The inverse problems mathematics behind X-ray tomography

Samuli Siltanen

Department of Mathematics and Statistics
University of Helsinki, Finland
samuli.siltanen@helsinki.fi
www.siltanen-research.net

Graduate school in inverse problems
Hiroshima University, Japan
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This is my industrial-academic background:

2009: Professor, University of Helsinki, Finland

2006: Professor, Tampere University of Technology, Finland

2005: R&D scientist at Palodex Group

2004: R&D scientist at GE Healthcare

2002: Postdoc at Gunma University, Japan

2000: R&D scientist at Instrumentarium Imaging

1999: PhD, Helsinki University of Technology, Finland
Kumpula Science Campus of University of Helsinki
Massive Open Online Courses (MOOC) at University of Helsinki, Finland

Our team: Tatiana Bubba, Rashmi Murthy, Luca Ratti, Teemu Tyni and Heli Virtanen.

We teach three MOOC courses in Helsinki:
▶ Applications of matrix computations
▶ Inverse Problems 1: convolution and deconvolution
▶ Inverse Problems 2: tomography and regularization

Today we give an overview of the course Inverse Problems 2.

You are welcome to study the courses!
Outline

What is an X-ray image?

Slice imaging: X-ray tomography

Are you a natural tomographer?

Filtered back-projection (FBP)

Sparse-angle tomography

Limited-angle tomography

Industrial case study: low-dose 3D dental X-ray imaging
We can see through a box of candy!

https://www.dropbox.com/s/e7i3exqc4sdpr1s/Sisu2.mp4?dl=0
X-ray images are very useful for doctors. For example, they can see fractures.
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An X-ray with intensity $I_0$ enters a homogeneous physical body.

The intensity $I_1$ of the X-ray when it exits the material is

$$I_1 = I_0 e^{-\mu s},$$

where $s$ is the length of the path of the X-ray inside the body and $\mu > 0$ is X-ray attenuation coefficient.
X-ray intensity attenuates inside matter, here shown with a homogeneous block

https://www.youtube.com/watch?v=lfXo2S1xXCQ
A digital X-ray detector counts how many photons arrive at each pixel.
Adding material between the source and detector reveals the exponential X-ray attenuation law.
We take logarithm of the photon counts to compensate for the exponential attenuation law.
Final calibration step is to subtract the logarithms from the empty space value (here 6.9)

<table>
<thead>
<tr>
<th>photon count</th>
<th>log</th>
<th>line integral</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>6.9</td>
<td>0.0</td>
</tr>
<tr>
<td>500</td>
<td>6.2</td>
<td>0.7</td>
</tr>
<tr>
<td>250</td>
<td>5.5</td>
<td>1.4</td>
</tr>
</tbody>
</table>
Formula for X-ray attenuation along a line: Beer-Lambert law

Let $f : [a, b] \to \mathbb{R}$ be a nonnegative function modelling X-ray attenuation along a line inside a physical body.

Beer-Lambert law connects the initial and final intensities:

$$I_1 = I_0 e^{-\int_a^b f(x)dx}.$$ 

We can also write it in the form

$$-\log(I_1/I_0) = \int_a^b f(x)dx,$$

where $I_0$ is known from calibration and $I_1$ from measurement.
Here is a 2D slice through a human head

Andrew Ciscel, Wikimedia commons
Now the attenuation process is more complicated because there are different tissues

https://youtu.be/lvUAOeS1sv8
After calibration we are observing how much attenuating matter the X-ray encounters in total.

https://youtu.be/RFArLtWEfsQ
This sweeping movement is the data collection mode of first-generation CT scanners

https://youtu.be/JHUz5oyeZb0
Data is collected by rotating the system around the patient

https://youtu.be/newxZbw7YAs
Modern CT scanners look like this.
Modern scanners rotate at high speed

https://commons.wikimedia.org/wiki/File:CT-Rotation.ogv
This is the inverse problem of tomography: we only know the data

https://youtu.be/pr8bXB0oAqI
This is an illustration of the standard reconstruction by filtered back-projection

https://youtu.be/tRD58lO1FKw
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Industrial case study: low-dose 3D dental X-ray imaging
Let’s warm up before the tests.
Here is tomographic data of a simple object:
Can you guess the shape of the object from the tomographic data?
Test: can you guess the image?

https://youtu.be/NishyJWhXDk
Alternatives
Solution

https://youtu.be/MkAQoF3YOwg
Test: can you guess the image?

https://youtu.be/ZJaek4nkcRA
Alternatives

S

B

W
Solution

https://youtu.be/YHpG5HqDmZk
Test: can you guess the image?

https://youtu.be/jP4AC7l8guo
Alternatives
https://youtu.be/epHR4x3up8I
Test: can you guess the image?

https://youtu.be/jP4AC7l8guo
Alternatives
Solution

https://youtu.be/epHR4x3up8I
Test: can you guess the image?

https://youtu.be/goddXsubZO8
Alternatives
Solution

https://youtu.be/RfKA3R2-pjk
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Industrial case study: low-dose 3D dental X-ray imaging
Reconstruction of a function from its line integrals was first invented by Johann Radon in 1917

\[
f(P) = -\frac{1}{\pi} \int_0^\infty \frac{d\bar{F}_p(q)}{q}
\]

Johann Radon (1887-1956)
Let us start with a crucial calculation

To reconstruct $f$ at a point $x$, the most obvious data related to $f(x)$ are the integrals over lines passing through $x$. Let us sum them all together, call the result $Tf(x)$ and see what we get by introducing polar coordinates:

$$Tf(x) = \int_0^\pi \int_{-\infty}^{\infty} f(x + t\theta)dt\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\infty} \frac{f(x + t\theta)}{t} t\,dt\,d\theta$$

$$= \int_{\mathbb{R}^2} \frac{f(x + y)}{|y|} dy$$

$$= \int_{\mathbb{R}^2} \frac{f(y)}{|x - y|} dy$$

$$= (f(y) * \frac{1}{|y|})(x),$$

where $*$ stands for convolution.
The Calderón operator $\Lambda$ is the inverse of $T$

Recall that Fourier transform converts convolution to multiplication $(\hat{g} \ast \hat{h} = \hat{g} \hat{h})$ and that

$$\frac{1}{|y|}(\xi) = \frac{1}{|\xi|}.$$ 

Furthermore, define the Calderón operator $\Lambda$ by

$$\Lambda f(x) := \mathcal{F}^{-1} |\xi| \hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} |\xi| \hat{f}(\xi) d\xi,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Note that $\Lambda$ can be thought of as a high-pass filter. Now we see that

$$\widehat{\Lambda T f}(\xi) = \frac{\hat{f}(\xi)}{|\xi|},$$

and thus

$$\Lambda T f = f.$$
Here is a simple example of tomographic data collection, with two discs as the target.

https://youtu.be/5DUGTXd26nA
The inverse problem of tomography is to recover the unknown target from the measured X-ray data

https://youtu.be/YhClb0MaB70
Summing up all the back-projections results in a blurred reconstruction
Summing up all the back-projections results in a blurred reconstruction
Summing up all the back-projections results in a blurred reconstruction.
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Summing up all the back-projections results in a blurred reconstruction
Summing up all the back-projections results in a blurred reconstruction
Summing up all the back-projections results in a blurred reconstruction
Here we use more directions, so the reconstruction quality is higher

https://youtu.be/5DUGTXd26nA
Final reconstruction involves filtering on top of the back-projection.

\[ \hat{f}(\xi) \]

Multiplication with “ice-cream cone”

\[ |\xi| \hat{f}(\xi) \]

FFT

IFFT
This is an illustration of the standard reconstruction by filtered back-projection.

https://youtu.be/tRD58IO1FKw
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We collected X-ray projection data of a walnut from 1200 directions.

Data collection: thanks to Keijo Hämäläinen and Aki Kallonen, University of Helsinki.

The data is openly available at http://fips.fi/dataset.php, thanks to Esa Niemi and Antti Kujanpää.
Reconstructions of a 2D slice through the walnut using filtered back-projection (FBP)

FBP with comprehensive data (1200 projections)

FBP with sparse data (20 projections)
Sparse-data reconstruction of the walnut using non-negative total variation regularization

Filtered back-projection

Constrained TV regularization
\[
\arg\min_{f \in \mathbb{R}_+^n} \{ \|Af - m\|_2^2 + \alpha \|\nabla f\|_1 \}
\]
Consider a simple example of a 2D square patient, whose internal structures consist of small squares.
Two horizontal X-rays give us two numbers: row sums of the $2 \times 2$ array of attenuations

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

- $8 \ (= 2 + 6)$
- $9 \ (= 2 + 7)$
Tomographic imaging requires collecting X-ray data along another direction as well.
“Direct problem” in this example is to compute row and column sums of a known interior

<table>
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<th></th>
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</tr>
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<tbody>
<tr>
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<td>7</td>
<td></td>
</tr>
</tbody>
</table>

Row sums: 8, 9
Column sums: 4, 13
“Inverse problem” in this example is to recover the interior numbers from the measurements.
With such a limited amount of data, the inverse problem has multiple solutions!
With such a limited amount of data, the inverse problem has multiple solutions!
With such a limited amount of data, the inverse problem has multiple solutions!
So-called “ghosts,” or targets with zero data, are the source of multiple solutions.
Adding a ghost does not change the data!

\[
\begin{bmatrix}
2 & 6 \\
2 & 7 \\
\end{bmatrix} + \begin{bmatrix}
2 & -2 \\
-2 & 2 \\
\end{bmatrix} = \begin{bmatrix}
4 & 4 \\
0 & 9 \\
\end{bmatrix}
\]
The forward problem in matrix form

\[
\begin{bmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 \\
  1 & 1 & 0 & 0 \\
  0 & 0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix}
= 
\begin{bmatrix}
  8 \\
  9 \\
  4 \\
  13 \\
\end{bmatrix}
\]

\[Ax = m\]
A spooky diversion: the origin of ghosts

\[
\begin{bmatrix}
2 & -2 \\
-2 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\
-2 \\
2 \\
-2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[\text{Ker}(A) = \left\{ \begin{bmatrix} t \\ -t \\ t \\ -t \end{bmatrix} : t \in \mathbb{R} \right\}\]
The continuous tomographic model needs to be approximated using a discrete model

Continuous model:

In this schematic setup we have 5 projection directions and a 10-pixel detector. Therefore, the number of data points is 50.

Discrete model:
The resolution of the discrete model can be freely chosen according to computational resources.

Continuous model:

In this schematic setup we have 5 projection directions and a 10-pixel detector. Therefore, the number of data points is 50.

The number of degrees of freedom in the three discrete models below are 16, 64 and 256, respectively.

Discrete models:
Discretize the unknown by dividing it into pixels

Target (unknown)  

32×32 pixel grid
System matrix $A$, given by the grid and X-rays

735×1024 system matrix $A$, only nonzero elements shown

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32×32 pixel grid
System matrix $A$, given by the grid and X-rays

- $735 \times 1024$ system matrix $A$
- only nonzero elements shown
- $32 \times 32$ pixel grid
System matrix $A$, given by the grid and X-rays

- 735×1024 system matrix $A$, only nonzero elements shown
- 32×32 pixel grid
System matrix $A$, given by the grid and X-rays

735×1024 system matrix $A$, only nonzero elements shown

32×32 pixel grid
System matrix $A$, given by the grid and X-rays

$735 \times 1024$ system matrix $A$, only nonzero elements shown

$32 \times 32$ pixel grid
System matrix $A$, given by the grid and X-rays

$735 \times 1024$ system matrix $A$, only nonzero elements shown

$32 \times 32$ pixel grid
System matrix $A$, given by the grid and X-rays

$735 \times 1024$ system matrix $A$, only nonzero elements shown

$32 \times 32$ pixel grid
Illustration of the ill-posedness of sparse tomography
Illustration of the ill-posedness of sparse tomography
Example of the action of $A^T$: point target
This is why $A^T$ is called the back-projection.
Example of the action of $A^T$: point sinogram
Here is another point sinogram
Naive reconstruction using the minimum norm solution from the normal equation \((A^T A)f^\dagger = A^T m\)
Singular value decomposition \( A = U^T D V \)

- 735×1024 system matrix \( A \), only nonzero elements shown

- Singular values of \( A \) (diagonal of \( D \))
Naive reconstruction using the minimum norm solution with non-negativity constraint

Original phantom, 32×32 resolution, values between 0 (black) and 0.44

Reconstruction: minimum value 0, maximum value 2.3
Non-negative TV regularization

\[
\arg\min_{f \in \mathbb{R}^n_+} \left\{ \|Af - m\|_2^2 + \alpha \|\nabla f\|_1 \right\}
\]

Original phantom sampled at 32 × 32 resolution

TV regularized reconstruction
Relative square norm error 7%
How can a reconstruction method pick out the correct image among all that match the data?
Consider these three candidates for reconstruction

<table>
<thead>
<tr>
<th>True target</th>
<th>Wrong data, good “tissue type”</th>
<th>Right data, bad “tissue type”</th>
</tr>
</thead>
<tbody>
<tr>
<td>2  6  7</td>
<td>3  3</td>
<td>4  4</td>
</tr>
<tr>
<td>2  6  7</td>
<td>3  3</td>
<td>4  4</td>
</tr>
</tbody>
</table>
Consider these three candidates for reconstruction

True target

Wrong data, good “tissue type”

Right data, bad “tissue type”
Penalty calculation for candidate 1 (true target).

First the penalty from (mis)matching X-ray data:

\[
\begin{align*}
(8 - 8)^2 &+ (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0.
\end{align*}
\]
Penalty calculation for candidate 1. Then the penalty from prior information.

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).

Prior penalty: |2 − 6|
Penalty calculation for candidate 1.
Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).
Prior penalty: \(|2 - 6| + |2 - 7|\)
Penalty calculation for candidate 1.

Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0\).

Prior penalty: \(|2 - 6| + |2 - 7| + |2 - 2|\)
Penalty calculation for candidate 1.
Then the penalty from prior information

\[
\text{Data penalty: } (8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0.
\]
\[
\text{Prior penalty: } |2 - 6| + |2 - 7| + |2 - 2| + |6 - 7| = 4 + 5 + 0 + 1 = 10.
\]
Penalty calculation for candidate 1. Total penalty is the sum of data and prior penalties.

\[
\begin{array}{|c|c|}
\hline
2 & 6 \\
\hline
2 & 7 \\
\hline
\end{array}
\]

\[
\text{data penalty} \quad 0 \\
+ \quad \text{prior penalty} \quad 10 \\
\hline
= \quad \text{total penalty} \quad €10
\]
Penalty calculation for candidate 1. Total penalty is the sum of data and prior penalties

\[
\begin{array}{c}
data \text{ penalty} & 0 \\
+ & \text{prior penalty} & 10 \\
\hline
= & \text{total penalty} & ¥10
\end{array}
\]
Penalty calculation for candidate 2. First the penalty from (mis)matching X-ray data.

\[
\begin{array}{cc}
3 & 3 \\
3 & 3 \\
\end{array}
\]

\[
(6 - 8)^2 \quad (6 - 9)^2 \\
(6 - 4)^2 \quad (6 - 13)^2 \\
\]

Data penalty: \(2^2 + 3^2 + 2^2 + 7^2 = 4 + 9 + 4 + 49 = 66.\)
Penalty calculation for candidate 2.
Then the penalty from prior information

Data penalty: \(2^2 + 3^2 + 2^2 + 7^2 = 4 + 9 + 4 + 49 = 66\).

Prior penalty: \(|3 - 3| + |3 - 3| + |3 - 3| + |3 - 3| = 0\).
Penalty calculation for candidate 2. Total penalty is the sum of data and prior penalties.

\[
\begin{array}{|c|c|}
\hline
3 & 3 \\
\hline 
3 & 3 \\
\hline
\end{array}
\]

\[
data \text{ penalty } & 66 \\
+ \text{ prior penalty } & 0 \\
\hline
= \text{ total penalty } & €66
\]
Penalty calculation for candidate 3.
First the penalty from (mis)matching X-ray data

\[
\begin{array}{cccc}
(8 - 8)^2 & 4 & 4 \\
(9 - 9)^2 & 0 & 9 \\
(4 - 4)^2 & (13 - 13)^2 & \\
\end{array}
\]

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0.\)
Penalty calculation for candidate 3.
Then the penalty from prior information

Data penalty: \((8 - 8)^2 + (9 - 9)^2 + (4 - 4)^2 + (13 - 13)^2 = 0.\)
Prior penalty: \(|4 - 4| + |0 - 9| + |4 - 0| + |4 - 9| = 0 + 9 + 4 + 5 = 18.\)
Penalty calculation for candidate 3. Total penalty is the sum of data and prior penalties.

\[
\begin{array}{c|c}
\text{data penalty} & 0 \\
+ \text{prior penalty} & 18 \\
\hline
= \text{total penalty} & €18 \\
\end{array}
\]
Which of candidates has smallest total penalty?

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<tbody>
<tr>
<td>2 6</td>
<td>3 3</td>
<td>4 4</td>
</tr>
<tr>
<td>2 7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Data penalty

<table>
<thead>
<tr>
<th>Data penalty</th>
<th>Prior penalty</th>
<th>Total penalty</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>€10</td>
</tr>
<tr>
<td>66</td>
<td>0</td>
<td>€66</td>
</tr>
<tr>
<td>0</td>
<td>18</td>
<td>€18</td>
</tr>
</tbody>
</table>
Which of candidates has smallest total penalty?

True target

\[ \text{data penalty} \quad 0 \quad + \quad \text{prior penalty} \quad 10 \quad = \quad \text{total penalty} \quad €10 \]

Wrong data, good “tissue type”

\[ \text{data penalty} \quad 66 \quad + \quad \text{prior penalty} \quad 0 \quad = \quad \text{total penalty} \quad €66 \]

Right data, bad “tissue type”

\[ \text{data penalty} \quad 0 \quad + \quad \text{prior penalty} \quad 18 \quad = \quad \text{total penalty} \quad €18 \]
The problem can be solved in general using optimization.

General target

Find numbers $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and $x_4 \geq 0$ such that the sum of these two penalties is as small as possible:

Data penalty: $\left(x_1 + x_3 - 8\right)^2 + \left(x_2 + x_4 - 9\right)^2 + \left(x_1 + x_2 - 4\right)^2 + \left(x_3 + x_4 - 13\right)^2$

Prior penalty: $|x_1 - x_3| + |x_2 - x_4| + |x_1 - x_2| + |x_3 - x_4|$

This method is called total variation regularization.
We can now formulate some solution methods mathematically

Total variation regularization:

$$\arg\min_{x \in \mathbb{R}^{N \times N}} \left\{ \|Ax - m\|_2^2 + \alpha \|\nabla x\|_1 \right\}$$

Minimum square-norm solution is the shortest vector satisfying the normal equations

$$A^T A x = A^T m$$
### Solutions from optimization methods

#### True target

<table>
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<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

- **Data penalty**: 0
- **Prior penalty**: 10
- **Total penalty**: €10

#### Total variation regularization

<table>
<thead>
<tr>
<th>2.5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>6</td>
</tr>
</tbody>
</table>

- **Data penalty**: 1.6
- **Prior penalty**: 7.0
- **Total penalty**: €8.6
Resolution
2×2
Resolution
4 × 4
Resolution
8×8
Resolution
16×16
Resolution
32×32
Resolution
$64 \times 64$
Resolution
128×128
Resolution
256×256
Resolution
512×512
Resolution

2×2

<table>
<thead>
<tr>
<th>3</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
Resolution
4 × 4
Resolution
8×8
Resolution
$16 \times 16$
Resolution
32×32
Resolution
64×64
Resolution
128×128
Resolution
$256 \times 256$
Resolution
512×512
Outline

What is an X-ray image?

Slice imaging: X-ray tomography

Are you a natural tomographer?

Filtered back-projection (FBP)

Sparse-angle tomography

Limited-angle tomography

Industrial case study: low-dose 3D dental X-ray imaging
Construction of limited-angle sinogram
Construction of limited-angle sinogram
Construction of limited-angle sinogram
Construction of limited-angle sinogram
Construction of limited-angle sinogram
Construction of limited-angle sinogram
Construction of limited-angle sinogram
Construction of limited-angle sinogram
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Construction of limited-angle sinogram
Construction of limited-angle sinogram
Construction of limited-angle sinogram
SVD reveals the ill-posedness of the limited-angle problem, see Davison 1983 and Louis 1986.

$735 \times 1024$ system matrix $A$, only nonzero elements shown

Singular values of $A$ (diagonal of $D$)
Filtered Back-Projection (FBP) reconstruction from limited-angle data

Original phantom sampled at 32×32 resolution

Filtered back-projection
Non-negative limited-angle TV regularization

\[
\underset{f \in \mathbb{R}_+^n}{\arg \min} \left\{ \|Af - m\|_2^2 + \alpha \|\nabla f\|_1 \right\}
\]

Original phantom sampled at 32\times32 resolution

TV reconstruction
Definition of the Radon transform

Let \( f(x) = f(x_1, x_2) \) be the X-ray attenuation coefficient. The classical model for tomographic data is the **Radon transform**

\[
Rf(\theta, s) = \int_{x \cdot \theta = s} f(x) \, dx = \int_{y \in \theta^\perp} f(s\theta + y) \, dy, \quad \theta \in S^1, s \in \mathbb{R},
\]

where \( S^1 \) is the unit circle, \( \theta^\perp \) is the orthogonal complement of the unit vector \( \theta \) and \( x \cdot \theta \) denotes vector inner product.

Note that \( f \) is defined on \( \mathbb{R}^2 \) and \( Rf \) is defined on \( S^1 \times \mathbb{R}^1 \).

Also: \( Rf(\theta, s) = Rf(-\theta, -s). \quad \theta \in [0, 2\pi]. \)
The Fourier slice theorem

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be smooth and compactly supported. Denote $R_\theta f(s) := R f(\theta, s)$ for $\theta \in S^1$ and $s \in \mathbb{R}$. Then

$$\hat{R_\theta f}(\xi) = \hat{f}(\xi \theta).$$

**Proof.** The change of coordinates $x = s\theta + y$ gives $s = \theta \cdot x$ and $dx = dy ds$. Calculate

$$\hat{R_\theta f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi s} R_\theta f(s) \, ds$$

$$= \int_{-\infty}^{\infty} e^{-i\xi s} \int_{\theta \perp} f(s\theta + y) \, dy \, ds$$

$$= \int_{\mathbb{R}^2} e^{-i\xi \theta \cdot x} f(x) \, dx$$

$$= \hat{f}(\xi \theta). \quad \square$$
Practically Dubious Theorem: Unique determination from limited-angle data

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be smooth and compactly supported. Let \( \varepsilon > 0 \). Then \( f \) is uniquely determined from the limited-angle sinogram

\[ Rf(\theta, s) \quad \text{with} \quad -\varepsilon < \theta < \varepsilon \quad \text{and} \quad s \in \mathbb{R}. \]

**Proof.** Let \( f \) and \( g \) be compactly supported smooth functions defined on \( \mathbb{R}^2 \) satisfying \( Rf(\theta, s) = Rg(\theta, s) \) for \(-\varepsilon < \theta < \varepsilon\). By the Fourier slice theorem we know that

\[ \hat{f}(\xi \theta) \equiv \hat{g}(\xi \theta) \]

in the open set \( C_\varepsilon := \{(\xi, \theta) \in \mathbb{R}^2 \mid \xi > 0, -\varepsilon < \theta < \varepsilon\} \). The Fourier transform of a compactly supported smooth function is analytic. So \( \hat{f} = \hat{g} \) on the open set \( C_\varepsilon \), and due to analyticity \( \hat{f} = \hat{g} \) on the whole frequency domain \( \mathbb{R}^2 \). Therefore \( f \equiv g \). \( \square \)
Limited angle measurement information looks like a bowtie in the frequency domain.

Range of measured angles is $(-\epsilon, \epsilon)$

Frequency domain
What do we know about the singular values of the Radon transform?

Roughly speaking,

- the full-angle Radon transform allows a singular system where the singular values decay as $d_n \sim 1/n$ when $n \to \infty$;
- the singular values of limited-angle Radon transform decay exponentially even if the interval of missing angles is just $(-\varepsilon, \varepsilon)$ with any $\varepsilon > 0$.

The details are available in Natterer 1986, Sections IV.3 and VI.2.

For more information, see Davison 1983 and Louis 1986.
Limited data gives only part of the wavefront set

See [Greenleaf & Uhlmann 1989], [Quinto 1993], and [Frikel & Quinto 2013]
Filtered Back-Projection (FBP) reconstruction from limited-angle data

Stable part of wavefront set

Filtered back-projection
Constrained total variation (TV) regularization
\[
\arg \min_{f \in \mathbb{R}^n_+} \left\{ \|Af - m\|_2^2 + \alpha \|\nabla f\|_1 \right\}
\]

Stable part of wavefront set

TV regularized reconstruction
**Experimental data**

**Mayo Clinic**: human abdomen scans provided by the Mayo Clinic for the AAPM Low-Dose CT Grand Challenge.

- 10 patients (2378 slices of size $512 \times 512$ with thickness 3mm)
- 9 patients for training (2134 slices) and 1 patient for testing (244 slices)
- Mayo-$60^\circ$: missing wedge of $60^\circ$

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\(^1\)We would like to thank Dr. Cynthia McCollough, the Mayo Clinic, the American Association of Physicists in Medicine (AAPM), and grant EB01705 and EB01785 from the National Institute of Biomedical Imaging and Bioengineering for providing the Low-Dose CT Grand Challenge data set.
Filtered back-projection fails to recover the invisible parts of boundaries.

Ground truth

Filtered back-projection
When we learn the invisible parts of boundaries we can recover them

Outline

What is an X-ray image?

Slice imaging: X-ray tomography

Are you a natural tomographer?

Filtered back-projection (FBP)

Sparse-angle tomography

Limited-angle tomography

Industrial case study: low-dose 3D dental X-ray imaging
Application: dental implant planning, where a missing tooth is replaced with an implant
This is the classical imaging procedure of the panoramic X-ray device

https://www.youtube.com/watch?v=QFTXegPxC4U
The resulting image shows a sharp layer positioned inside the dental arc.
Nowadays, a digital panoramic imaging device is standard equipment at dental clinics.

A panoramic dental image offers a general overview showing all teeth and other structures simultaneously.

Panoramic images are not suitable for dental implant planning because of unavoidable geometric distortion.
We reprogram the panoramic X-ray device so that it collects projection data by scanning

https://www.youtube.com/watch?v=motthjiP8ZQ
We reprogram the panoramic X-ray device so that it collects projection data by scanning.

Number of projection images: 11
Angle of view: 40 degrees
Image size: $1000 \times 1000$ pixels

The unknown vector $f$ has 7 000 000 elements.
Standard Cone Beam CT reconstruction delivers 100 times more radiation than VT imaging.

Kolehmainen, Vanne, S, Järvenpää, Kaipio, Lassas & Kalke 2006
Kolehmainen, Lassas & S 2008
Cederlund, Kalke & Welander 2009
Hyvönen, Kalke, Lassas, Setälä & S 2010
U.S. patent 7269241, thousands of VT units in use
The VT device was developed in 2001–2012 by

Nuutti Hyvönen
Seppo Järvenpää
Jari Kaipio
Martti Kalke
Petri Koistinen
Ville Kolehmainen
Matti Lassas
Jan Moberg
Kati Niinemäki
Juha Pirttilä
Maaria Rantala
Eero Saksman
Henri Setälä
Erkki Somersalo
Antti Vanne
Simopekka Vänskä
Richard L. Webber
Open CT datasets:
- Finnish Inverse Problems Society (FIPS) dataset page

Matrix-based parallel-beam reconstruction algorithms:
FIPS Computational Blog
- Truncated SVD
- Total Variation regularization

Matrix-free large-scale reconstruction algorithms:
- Matlab page of Mueller-S 2012 book
- ASTRA toolbox
- TVReg: Software for 3D Total Variation Regularization

Inverse problems software:
- https://github.com/odlgroup/odl
Thank you for your attention!