Inversion methods for three-dimensional medical X-ray imaging

Samuli Siltanen
Department of Mathematics
Tampere University of Technology
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1. X-rays and the inverse problem of tomography

2. Low-dose imaging using Bayesian inversion

3. Limited-angle dental X-ray tomography

4. Discretization-invariance

5. Besov space priors and 3D mammography
X-ray images as measurements

X-ray source

Detector

250  500  1000

Logarithm  5.5  6.2  6.9
Density     1.4  0.7  0.0
Every X-ray measures the sum of attenuation through tissue
Direct problem of tomography is to find the radiographs from given tissue.
Inverse problem of tomography is to find the tissue from radiographs.

9 unknowns, 11 linear equations.
The limited angle problem is harder than the full angle problem.

- 9 unknowns, 11 linear equations
- 9 unknowns, 6 linear equations
In limited angle 3D imaging there are many tissues matching the radiographs

\[
\begin{array}{c|c|c}
8\sqrt{2} & 5 & 6 \\
9\sqrt{2} & 1 & 5 \\
1\sqrt{2} & 4 & 4 \\
1 & 3 & 4 \\
1 & 0 & 2 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
13 & 4 & 0 \\
8 & 0 & -1 \\
3 & 9 & 1 \\
3 & 1 & 0 \\
3 & 0 & 0 \\
\end{array}
\]

\textit{a priori} information is needed!
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To explain low-dose 3D imaging, let us start by explaining traditional 3D imaging first.

X-ray attenuation data is collected from 180 directions separately for each two-dimensional slice.

Using a reconstruction algorithm, inner structure in the slice is revealed.

This is called computerized tomography (CT).
Reconstruction of a function from its line integrals was first invented by

Johann Radon (1887-1956).

This is the famous inversion formula from 1917 for the Radon transform \( Rf \) of a function \( f \):

\[
f(x) = \frac{1}{4\pi^2} \int_{S^1} \int_{\mathcal{R}} \frac{d}{ds}(Rf)(\theta, s) \frac{1}{x \cdot \theta - s} \, ds \, d\theta
\]
Filtered back-projection (FBP) is mathematical technology used on a daily basis in hospitals around the world. The quality of 3D reconstruction using FBP is excellent. Nobel prize was awarded to Hounsfield and Cormack 1979.

However, a comprehensive data set is mandatory for FBP.
A series of projects started in 2001 aiming for a new type of low-dose 3D imaging

The goal was a mathematical algorithm with

**Input:** small number of digital X-ray images taken with any X-ray device

**Output:** three-dimensional reconstruction with quality good enough for the task at hand

Products of Instrumentarium Imaging in 2001:
Essential history of the three projects:

**Academic members:** Inverse problems research groups in University of Helsinki, University of Kuopio, Helsinki University of Technology and Tampere University of Technology

**Industrial members:**
- **2001-2002** Instrumentarium Imaging and Invers Ltd
- **2003-2004** GE Healthcare Finland
- **2005-2007** PaloDEx Group

**Funding** by TEKES and the companies.

**Outcome:** 12 peer-reviewed articles, 3 patents, algorithms for 3 commercial products
We write the reconstruction problem in matrix form and assume Gaussian noise

\[ m = Ax + \epsilon \]

with Gaussian noise \( \epsilon \) of standard deviation \( \sigma \) leads to the following likelihood distribution:

\[
p(m|x) = p_\epsilon(Ax - m) \sim \exp\left(-\frac{1}{2\sigma^2} \|Ax - m\|_2^2\right)
\]
Bayes formula combines measured data and *a priori* information together

We reconstruct the most probable 3D tissue in light of
1. Available radiographs and
2. Physiological *a priori* information

Bayes formula gives the *posterior distribution* $p(x|m)$:

$$p(x|m) \sim p(x)p(m|x)$$

*Prior distribution, or tissue model*  
*Likelihood distribution, or measurement model*

We recover $x$ as a point estimate from $p(x|m)$
Bayesian inversion algorithms are flexible and widely applicable

Algorithms can be tailored to any measurement geometry.

Bayes formula leads to modular software:

- **measurement model (likelihood), and**
- **tissue model (prior)**

...can be designed independently

Following type of computation is needed to estimate \( x \):

- **MAP estimate**: large-scale optimization
- **Conditional mean**: integration in high-dimensional space
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Three kinds of *a priori* information are available in dental 3D imaging

1. Attenuation coefficient is non-negative (X-radiation does not intensify inside tissue).

2. Different tissue types (bone, gum, pulp chamber) are approximately homogeneous.

3. There are clear boundaries between tissues.
We build a prior distribution for dental tissue using total variation prior

Positivity constraint:

\[ p_+(x) = \begin{cases} 
0 & \text{if } x_j < 0 \text{ for some } j \\
1 & \text{otherwise}
\end{cases} \]

Approximate total variation penalty:

\[ p_{\text{TV}}(x) = \exp\left(\alpha \sum_{\text{neighbors}} |x_\ell - x_k|_\beta \right) \]

\[ |t|_\beta = \frac{1}{\beta} \log(\cosh(\beta t)) \]
Computation of the MAP estimate

Large scale optimization problem:

\[ x_{\text{MAP}} = \arg \min_{x_j \geq 0} \left\{ \frac{1}{2 \sigma^2} \| Ax - m \|_2^2 + \alpha \sum_{\text{neighbors}} |x_\ell - x_k| / \beta \right\} \]

We use the gradient method of Barzilai & Borwein, which is a modification of Euler’s steepest descent method.

\[ -\frac{1}{\alpha_j} \nabla \Phi(x_{j-1}) \rightarrow x_j \]

\[ -\frac{1}{\alpha_j} \nabla \Phi(x_j) \rightarrow x_{j+1} \]

Step size differs from Euler’s:

\[ \alpha_j = \frac{(x_j - x_{j-1}) \cdot (\nabla \Phi(x_j) - \nabla \Phi(x_{j-1}))}{(x_j - x_{j-1}) \cdot (x_j - x_{j-1})} \]
Experimental setting

X-ray source “Focus”
X-ray source positions

Tooth donated to science by Helena Sarlin, thanks!
The projection images look like this
Horizontal slices:  

<table>
<thead>
<tr>
<th></th>
<th>truth</th>
<th>Bayes</th>
<th>tomo</th>
</tr>
</thead>
<tbody>
<tr>
<td>full angle</td>
<td>limited angle</td>
<td>limited angle</td>
<td></td>
</tr>
</tbody>
</table>
Some parts of the boundary are strongly visible in projection data.

Visible parts of boundary

Indetectable parts of boundary

Microlocal analysis of recoverable singularities is available in Quinto (1993) and Ramm & Katsevich (1996).
Vertical slices:  

<table>
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2D projection radiograph is not enough for dental implant planning
Panoramic X-ray device rotates around the head and produces a general picture

Panoramic imaging was invented by Yrjö Paatero in 1950’s.

Nowadays a panoramic device is standard equipment at every dental clinic around the world.

In our project, we reprogrammed the device so that it collects limited-angle data.
We consider the following limited angle experiment with the panoramic x-ray device:

11 projection images of the mandibular area

40 degrees aperture

1000 x 1000 pixels per image formed by a scanning movement
Limited angle reconstruction can be used for locating the mandibular nerve

This is core technology for the VT product of PaloDEx Group
Kolehmainen, Lassas and S (2008)
Cederlund, Kalke and Welander (2009)
Hyvönen, Kalke, Lassas, Setälä, Siltanen (submitted)
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Continuum model for tomography:

\[ M = AU + \mathcal{E} \]
Practical measurement model:

\[ M_k = A_k U + \mathcal{E}_k = P_k A U + P_k \mathcal{E} \]
Computational model:

\[ M_{kn} = P_k AU_n + \mathcal{E}_k \]
$k=8$
$n=48$
$k=8$
$n=156$
$k=8$

$n=440$
$k = 16$
$n = 440$
This is the central idea of studying discretization-invariance:

The numbers $n$ and $k$ are independent.

For the Bayesian inversion strategy to work, the conditional mean estimates must converge as $n$ or $k$ or both tend to infinity.
We arrived at these results suggesting that Besov space priors are the way to go

**Counterexample** (Lassas and S 2004)
Bayesian inversion using **Total variation prior** is not discretization-invariant.

**Theorem** (Lassas, Saksman and S 2008)
Bayesian inversion using $B^{1}_{11}(\mathbb{T}^2)$ **Besov prior** is discretization-invariant.
Our results continue the tradition of infinite-dimensional statistical inversion

1970 Franklin
1984 Mandelbaum
1989 Lehtinen, Päivärinta and Somersalo
1991 Fitzpatrick
1995 Luschgy
2002 Lasanen
2005 Piironen

We achieve discretization invariance for Gaussian and some non-Gaussian prior distributions. Furthermore, we consider realistic measurements.
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Wavelet transform divides a function into details at different scales
We introduce a convenient renumbering of the basis functions

\[
f(x) = \sum_{\ell=1}^{\infty} c_\ell \psi_\ell(x)
\]
Besov space norms can be written in terms of wavelet coefficients

The function

\[ f(x) = \sum_{\ell=1}^{\infty} c_\ell \psi_\ell(x) \]

belongs to \( B_{pq}^s(\mathbb{T}^d) \) if and only if

\[
2^j s 2^{dj \left( \frac{1}{2} - \frac{1}{p} \right)} \left( \sum_{\ell=2^j d}^{2^{(j+1)d} - 1} |c_\ell|^p \right)^{1/p} \in \ell^q(\mathbb{N}).
\]

In particular, \( f \in B_{11}^1(\mathbb{T}^2) \) if and only if

\[
\sum_{\ell=1}^{\infty} |c_\ell| < \infty.
\]
Computation of the CM estimate reduces to sampling from well-known densities

\[ B_{11}^1(\mathbb{T}^2) \text{ prior: write } U \text{ in wavelet basis as} \]

\[ U = \sum_{\ell=1}^{\infty} X_{\ell} \psi_{\ell} \]

with each \( X_{\ell} \) distributed independently \( \sim \exp(-|x|) \).

Posterior distribution of \( U_n \) takes the following form in terms of wavelet coefficients \( x_1, \ldots, x_n \):

\[ C \exp \left( -\frac{1}{2} \| M_k(\omega_0) - A \sum_{\ell=1}^{n} x_{\ell} \psi_{\ell} \|_{L^2(\mathbb{T}^2)}^2 - \alpha \sum_{\ell=1}^{n} |x_{\ell}| \right) \]

Direct and inverse wavelet transforms are easy and quick to compute.
Limited angle tomography results for X-ray mammography

Rantala et al. (2006)
Thanks to GE Healthcare

Tomosynthesis

Besov prior
Conclusions:

Low-dose three-dimensional X-ray imaging is a novel tool for doctors. Complicated and expensive devices are replaced by mathematical computation.

Preprints available at www.siltanen-research.net
Local tomography results for dental X-ray imaging

\( \Lambda \)-tomography  \quad \text{MAP with } B^{1/2}_{3/2,3/2} \text{ prior}

Thanks to Palodex Group
Empirical Bayes methodology for specifying Besov prior parameters

Figure 16: Reconstructions from in vitro dental data. From left: $s = 0, s = 0.4, s = 0.8, s = 1.2$.

Vänskä, Lassas and S (2008)
Thanks to Palodex Group