Computational inversion based on complex geometrical optics solutions

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Outline

Electrical impedance tomography

The D-bar method for EIT

Electrical inclusion detection
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Electrical impedance tomography

The D-bar method for EIT

Electrical inclusion detection
Electrical impedance tomography (EIT) is an emerging medical imaging technique. Feed electric currents through electrodes. Measure the resulting voltages. Repeat the measurement for several current patterns.

Reconstruct distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient’s inner structure. EIT is a harmless and painless imaging method suitable for long-term monitoring.
This talk concentrates on applications of EIT to chest imaging

Applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let \( \Omega \subset \mathbb{R}^2 \) be the unit disc and let conductivity \( \sigma : \Omega \rightarrow \mathbb{R} \) satisfy

\[
0 < M^{-1} \leq \sigma(z) \leq M.
\]

Applying voltage \( f \) at the boundary \( \partial \Omega \) leads to the elliptic PDE

\[
\begin{cases}
\nabla \cdot (\sigma \nabla u) = 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = f.
\end{cases}
\]

Boundary measurements are modelled by the Dirichlet-to-Neumann map

\[
\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega}.
\]

Calderón’s problem is to recover \( \sigma \) from the knowledge of \( \Lambda_\sigma \). It is a nonlinear and ill-posed inverse problem.
Why is Calderón’s problem nonlinear?

Define a quadratic form $\mathcal{P}_\sigma$ for functions $f : \partial \Omega \to \mathbb{R}$ by

$$
\mathcal{P}_\sigma(f) = \int_\Omega \sigma |\nabla u|^2 \, dz, \tag{1}
$$

where $u$ is the solution of the Dirichlet problem

$$
\begin{cases}
    \nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
    u|_{\partial \Omega} = f.
\end{cases}
$$

Now the map $\sigma \mapsto \mathcal{P}_\sigma$ is nonlinear because $u$ depends on $\sigma$ in (1). Physically, $\mathcal{P}_\sigma(f)$ is the power needed for maintaining the voltage potential $f$ on the boundary $\partial \Omega$. Integrate by parts in (1):

$$
\mathcal{P}_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) \, ds = \int_{\partial \Omega} f \left( \Lambda_\sigma f \right) \, ds.
$$

Thus the map $\sigma \mapsto \Lambda_\sigma$ cannot be linear in $\sigma$. 
We illustrate the ill-posedness of Calderón’s problem using a simulated example.
We apply the voltage distribution $f(\theta) = \cos \theta$ at the boundary of the two different phantoms.
The measurement is the distribution of current through the boundary \( \sigma_1 \) and \( \sigma_2 \).
The measurements are very similar, although the conductivities are quite different.
Let us apply the more oscillatory distribution 
\[ f(\theta) = \cos 2\theta \] of voltage at the boundary
The measurement is again the distribution of current through the boundary
The current distribution measurements are again very similar

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \quad \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data.
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data.
Many different types of reconstruction methods have been suggested for EIT in the literature

- **Linearization:** Barber, Bikowski, Brown, Calderón, Cheney, Isaacson, Mueller, Newell
- **Iterative regularization:** Dobson, Gehre, Hua, Jin, Kaipio, Kindermann, Kluth, Leitão, Lechleiter, Lipponen, Maass, Neubauer, Rieder, Rondi, Santosa, Seppänen, Tompkins, Webster, Woo
- **Bayesian inversion:** Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen
- **Resistor network methods:** Borcea, Druskin, Mamonov, Vasquez
- **Layer stripping:** Cheney, Isaacson, Isaacson, Somersalo
- **D-bar methods:** Astala, Bikowski, Bowerman, Delbary, Hansen, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Perämäki, Saulnier, S, Tamasan
- **Teichmüller space methods:** Kolehmainen, Lassas, Ola, S
- **Methods for partial information:** Alessandrini, Ammari, Bilotta, Brühl, Eckel, Erhard, Gebauer, Hanke, Harrach, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kress, Kwon, Lechleiter, Lim, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others
The forward map $F : X \supset \mathcal{D}(F) \to Y$ of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \to X$ that inverts $F$ approximately.
A regularization strategy needs to be constructed so that the assumptions below are satisfied.

A set $\Gamma_\alpha : Y \to X$ of continuous maps parameterized by $\alpha > 0$ is a regularization strategy for $F$ if for each fixed $\sigma \in \mathcal{D}(F)$ we have

$$\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\sigma) - \sigma \|_X = 0.$$ 

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called admissible if $\alpha(\delta) \to 0$ as $\delta \to 0$, and for any fixed $\sigma \in \mathcal{D}(F)$ the following holds:

$$\sup_{\Lambda_\sigma^\delta} \{ \| \Gamma_{\alpha(\delta)}(\Lambda_\sigma^\delta) - \sigma \|_X : \| \Lambda_\sigma^\delta - \Lambda_\sigma \|_Y \leq \delta \} \to 0 \text{ as } \delta \to 0.$$
Outline

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Electrical inclusion detection
# History of CGO-based methods for real 2D EIT

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Nachman’s 1996 uniqueness proof in 2D uses complex geometric optics (CGO) solutions

Define a potential $q$ by setting $q(z) \equiv 0$ for $z$ outside $\Omega$ and

$$q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for } z \in \Omega.$$ 

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(-\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz}\psi(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2),$$

where $\tilde{p} > 2$ and $ikz = i(k_1 + ik_2)(x + iy)$. 

The CGO solutions are constructed using a generalized Lippmann-Schwinger equation

Define \( \mu(z, k) = e^{-ikz} \psi(z, k) \). Then \((-\Delta + q)\psi = 0\) implies

\[
(-\Delta - 4ik\overline{\partial}_z + q)\mu(\cdot, k) = 0,
\]

where the D-bar operator is defined by \(\overline{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})\).

A solution of (2) satisfying \(\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)\) can be constructed using the Lippmann-Schwinger type equation

\[
\mu = 1 - g_k \ast (q\mu),
\]

where \(g_k\) satisfies \((-\Delta - 4ik\overline{\partial}_z)g_k = \delta\) and is defined by

\[
g_k(z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz \cdot \xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} \ d\xi_1 d\xi_2.
\]
The Faddeev fundamental solution $g_1(z)$ has a logarithmic singularity at $z = 0$. It is enough to know $g_1(z)$ because of the relation $g_k(z) = g_1(kz)$. 

![Real part of $g_1(z)$](image1)

![Imaginary part of $g_1(z)$](image2)
The conductivity $\sigma$ can be recovered from the functions $\mu(z, k)$ at $k = 0$

Recall that

$$(-\Delta - 4ik\overline{\partial}z + q)\mu(\cdot, k) = 0$$

with the asymptotics

$$\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2).$$

Substituting $k = 0$ gives

$$(-\Delta + \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}})\mu(\cdot, 0) = 0, \quad (3)$$

and setting $\mu(z, 0) = \sqrt{\sigma(z)}$ gives the unique solution of (3) satisfying $\mu(z, 0) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$. 

The crucial intermediate object in the proof is the non-physical scattering transform $t(k)$

We denote $z = x + iy \in \mathbb{C}$ or $z = (x, y) \in \mathbb{R}^2$ whenever needed. The scattering transform $t : \mathbb{C} \to \mathbb{C}$ is defined by

$$t(k) := \int_{\mathbb{R}^2} e^{i\overline{k}z} q(z) \psi(z, k) \, dx \, dy.$$  \hspace{1cm} (4)

Sometimes (4) is called the nonlinear Fourier transform of $q$. This is because asymptotically $\psi(z, k) \sim e^{ikz}$ as $|z| \to \infty$, and substituting $e^{ikz}$ in place of $\psi(z, k)$ into (4) results in

$$\int_{\mathbb{R}^2} e^{i(kz + \overline{k}z)} q(z) \, dx \, dy = \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx \, dy$$

$$= \widehat{q}(-2k_1, 2k_2).$$
### Infinite-precision data:

Solve boundary integral equation
\[ \psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]
for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:
\[ t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu(z, k) \]
with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

### Practical data:

Solve boundary integral equation
\[ \psi^\delta(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\delta - \Lambda_1)\psi^\delta \]
for all \( 0 < |k| < R = -\frac{1}{10} \log \delta \).

Evaluate the scattering transform:
\[ t^\delta_R(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_\delta - \Lambda_1)\psi^\delta(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial k} \mu_R(z, k) = \frac{t^\delta_R(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu_R(z, k) \]
with \( \mu_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma_{1/R(\delta)}(\Lambda_\sigma) := (\mu_R(z, 0))^2 \).
We define spaces for our regularization strategy

Model space $X = L^\infty(\Omega)$

Data space $Y$

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ with

- $\|\sigma\|_{C^2(\Omega)} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators $A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying

- $A(1) = 0$,
- $\int_{\partial\Omega} A(f) \, ds = 0$. 
Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

There exists a constant $0 < \delta_0 < 1$, depending only on $M$ and $\rho$, with the following properties. Let $\sigma \in D(F)$ be arbitrary and assume given noisy data $\Lambda^\delta \sigma$ satisfying

$$\| \Lambda^\delta \sigma - \Lambda_\sigma \|_Y \leq \delta < \delta_0.$$

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\| \Gamma_{\alpha(\delta)}(\Lambda^\delta_\sigma) - \sigma \|_{L^\infty(\Omega)} \leq C(-\log \delta)^{-1/14}.$$
Let us analyze how the regularization works using a simulated heart-and-lungs phantom.
This is how the actual scattering transform looks like in the disc $|k| < 10$, computed by knowing $\sigma$.
Scattering transform in the disc $|k| < 10$, here computed from noisy measurement $\Lambda^\delta_\sigma$.

Real part of $t(k)$

Imaginary part
Regularized reconstructions from simulated data with noise amplitude $\delta = \| \Lambda^\delta - \Lambda_\sigma \|_Y$

$\delta \approx 10^{-6}$, $\delta \approx 10^{-5}$, $\delta \approx 10^{-4}$, $\delta \approx 10^{-3}$, $\delta \approx 10^{-2}$

$R = 6.7$, $5.9$, $4.3$, $3.5$, $2.5$

12%, 12%, 14%, 19%, 52%

The percentages are the relative square norm errors in the reconstructions.
The method works for real data as well, including laboratory phantoms and *in vivo* human data

Saline and agar phantom

Reconstruction \((R = 4)\)

[Isaacson, Mueller, Newell & S 2006]
[Montoya & Mueller 2012]
Unknown boundary shape can be estimated from EIT data using Teichmüller space methods

[Kolehmainen, Lassas, Ola & S 2012]
Outline

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Electrical inclusion detection
This is a brief history of the enclosure method

2000 Ikehata
2000 Ikehata & S
2000 Brühl & Hanke
2002 Ikehata
2002 Ikehata & Ohe
2004 Ikehata & S
2007 Ide, Isozaki, Nakata, S & Uhlmann
2008 Ikehata & Ohe
2008 Uhlmann & Wang
2010 Ide, Isozaki, Nakata & S
2010 Zhou
2012 Ikehata, Niemi & S

Enclosure method for EIT
Implementation for 2D EIT
Regularization analysis
Polygonal cavities in 2D EIT
Regularized cone probing in 2D EIT
Spherical probing in 2D EIT
Locating cracks in 2D EIT
General 2D probing
Spherical probing in 3D EIT
Maxwell’s equations
Inverse obstacle scattering in 2D
Different probing strategies

Half-space

Cone

Spherical
We denote the inclusion set by $\Omega_1$.

The background conductivity is $\sigma_0 \equiv 1$. Assume that $\sigma(x) = \sigma_0(x)$ for $x \notin \Omega_1$ and $\sigma(x) > \sigma_0(x)$ on $\Omega_1$.

Take $x_0$ from the outside of the convex hull of $\Omega$, and let $R > 0$.

Then there exists a smooth solution $u_\tau(x)$ of $\nabla \cdot (\sigma_0(x) \nabla u_\tau(x)) = 0$ on $\Omega$ with the following special properties.
Main theorem of spherical probing, part 2

Hyperbolic transformation: 
\[ y_1 = \frac{x_1^2 + x_2^2 - r^2}{(x_1 + r)^2 + x_2^2}, \quad y_2 = \frac{2x_2r}{(x_1 + r)^2 + x_2^2} \]

\((x_1, x_2)\)-plane: \(u_\tau(x)\)
\((y_1, y_2)\)-plane: \(\exp(-\tau y_1 + i\tau y_2)\)
Main theorem of spherical probing, part 3

Let $f_\tau = u_\tau|_{\partial\Omega}$. There is a $0 < \delta = \delta(\Omega_1, x_0, R)$ such that

$$\langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau, f_\tau \rangle < Ce^{-\delta \tau}$$

and

$$\langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau, f_\tau \rangle > Ce^{\delta \tau}$$
Practical scheme for detecting inclusions

Take $0 < \tau_1 < \tau_2$.

\[ \langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau, f_\tau \rangle < Ce^{-\delta\tau} \]

\[ \langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau_1, f_\tau_1 \rangle > \langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau_2, f_\tau_2 \rangle \]

\[ \langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau_1, f_\tau_1 \rangle < \langle (\Lambda_\sigma - \Lambda_{\sigma_0})f_\tau_2, f_\tau_2 \rangle \]
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## Spherical probing from ideal and noisy EIT data

<table>
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<tr>
<th>Conductivity</th>
<th>Maximal region</th>
<th>Reconstruction (ideal data)</th>
<th>Reconstruction (noisy data)</th>
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<tbody>
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<td><img src="image1" alt="Conductivity" /></td>
<td><img src="image2" alt="Maximal region" /></td>
<td><img src="image3" alt="Reconstruction (ideal data)" /></td>
<td><img src="image4" alt="Reconstruction (noisy data)" /></td>
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<tr>
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<td><img src="image8" alt="Reconstruction (noisy data)" /></td>
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<tr>
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<td><img src="image11" alt="Reconstruction (ideal data)" /></td>
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Spherical probing from ideal and noisy EIT data in 3D (Ide, Isozaki, Nakata & S 2010)
In this book, we explore inverse problems and their applications in various fields. The book explains how to identify ill-posed inverse problems arising in practice and how to design computational solution methods for them. Additionally, it provides computational approaches in a hands-on fashion, with related codes available online.

The guiding linear inversion example is the problem of image deblurring, X-ray tomography, and backward parabolic problems, including heat transfer, and electrical impedance tomography. This example serves as the guiding nonlinear inversion example.

The book's nonlinear material combines analytic and geometric research traditions and the regularization-based school of thought in a fruitful manner, paving the way to new theorems and algorithms for nonlinear inverse problems. Furthermore, it is the only mathematical textbook with a thorough treatment of electrical impedance tomography and these sections are suitable for beginning and experienced researchers in mathematics, engineering, and computer science.

Linear and Nonlinear Inverse Problems with Practical Applications is well-suited for students in mathematics, engineering, physics, or computer science who wish to learn computational inversion (inverse problems). Professors will find that the exercises and project work topics make this a suitable textbook for advanced undergraduate and graduate courses on inverse problems. Researchers developing large-scale inversion methods for linear or nonlinear inverse problems, as well as engineers working in research and development departments at high-tech companies and in electrical impedance tomography, will also find this a valuable guide.

Jennifer L. Mueller is a Professor of Mathematics and Biomedical Engineering at Colorado State University in Fort Collins, Colorado. Prior to that, she was an NSF Postdoctoral Fellow at Rensselaer Polytechnic Institute in Troy, New York, where she began working in electrical impedance imaging. Clinical applications are a strong motivating factor in her research in reconstruction algorithms for medical imaging. She has served as Vice Chair and Program Director for the SIAM Activity Group on Imaging Science.

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