty} & \quad 10 \\
\hline
= \text{total penalty} & \quad \$10
\end{align*}
\]
Penalty calculation for candidate 2.
First the penalty from (mis)matching X-ray data

\[
\begin{array}{|c|c|}
\hline
3 & 3 \\
\hline
3 & 3 \\
\hline
\end{array}
\]

\[(6 - 8)^2 \quad (6 - 9)^2 \]

\[(6 - 4)^2 \quad (6 - 13)^2 \]

Data penalty: \(2^2 + 3^2 + 2^2 + 7^2 = 4 + 9 + 4 + 49 = 66.\)
Penalty calculation for candidate 2.
Then the penalty from prior information

Data penalty: \[2^2 + 3^2 + 2^2 + 7^2 = 4 + 9 + 4 + 49 = 66.\]
Prior penalty: \[|3 - 3| + |3 - 3| + |3 - 3| + |3 - 3| = 0.\]
Penalty calculation for candidate 2. Total penalty is the sum of data and prior penalties

\[
\begin{array}{cc}
3 & 3 \\
3 & 3 \\
\end{array}
\]

\[
data\text{ penalty} \quad 66 \\
+ \quad \text{prior penalty} \quad 0 \\
\hline
= \text{total penalty} \quad $66
\]
Penalty calculation for candidate 3.
First the penalty from (mis)matching X-ray data

\[(8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0.\]
Penalty calculation for candidate 3.
Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).

Prior penalty: \(|4 - 4| + |0 - 9| + |4 - 0| + |4 - 9| = 0 + 9 + 4 + 5 = 18\).
Penalty calculation for candidate 3. Total penalty is the sum of data and prior penalties.

\[
\begin{array}{cc}
4 & 4 \\
0 & 9
\end{array}
\]

\[
data \text{ penalty} \quad 0 \\
+ \text{ prior penalty} \quad 18 \\
\hline
= \text{ total penalty} \quad $18
\]
Which of candidates has smallest total penalty?

True target

\[
\begin{array}{cc}
2 & 6 \\
2 & 7 \\
\end{array}
\]

\[
data \text{ penalty } 0 \\
+ \text{ prior penalty } 10 \\
= \text{ total penalty } $10
\]

Wrong data, good “tissue type”

\[
\begin{array}{cc}
3 & 3 \\
3 & 3 \\
\end{array}
\]

\[
data \text{ penalty } 66 \\
+ \text{ prior penalty } 0 \\
= \text{ total penalty } $66
\]

Right data, bad “tissue type”

\[
\begin{array}{cc}
4 & 4 \\
0 & 9 \\
\end{array}
\]

\[
data \text{ penalty } 0 \\
+ \text{ prior penalty } 18 \\
= \text{ total penalty } $18
\]
Which of candidates has smallest total penalty?

True target

\[
\begin{align*}
\text{data penalty} & \quad 0 \\
+ \text{prior penalty} & \quad 10 \\
= \text{total penalty} & \quad $10
\end{align*}
\]

Wrong data, good “tissue type”

\[
\begin{align*}
\text{data penalty} & \quad 66 \\
+ \text{prior penalty} & \quad 0 \\
= \text{total penalty} & \quad $66
\end{align*}
\]

Right data, bad “tissue type”

\[
\begin{align*}
\text{data penalty} & \quad 0 \\
+ \text{prior penalty} & \quad 18 \\
= \text{total penalty} & \quad $18
\end{align*}
\]
The problem can be solved in general using optimization.

General target

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_2$</td>
<td>$X_4$</td>
</tr>
</tbody>
</table>

Find numbers $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and $x_4 \geq 0$ such that the sum of these two penalties is as small as possible:

Data penalty: $(x_1 + x_3 - 8)^2 + (x_2 + x_4 - 9)^2 + (x_1 + x_2 - 4)^2 + (x_3 + x_4 - 13)^2$

Prior penalty: $|x_1 - x_3| + |x_2 - x_4| + |x_1 - x_2| + |x_3 - x_4|$

This method is called total variation regularization.
Solutions from optimization methods

True target

\[\begin{array}{cc}
2 & 6 \\
2 & 7 \\
\end{array}\]

\[
data \text{ penalty} \quad 0 \\
+ \text{ prior penalty} \quad 10 \\
= \text{ total penalty} \quad $10
\]

Total variation regularization

\[\begin{array}{cc}
2.5 & 6 \\
2.5 & 6 \\
\end{array}\]

\[
data \text{ penalty} \quad 1.6 \\
+ \text{ prior penalty} \quad 7.0 \\
= \text{ total penalty} \quad $8.6
\]
### Solutions from optimization methods

<table>
<thead>
<tr>
<th>True target</th>
<th>Total variation regularization</th>
<th>Minimum square-norm solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 6</td>
<td>2.5 6</td>
<td>1.75 6.25</td>
</tr>
<tr>
<td>2 7</td>
<td>2.5 6</td>
<td>2.25 6.75</td>
</tr>
</tbody>
</table>

- Data penalty + Prior penalty = Total penalty
  - **True target**
    - Data penalty: 0
    - Prior penalty: 10
    - Total penalty: $10
  - **Total variation regularization**
    - Data penalty: 1.6
    - Prior penalty: 7.0
    - Total penalty: $8.6
  - **Minimum square-norm solution**
    - Data penalty: 0
    - Prior penalty: 10
    - Total penalty: $10
Solutions from optimization methods, in grayscale

**True target**

```
data penalty  0  
+ prior penalty 10  
= total penalty $10
```

**Total variation regularization**

```
data penalty  1.6  
+ prior penalty 7.0  
= total penalty $8.6
```

**Minimum square-norm solution**

```
data penalty  0  
+ prior penalty 10  
= total penalty $10
```
Resolution
2×2
Resolution
4×4
Resolution
8×8
Resolution
$16 \times 16$
Resolution
64×64
Resolution
128×128
Resolution
256 $\times$ 256
Resolution
512×512
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X-ray vision without X-rays
Application: dental implant planning, where a missing tooth is replaced with an implant.
This is the classical imaging procedure of the panoramic X-ray device

https://www.youtube.com/watch?v=QFTXegPxC4U
The resulting image shows a sharp layer positioned inside the dental arc.
Nowadays, a digital panoramic imaging device is standard equipment at dental clinics.

A panoramic dental image offers a general overview showing all teeth and other structures simultaneously.

Panoramic images are not suitable for dental implant planning because of unavoidable geometric distortion.
We reprogram the panoramic X-ray device so that it collects projection data by scanning

https://www.youtube.com/watch?v=motthjiP8ZQ
We reprogram the panoramic X-ray device so that it collects projection data by scanning.

Number of projection images: 11
Angle of view: 40 degrees
Image size: $1000 \times 1000$ pixels

The unknown vector $f$ has 7,000,000 elements.
Standard Cone Beam CT reconstruction delivers 100 times more radiation than VT imaging

Kolehmainen, Vanne, S, Järvenpää, Kaipio, Lassas & Kalke 2006
Kolehmainen, Lassas & S 2008
Cederlund, Kalke & Welander 2009
Hyvönen, Kalke, Lassas, Setälä & S 2010
U.S. patent 7269241, thousands of VT units in use
The VT device was developed in 2001–2012 by

Nuutti Hyvönen
Seppo Järvenpää
Jari Kaipio
Martti Kalke
Petri Koistinen
Ville Kolehmainen
Matti Lassas
Jan Moberg
Kati Niinimäki
Juha Pirttilä
Maaria Rantala
Eero Saksman
Henri Setälä
Erkki Somersalo
Antti Vanne
Simopekka Vänskä
Richard L. Webber
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Tomography appears in adaptive optics

- Modern telescope imaging suffers from turbulence in the atmosphere ⇒ blurring of images
- **Adaptive optics** corrects the perturbed incoming light in real-time
- Major challenge in wide-field AO: atmospheric tomography

Helin, Kindermann, Lehtonen & Ramlau 2018
Yudytskiy, Helin & Ramlau 2014
The mathematics of X-ray tomography can be used for recovering the ozone layer

European Space Agency
Finnish Meteorological Institute
Envisat and GOMOS projects
Thanks to Johanna Tamminen!
Cosmic muon imaging revealed a secret chamber inside the Pyramid of Cheops
Imaging with neutrons opens up new possibilities as water attenuates but metal is transparent

Video:
Anders Kaestner
Neutron Imaging and Activation Group,
Paul Scherrer Institute
Electron transmission cryotomography reveals the swimming engine of *Treponema primitia* bacteria

[Murphy, Leadbetter & Jensen 2016]
Thank you for your attention!