Classifying Stroke Using Electrical Impedance Tomography

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Finland

UNIVERSITY OF JYVÄSKYLÄ

TAMPERE UNIVERSITY OF TECHNOLOGY

FINNISH METEOROLOGICAL INSTITUTE

EUROPEAN UNION

ACADEMY OF FINLAND

Aalto University
Links to open computational resources

Open EIT datasets:
- Finnish Inverse Problems Society (FIPS) dataset page

Reconstruction algorithms: FIPS Computational Blog
- The D-bar Method for EIT—Simulated Data
- The D-bar Method for EIT—Experimental Data

For these slides see: http://www.siltanen-research.net/talks.html
Outline

Electrical impedance tomography (EIT)

Complex geometric optics (CGO) solutions, D-bar method

Application of EIT to stroke

Virtual Hybrid Edge Detection (VHED)
  The scattering series
  Filtered back-projection theorem

Combining machine learning with VHED
This section concentrates on applications of EIT to chest imaging

Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
Note that EIT data collection involves applying several current patterns

Saline and agar phantom  Apply current pattern $\cos \theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern \( \cos 2\theta \)

Measure the resulting voltages at the 32 electrodes.
Note that EIT data collection involves applying several current patterns

Saline and agar phantom  
Apply current pattern $\cos 3\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 4\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 5\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 16\theta$

Measure the resulting voltages at the 32 electrodes
The D-bar method works for real EIT data, such as laboratory phantoms and *in vivo* human data.

Saline and agar phantom

Reconstruction \((R = 4)\)

[Isaacson, Mueller, Newell & S 2004]
[Montoya 2012]
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón.

Let \( \Omega \subset \mathbb{R}^2 \) be the unit disc and let conductivity \( \sigma : \Omega \to \mathbb{R} \) satisfy

\[
0 < M^{-1} \leq \sigma(z) \leq M.
\]

Applying voltage \( f \) at the boundary \( \partial \Omega \) leads to the elliptic PDE

\[
\begin{cases}
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
u \big|_{\partial \Omega} = f.
\end{cases}
\]

Boundary measurements are modelled by the Dirichlet-to-Neumann map

\[
\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial n} \bigg|_{\partial \Omega}.
\]

Calderón’s problem is to recover \( \sigma \) from the knowledge of \( \Lambda_\sigma \). It is a nonlinear and ill-posed inverse problem.
We illustrate the ill-posedness of EIT using a simulated example
We apply the voltage distribution \( f(\theta) = \cos \theta \) at the boundary of the two different phantoms.
The measurement is the distribution of current through the boundary.
The current data are very similar, although the conductivities are quite different.
Let us apply the more oscillatory distribution \( f(\theta) = \cos 2\theta \) of voltage at the boundary.
The measurement is again the distribution of current through the boundary

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \]

\[ \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
The current distribution measurements are almost the same

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \quad \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data.
We simulate current-to-voltage data for constructing the Dirichlet-to-Neumann matrix.

Define Fourier basis functions $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$.

$$
\begin{array}{cccc}
\mathcal{R} \exp(-i2\theta) & \mathcal{R} \exp(-i\theta) & \mathcal{R} \exp(i\theta) & \mathcal{R} \exp(i2\theta) \\
\exp(-i2\theta) & \langle \mathcal{R} \varphi_2, \varphi_2 \rangle & \langle \mathcal{R} \varphi_1, \varphi_2 \rangle & \langle \mathcal{R} \varphi_2, \varphi_2 \rangle \\
\exp(-i\theta) & \langle \mathcal{R} \varphi_2, \varphi_1 \rangle & \langle \mathcal{R} \varphi_1, \varphi_1 \rangle & \langle \mathcal{R} \varphi_2, \varphi_1 \rangle \\
\exp(i\theta) & \langle \mathcal{R} \varphi_2, \varphi_1 \rangle & \langle \mathcal{R} \varphi_1, \varphi_1 \rangle & \langle \mathcal{R} \varphi_2, \varphi_1 \rangle \\
\exp(i2\theta) & \langle \mathcal{R} \varphi_2, \varphi_2 \rangle & \langle \mathcal{R} \varphi_1, \varphi_2 \rangle & \langle \mathcal{R} \varphi_2, \varphi_2 \rangle \\
\end{array}
$$

Inverting the above matrix gives the DN matrix up to constant functions.
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Combining machine learning with VHED
This part is a joint work with

- **David Isaacson**, Rensselaer Polytechnic Institute, USA
- **Kim Knudsen**, Technical University of Denmark
- **Matti Lassas**, University of Helsinki, Finland
- **Jon Newell**, Rensselaer Polytechnic Institute, USA
- **Jennifer Mueller**, Colorado State University, USA
There exists a nonlinear Fourier transform adapted to electrical impedance tomography.
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements.

\[ \Lambda_\sigma \rightarrow \text{BIE} \rightarrow \text{Ideal measurement} \rightarrow \text{Nonlinear IFFT} \]

[Nachman 1996]
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies.

Practical measurement → BIE → Nonlinear IFFT
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
The D-bar method is a regularization strategy for reconstructing the full conductivity distribution.
<table>
<thead>
<tr>
<th><strong>Infinite-precision data:</strong></th>
<th><strong>Practical data:</strong></th>
</tr>
</thead>
</table>
| Solve boundary integral equation  \(|\partial \Omega| = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi  \)
for every complex number \(k \in \mathbb{C} \setminus 0\). | Solve boundary integral equation  \(|\partial \Omega|^\varepsilon| = e^{ikz} - S_k(\Lambda_\sigma^\varepsilon - \Lambda_1)\psi^\varepsilon  \)
for all \(0 < |k| < R = -\frac{1}{10} \log \varepsilon\). |
| Evaluate the scattering transform:  \(t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds\). | For \(|k| \geq R\) set \(t_R^\varepsilon(k) = 0\). For \(|k| < R\)  \(t_R^\varepsilon(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k) \, ds\). |
| Fix \(z \in \Omega\). Solve D-bar equation  \(\frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu(z, k)\)  
with \(\mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})\). | Fix \(z \in \Omega\). Solve D-bar equation  \(\frac{\partial}{\partial k} \mu_R^\varepsilon(z, k) = \frac{t_R^\varepsilon(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu_R^\varepsilon(z, k)\)  
with \(\mu_R^\varepsilon(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})\). |
| Reconstruct: \(\sigma(z) = (\mu(z, 0))^2\). | Set \(\Gamma_{1/R(\varepsilon)}(\Lambda_\sigma^\varepsilon) := (\mu_R^\varepsilon(z, 0))^2\). |
We define spaces for our regularization strategy

Model space $X = L^\infty(\Omega)$

Data space $Y$

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ with

- $\|\sigma\|_{C^2(\overline{\Omega})} \leq M,$
- $\sigma(z) \geq M^{-1},$
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1.$

Bounded linear operators $A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying

- $A(1) = 0,$
- $\int_{\partial\Omega} A(f) \, ds = 0.$
Nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

Fix a conductivity $\sigma \in \mathcal{D}(F)$. Assume given noisy data $\Lambda_\sigma^\varepsilon$ satisfying

$$\|\Lambda_\sigma^\varepsilon - \Lambda_\sigma\|_Y \leq \varepsilon.$$  

Then $\Gamma_\alpha$ with the choice

$$R(\varepsilon) = -\frac{1}{10} \log \varepsilon, \quad \alpha(\varepsilon) = \frac{1}{R(\varepsilon)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\varepsilon)}(\Lambda_\sigma^\varepsilon) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \varepsilon)^{-1/14}.$$
Here are the D-bar reconstructions from simulated EIT data using frequency cutoff $R = 4$.
The difference image shows clearly where the two patients are not the same.
D-bar images can be sharpened by Deep Learning

[Hamilton & Hauptmann 2017]
CGO solutions can serve as nonlinear features in machine learning

https://www.youtube.com/watch?v=onGMu7gweg

[Hamilton, Hauptmann & S 2014]
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Motivation of this study: imaging stroke with EIT

Ischemic stroke: low conductivity. CT image from Jansen 2008

Hemorrhagic stroke: high conductivity. CT image from Nakano et al. 2001

Same symptoms in both cases!

Difficulties: resistive skull layer and unknown background. However, see
- Romsauerova, McEwan, Horesh, Yerworth, Bayford & Holder 2006
- Shi, You, Xu, Wang, Fu, Liu & Dong 2009
- Bayford & Tizzard 2012
- Malone, Jehl, Arridge, Betcke & Holder 2014
- Boverman, Kao, Wang, Ashe, Davenport & Amm 2016
Brain EIT imaging is based on covering the head partly by electrodes

<table>
<thead>
<tr>
<th>Tissue</th>
<th>Ωcm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cortex</td>
<td>229</td>
</tr>
<tr>
<td>White matter</td>
<td>344</td>
</tr>
<tr>
<td>Blood</td>
<td>125</td>
</tr>
<tr>
<td>CS fluid</td>
<td>69</td>
</tr>
<tr>
<td>Scalp</td>
<td>490</td>
</tr>
<tr>
<td>Skull</td>
<td>6500</td>
</tr>
</tbody>
</table>

The current activity was initiated by Alex Ross from GE. He is a former student of David Isaacson.
The idea would be to equip every ambulance with an EIT device for classifying strokes.
Another important application of stroke-EIT is monitoring a patient in an intensive care unit.
We have a collaboration network in place for the stroke-EIT project

Project funded for 2017–2020
- Jari Hyttinen & Antti Paldanius (U Tampere)
- Ville Kolehmainen, Asko Hänninen & Jussi Toivanen (U Eastern Finland)
- S, Matti Lassas, Minh Mach & Rashmi Murthy (U Helsinki)

Finnish collaboration:
Stefan Björkman (U Helsinki)
Valentina Candiani (Aalto U)
Antti Hannukainen (Aalto U)
Nuutti Hyvönen (Aalto U)

International collaboration:
Juan Pablo Agnelli (U Córdoba)
Melody Alsaker (Gonzaga U)
Aynur Çöl (Sinop U)
Sarah Hamilton (Marquette U)
Andreas Hauptmann (UCL)
Jennifer Mueller (CSU),
Toshiaki Yachimura (Tohoku)

Nina Forss
Daniel Strbian
We can test our algorithms with realistic head phantoms.

Ville Kolehmainen, Asko Hänninen, Tuomo Savolainen, Jussi Toivanen
University of Eastern Finland
We consider three simulated 2D stroke phantoms: here healthy brain
We consider three simulated 2D stroke phantoms: here ischemic stroke
We consider three simulated 2D stroke phantoms: here hemorrhagic stroke
New result: inverse scattering methods can transform EIT into “X-ray tomography”

Video:

https://www.youtube.com/watch?v=37yOCfBfRJk

[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]
New result: inverse scattering methods can transform EIT into “X-ray tomography”

[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]
My team in Helsinki studies possibilities of using VHED as a nonlinear feature for AI.

<table>
<thead>
<tr>
<th></th>
<th>Margin 0.5</th>
<th>Margin 0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>ND matrix</td>
<td>0.982</td>
<td>0.979</td>
</tr>
<tr>
<td>DN matrix</td>
<td>0.985</td>
<td>0.970</td>
</tr>
<tr>
<td>VHED</td>
<td>0.999</td>
<td>0.997</td>
</tr>
</tbody>
</table>

[Agnelli, Çöl, Murthy and Siltanen, unpublished results]
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The results in this part are a joint work with

Allan Greenleaf, University of Rochester, NY, USA
Matti Lassas, University of Helsinki, Finland
Matteo Santacesaria, University of Helsinki, Finland
Gunther Uhlmann, University of Washington, USA
This project started in 2000
We consider exponentially behaving Complex Geometric Optics (CGO) solutions

Denote $x = (x_1, x_2) \in \mathbb{R}^2$ and $k = i\tau\theta$ where

$$\theta = \theta_1 + i\theta_2 \in \mathbb{C} \quad \text{with } |\theta| = 1.$$ 

Let $z = x_1 + ix_2 \in \mathbb{C}$ and

$$\eta = \eta_R + i\eta_I = (\theta_1 + i\theta_2, -\theta_2 + i\theta_1) \in \mathbb{C}^2,$$

so that $z\theta = x_1\theta_1 - x_2\theta_2 + i(x_1\theta_2 + x_2\theta_1) = x \cdot \eta$. Note that $\eta \cdot \eta = 0$. We consider solutions of the conductivity equation

$$\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega,$$

with a strictly positive conductivity $\sigma \in L^\infty(\Omega)$, of the form

$$u(x) = e^{i\tau\theta z} w(x, \tau) = e^{i\tau\eta \cdot x} w(x, \tau).$$
Since \( u(x) = e^{i\tau \eta \cdot x} w(x, \tau) \) satisfies the conductivity equation,

\[
\begin{align*}
0 &= \frac{1}{\sigma(x)} \nabla \cdot (\sigma(x) \nabla u(x)) \\
&= (\Delta + \frac{1}{\sigma}(\nabla\sigma) \cdot \nabla)(e^{i\tau \eta \cdot x} w(x, \tau)) \\
&= \left( \Delta w(x, \tau) + 2i\tau\eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla\sigma \right) \cdot (\nabla + i\tau \eta) w(x, \tau) \right) e^{i\tau \eta \cdot x}.
\end{align*}
\]

Hence, we have

\[
\Delta w(x, \tau) + 2i\tau\eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla\sigma \right) \cdot (\nabla + i\tau \eta) w(x, \tau) = 0.
\]
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

Let \( \hat{w}(x, t) \) be the Fourier transform of \( w(x, \tau) \) in the \( \tau \) variable:

\[
\hat{w}(x, t) = \mathcal{F}w(x, t) = \int_{\mathbb{R}} e^{-it\tau} w(x, \tau) \, d\tau.
\]

We call \( t \) the pseudo-time corresponding to complex frequency \( \tau \). Then equation

\[
\Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot \left( \nabla + i\tau \eta \right) w(x, \tau) = 0
\]
yields

\[
\Delta \hat{w}(x, t) + 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot \left( \nabla + \eta \frac{\partial}{\partial t} \right) \hat{w}(x, t) = 0.
\]
Complex principal type operator leads to singularities propagating along leaves

In the equation

\[
\Delta \hat{w}(x, t) - 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \frac{1}{\sigma} (\nabla \sigma) \cdot (\nabla - \eta \frac{\partial}{\partial t}) \hat{w}(x, t) = 0
\]

the principal part is

\[
\Delta - 2\eta \frac{\partial}{\partial t} \cdot \nabla,
\]

which is a complex principal type operator in the sense of Duistermaat and Hörmander.

For a real principal type operator the characteristic singularities propagate along one-dimensional rays. For instance, for the wave equation the light-like singularities propagate along light rays.

For a complex principal type operator the characteristic singularities propagate along two-dimensional surfaces called leaves.
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
We use the Beltrami-type complex geometric optics (CGO) solutions

Set $\mu := (1 - \sigma)/(1 + \sigma)$. Write $f = u + iv$ and note that

$$\overline{\partial}_z f_\mu = \mu \overline{\partial}_z f_\mu \iff \nabla \cdot \sigma \nabla u = 0 \text{ and } \nabla \cdot \sigma^{-1} \nabla v = 0.$$

The CGO solutions of [Astala-Päivärinta 2006] have the form

$$f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)),$$
$$f_{-\mu}(z, k) = e^{ikz}(1 + \omega^-(z, k)),$$

with the asymptotic condition

$$\omega^\pm(z, k) = \mathcal{O}\left(\frac{1}{z}\right) \text{ as } |z| \to \infty.$$

Here $ikz = i(k_1 + ik_2)(x + iy)$ and $\overline{\partial}_z = \frac{1}{2} (\partial_x + i\partial_y)$. 
This is a brief history of computational solution methods for the Beltrami CGO solutions

1987 Sylvester and Uhlmann: Introduction of CGO solutions
2000 S, Mueller and Isaacson: Numerical CGOs
2006 Astala and Päivärinta: Original Beltrami-type construction

2010 Astala, Mueller, Päivärinta and S: First numerical solution method

2011 Astala, Mueller, Päivärinta, Perämäki and S: Novel EIT reconstruction method

2012 Huhtanen and Perämäki: Preconditioned Krylov subspace method for real-linear systems

2014 Astala, Päivärinta, Reyes and S: Computational high-frequency experiments
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

In \( f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)) \), write the complex parameter in the form \( k = \tau e^{i\varphi} \) with \( \tau \in \mathbb{R} \). Denote \( \omega^+(z, \tau, e^{i\varphi}) = \omega^+(z, k) \).

Fourier transform \( \omega^+(z, \tau, e^{i\varphi}) \) in the \( \tau \) variable:

\[
\hat{\omega}^+(z, t, e^{i\varphi}) = \mathcal{F}_{\tau \rightarrow t}(\omega^+(z, \tau, e^{i\varphi})) = \int_{-\infty}^{\infty} e^{-it\tau} \omega^+(z, \tau, e^{i\varphi}) \, d\tau.
\]

We call \( t \) the pseudo-time.
Let us choose a simple rotationally symmetric conductivity for a test case

\[ \sigma(x, y) \]
$\hat{\omega}^+(-1, 2t, 1)$

Profile $\sigma(x, 0)$
$\hat{\omega}^+(-1, 2t, 1)$

$[\hat{\omega}^+ - \hat{\omega}^-](-1, 2t, 1)$
We define an averaging operator

**Definition** (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018)
Define operator $T^\pm$ by complex contour integral:

$$T^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi} \int_{0}^{2\pi} \hat{\omega}^\pm(e^{i\gamma}, t, e^{i\varphi})e^{i\gamma} d\gamma.$$
\[ \hat{\omega}^+(-1, 2t, 1) \]

\[ [\hat{\omega}^+ - \hat{\omega}^-](-1, 2t, 1) \]

\[ [T^+ - T^-] \mu(2t, 1) \]
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Define the Cauchy and Beurling transforms by

\[ Pf(z) = \frac{1}{\partial - 1} f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\lambda)}{\lambda - z} d\lambda_1 d\lambda_2, \]

\[ Sg(z) = \partial \bar{\partial}^{-1} g(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\lambda - z| > \varepsilon} \frac{g(\lambda)}{(\lambda - z)^2} d\lambda_1 d\lambda_2. \]

Also, set

\[ e_k(z) = \exp(i(kz + \bar{k}\bar{z})), \]

\[ \alpha(z, k) = -i \bar{k} e_{-k}(z) \mu(z), \]

\[ \nu(z, k) = e_{-k}(z) \mu(z), \]

and define the operator \( A \) by

\[ A := (\bar{\nu} S - \bar{\alpha} P). \]
We introduce a new scattering series

Huhtanen and Perämäki (2012) modified the original construction of Astala and Päivärinta (2006) for computational purposes. We use the 2012 technique for the construction of a novel scattering series

\[ \omega = \sum_{n=1}^{\infty} \omega_n, \]

where \( A := (-\bar{\nu}S - \bar{\alpha}P) \) and

\[ \omega_n = -\bar{\partial}_z^{-1} u_n, \quad u_n = -Au_{n-1}, \quad u_1 = -\bar{\alpha}. \]

The single scattering term \( \omega_1 = \bar{\partial}_z^{-1} \alpha \) determines singularities of \( \mu \).

The terms \( \omega_n \) with \( n > 1 \) arise from multiple scattering.
Let us consider a rotationally symmetric conductivity with jump along the circle $|z| = 0.2$. 

\[ k = \tau e^{i\varphi} \]

\[ \varphi = 0 \]
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
Here are the two lowest-order odd terms in the scattering series, with subtraction
Detail from the previous slide, with 70-fold magnification of the function inside green circle.
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Recovery by complex “filtered back-projection”

**Theorem.** (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018) Define averaged operators $T_j^\pm$ for $j = 1, 2, 3, \ldots$ by the complex contour integral:

$$T_j^\pm \mu(t, e^{i\phi}) = \frac{1}{2\pi i} \int_{\partial \Omega} \hat{\omega}_j^\pm(z, t, e^{i\phi}) dz,$$

Then we have a filtered back-projection formula

$$(-\Delta)^{-1/2}(T_1^\pm)^* T_1^\pm \mu = \mu.$$
Simple example of tomographic imaging with a double-disc target

https://youtu.be/5DUGTXd26nA
We can back-project the measured data into the image, integrating over all directions

https://youtu.be/5DUGTXd26nA
Final FBP reconstruction involves filtering on top of the back-projection.

\[ \text{Multiplication by } |\xi| \]
(Calderón’s operator)
**FBP-type reconstruction algorithm for EIT**

**Step 1.** Given the measurement $\Lambda_\sigma$, follow \[Astala, Mueller, Päivärinta, Perämäki & S 2011\] to compute both $\omega^+(x, k)$ and $\omega^-(x, k)$ for $x \in \partial \Omega$ by solving the boundary integral equation derived in \[Astala & Päivärinta 2006\].

*Note:* In practice this can only be done in a disc $|k| < R$ with $R$ depending on measurement noise amplitude.

**Step 2.** Write $k = \tau e^{i\varphi}$ and compute the partial Fourier transform to get $\hat{\omega}^\pm(z, t, e^{i\varphi})$.

*Note:* In practice the Fourier transform needs to be windowed.

**Step 3.** Reconstruct $\sigma = (\mu - 1)/(\mu + 1)$ approximately as $(\widetilde{\mu} - 1)/(\widetilde{\mu} + 1)$ using formula $\widetilde{\mu} = (\widetilde{\mu}^+ - \widetilde{\mu}^-)/2$ with

$$\widetilde{\mu}^\pm := \Delta^{-1/2} (T_1^\pm)^* T^\pm \mu.$$
Conductivity

Filtered back-projection
Conductivity Filtered back-projection “Λ-tomography”
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Minh Mach, University of Helsinki, Finland
Rashmi Murthy, University of Helsinki, Finland
Matteo Santacesaria, University of Helsinki, Finland
Gunther Uhlmann, University of Washington, USA
Toshiaki Yachimura, Tohoku University, Japan
We can see the difference in conductivity reflected in the VHED projections (blue and red graphs)
Given unrealistic-precision EIT measurements on full boundary we can classify the stroke easily.
Practical EIT measurements blur the information due to heavily windowed Fourier transform.
We simulate a set of 5000 conductivities with ischemic/hemorrhagic stroke on right hemisphere.
We simulate a set of 5000 conductivities with ischemic/hemorrhagic stroke on right hemisphere
We simulate a set of 5000 conductivities with ischemic/hemorrhagic stroke on right hemisphere
The conductivities have random parameters

Boundary: \( \sigma(x) = 1 \)

Skull: \( \sigma(x) \in [0.45, 0.55] \)

Stroke: \( \sigma(x) \in [3, 4] \)

Background: \( \sigma(x) \in [0.7, 0.8] \)

Background: \( \sigma(x) \in [0.8, 1.1] \)
Preliminary results on using VHED as a nonlinear feature for machine learning

We trained each Fully Connected Neural Network (FCNN) using the 5000 disc inclusions simulated in Experiment 1 and then we tested each network using 3500 samples corresponding to disc inclusions with the properties described in Experiment 2.

We repeated this process 10 times and then we computed the average accuracy of each network in each data set.

<table>
<thead>
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<td>DN matrix</td>
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<td>0.970</td>
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<tr>
<td>VHED</td>
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<td>0.997</td>
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</tbody>
</table>

[Agnelli, Çöl, Murthy and Siltanen, unpublished results]
Practical challenges in applying VHED

VHED works with ideal simulated data and simple digital phantoms. However, these issues must be solved before it can be applied to stroke classification:

**Data is noisy.** We know the Fourier transform of the desired function only in an interval $[-R, R]$ with $R \approx 4$.

**Anatomy is complicated.** Need to be tested with realistic phantoms.

**We can only measure on a part of the boundary.** Some progress is reported in [Hauptmann, Santacesaria and S 2017].

**Measurements are done using a finite number of electrodes.** Recovering CGO solutions from electrode data needs new research.

**People are three-dimensional.** VHED needs to be extended to 3D.
Thank you for your attention!