Inverse scattering methods for electrical impedance tomography and the Novikov-Veselov equation

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http://wiki.helsinki.fi/display/inverse/Home
Outline

**Electrical Impedance Tomography (EIT)**

Regularized D-bar method for EIT

The CGO sinogram

The Novikov-Veselov equation
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = f. \end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \frac{\partial u}{\partial n}|_{\partial \Omega}.$$
We illustrate the ill-posedness of EIT using a simulated example.
We apply the voltage distribution $f(\theta) = \cos \theta$ at the boundary of the two different phantoms $\sigma_1$, $\sigma_2$, $u_1$, and $u_2$. 
The measurement is the distribution of current through the boundary

\[ \sigma_1 \partial u_1 / \partial \vec{n} \]

\[ \sigma_2 \partial u_2 / \partial \vec{n} \]
The current data are very similar, although the conductivities are quite different.

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \quad \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
Let us apply the more oscillatory distribution $f(\theta) = \cos 2\theta$ of voltage at the boundary.
The measurement is again the distribution of current through the boundary.
The current distribution measurements are almost the same.
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data.
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data

$$\sigma_1$$

$$\sigma_2$$

$$\cos \theta$$

$$\cos 2\theta$$

$$\cos 3\theta$$

$$\cos 4\theta$$

$$\cos 5\theta$$

$$\cos 6\theta$$
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This part of the talk is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA

Kim Knudsen, Technical University of Denmark

Matti Lassas, University of Helsinki, Finland

Jon Newell, Rensselaer Polytechnic Institute, USA

Jennifer Mueller, Colorado State University, USA
There exists a nonlinear Fourier transform adapted to electrical impedance tomography.
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

\[ \Lambda_{\sigma} \rightarrow \text{BIE} \rightarrow \text{Ideal measurement} \rightarrow \text{Nonlinear IFFT} \]

[Nachman 1996]
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies.
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
There is currently only one regularized method for reconstructing the full conductivity distribution.

Practical measurement → BIE → Lowpass → Nonlinear IFFT

[S, Mueller & Isaacson 2000]
[Knudsen, Lassas, Mueller & S 2009]
Recall these phantoms. Can we distinguish between them using the D-bar method?

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
Here are the D-bar reconstructions from simulated EIT data using frequency cutoff $R = 4$.
The difference image shows clearly where the two patients are not the same.
This is a brief history of the two-dimensional regularized D-bar method for EIT

1966 Faddeev: Complex geometric optics (CGO) solutions

1987 Sylvester and Uhlmann: CGO solutions for inverse boundary-value problems; uniqueness for 3D EIT with smooth conductivities and infinite-precision data

1988 R. G. Novikov: Outline of the core ideas of the D-bar method; no rigorous proof

1996 Nachman: Uniqueness and reconstruction for 2D EIT with $C^2$ conductivities and infinite-precision data

2000 S, Mueller and Isaacson: Numerical implementation of Nachman’s proof using a Born approximation

2006 Isaacson, Mueller, Newell and S: Application of the D-bar method to EIT data measured from a human subject

2009 Knudsen, Lassas, Mueller and S: Regularization proof
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Nachman’s 1996 uniqueness proof in 2D relies on complex geometric optics solutions

Define a potential $q$ by setting $q(z) \equiv 0$ for $z$ outside $\Omega$ and

$$q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}}$$

for $z \in \Omega$.

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(−\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz}\psi(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2),$$

where $\tilde{p} > 2$ and $ikz = i(k_1 + ik_2)(x + iy)$. 
Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

\[ \varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}. \]

We invert the linear operator appearing in the equation

\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = [I + S_k(\Lambda^\delta - \Lambda_1)]^{-1} e^{ikz}|_{\partial\Omega} \]

as a matrix in \( \text{span}(\{\varphi_n\}_{n=-N}^N) \).

The single-layer operator

\[ (S_k \phi)(z) = \int_{\partial\Omega} G_k(z-w)\phi(w) \, ds(w) \]

uses Faddeev’s Green’s function.
What is this so-called CGO sinogram?

The CGO sinogram is a collection of traces of the modified exponential functions used in the nonlinear Fourier transform.

Define the CGO sinogram by setting $\mu(z, k) = e^{-ikz}\psi(z, k)$ and

$$S_\sigma(\theta, \varphi, R) := \mu(e^{i\theta}, Re^{i\varphi}),$$

where both $\theta$ and $\varphi$ range in the interval $[0, 2\pi)$.

The radius $R > 0$ should be small enough for the solution of the boundary integral equation to be stable.
The CGO sinogram is more intuitive geometrically than the DN matrix: here a simple example.
The CGO sinogram is a promising choice for likelihood models in Bayesian inversion.

For recent uses of CGO sinogram, see

**Hamilton, Hauptmann and S (2014):**
Data-driven edge-preserving D-bar method for EIT based on Ambrosio-Tortorelli flow

**Hamilton, Reyes, S and Zhang (submitted):**
A Hybrid Segmentation and D-bar Method for Electrical Impedance Tomography.
Outline

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The Novikov-Veselov equation
Report on waves, J. Scott Russell (1845), British Association for the Advancement of Science
John Scott Russell (1808–1882)  
Heriot-Watt University 1995
Korteweg and de Vries formulated in 1895 an equation for waves in shallow water

\[ u_\tau + u_{xxx} + 6uu_x = 0, \quad x \in \mathbb{R}, \quad \tau \geq 0 \]

Assumptions: wave height is small compared to the depth, which in turn is small compared to the length of the wave.

The KdV equation is a nonlinear, dispersive wave equation. It allows solitary wave solutions observed by Russell and studied by Boussinesq (1871) and lord Rayleigh (1876).
Gardner, Greene, Kruskal and Miura (1967) found a striking connection between the KdV equation and Schrödinger scattering.

\[
(\lambda_n, c_n, R(k)) \mapsto (\lambda_n, c_ne^{4k^3\tau}, R(k)e^{8ik^3\tau})
\]

The inverse scattering step is due to

1946 Borg
1949 Levinson
1951 Gelfand-Levitan
1952 Marchenko
1953 Krein
Novikov-Veselov equation is the most natural 2D generalization of the KdV equation

Korteweg-de Vries equation, dimension (1+1):

\[ \dot{q} + \frac{\partial^3 q}{\partial x^3} + 6q \frac{\partial q}{\partial x} = 0. \]

Kadomtsev-Petviashvili equation, dimension (2+1):

\[ \frac{\partial}{\partial x} \left( \dot{q} + \frac{\partial^3 q}{\partial x^3} + 6q \frac{\partial q}{\partial x} \right) = \pm \frac{\partial^2 q}{\partial y^2}. \]

Novikov-Veselov equation, dimension (2+1):

\[ \dot{q} + \partial_z^3 + \overline{\partial}_z^3 - 3\partial_z(qv) - 3\overline{\partial}_z(q\overline{v}) = 0, \quad \overline{\partial}_z q = \partial_z v. \]

Here \( z = x + iy \) and \( \overline{\partial}_z = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \)
This is another comparison between the KdV and Novikov-Veselov equations

Korteveg-de Vries equation:

\[ \dot{q} = -\frac{\partial}{\partial x} \left( \frac{\partial^2 q}{\partial x^2} \right) - 6q \frac{\partial q}{\partial x}. \]

Novikov-Veselov equation:

\[ \dot{q} = -\frac{\partial}{\partial x} \left( -\frac{\partial^2 q}{\partial x^2} + 3 \frac{\partial^2 q}{\partial y^2} \right) + 6 \text{Re}\left[ (\frac{\partial q}{\partial x}) Sq \right] + 6 \text{Re}[q \frac{\partial}{\partial x}(Sq)] \]
The inverse scattering method is one way to solve the Novikov-Veselov equation:

\[ \text{t}_0(k) \xrightarrow{\exp(i\tau(k^3 + \bar{k}^3))} \text{t}_\tau(k) \]

\[ \mathcal{T} \quad Q \quad \mathcal{T} \quad Q \]

\[ q_0(z) \xrightarrow{\text{nonlinear NV evolution}} q^{\text{IS}}_\tau(z) \]

\[ q_0(z) \xrightarrow{\text{nonlinear NV evolution}} q^{\text{NV}}_\tau(z), \]
The direct and inverse nonlinear Fourier transforms $\mathcal{T}$ and $\mathcal{Q}$ are defined as follows:

The direct transform $q_\tau \mapsto \mathcal{T} q_\tau$ is given by

$$(\mathcal{T} q_\tau)(k) = \int_{\mathbb{R}^2} e^{i k z} q_\tau(z) \psi_\tau(z, k) dz,$$

where $(-\Delta + q_\tau) \psi_\tau(\cdot, k) = 0$ and $\psi_\tau(z, k) \sim e^{ikz}$ as $|z| \to \infty$.

The inverse transform $t_\tau \mapsto \mathcal{Q} t_\tau$ is given by

$$(\mathcal{Q} t_\tau)(z) = \frac{i}{\pi^2} \overline{\partial_z} \int_{\mathbb{C}} \frac{t_\tau(k)}{k} e^{-ikz} \overline{\psi_\tau(z, k)} dk,$$

where $\psi_\tau(z, k) = e^{ikz} \mu_\tau(z, k)$ and $\mu_\tau$ satisfies the D-bar equation

$$\frac{\partial}{\partial k} \mu_\tau(z, k) = \frac{t_\tau(k)}{4\pi k} e^{-i(kz+\overline{k}z)} \overline{\mu_\tau(z, k)}, \quad \mu_\tau(z, \cdot) \sim 1.$$
Zero-energy inverse scattering & NV equation

1984 Novikov & Veselov: Periodic case.


1993 Tsai: Formal analysis assuming no exceptional points.

1996 Nachman: Conductivity-type $q_0$ have no exceptional points.

2007 Lassas, Mueller & S: Inverse scattering evolution $q^{IS}_T$ well-defined for conductivity-type initial data $q_0$.

2011 Lassas, Mueller, S & Stahel: Evolution $q^{IS}_T$ preserves conductivity-type, numerical evidence for $q^{IS}_T = q^{NV}_T$.

2012 Perry: $q^{IS}_T = q^{NV}_T$ holds for conductivity-type $q_0$.

2013 Music, Perry & S: Supercritical exceptional points exist.

2014 Music: Subcritical $q_0$ have no exceptional points.


soon Music & Perry: Global existence for critical and subcritical $q_0$. 
Let’s look at an example. Here is a smooth and rotationally symmetric conductivity function $\sigma(z)$. 
This is the initial potential \( q_0(z) = \sigma^{-1/2}(z) \Delta \sigma^{1/2}(z) \).
This is the initial scattering transform $t_0(k)$.
This is the Novikov-Veselov evolution
All Matlab codes freely available on a website!

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Hexagonal storm on Saturn imaged by NASA’s Cassini probe
Hexagons in optical experiments

Papoff, D’Alessandro, Oppo and Firth 1993: Local and global effects of boundaries on optical-pattern formation in Kerr media
Hexagons in optical experiments

Ackemann, Logvin, Heuer and Lange 1995: Transition between Positive and Negative Hexagons in Optical Pattern Formation
Optical aberrations

**Table 1.** Orthonormal Zernike circle polynomials $Z_j(\rho, \theta)$. The indices $j$, $n$, and $m$ are defined as the polynomial number, radial degree, and azimuthal frequency, respectively. The polynomials $Z_j$ are ordered such that even $j$ corresponds to a symmetric polynomial defined by $\cos m\theta$, while odd $j$ corresponds to an antisymmetric polynomial given by $\sin m\theta$. For a given $n$, a polynomial with a lower value of $m$ is ordered first. $x = \rho \cos \theta$, $y = \rho \sin \theta$, $0 \leq \rho \leq 1$, $0 \leq \theta < 2\pi$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$n$</th>
<th>$m$</th>
<th>$Z_j(\rho, \theta)$</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>Piston</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>$2\rho \cos \theta$</td>
<td>$x$ tilt</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>$2\rho \sin \theta$</td>
<td>$y$ tilt</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0</td>
<td>$\sqrt{3(2\rho^2 - 1)}$</td>
<td>Defocus</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
<td>$\sqrt{6\rho^2 \sin 2\theta}$</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>2</td>
<td>$\sqrt{6\rho^2 \cos 2\theta}$</td>
<td>Astigmatism</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>1</td>
<td>$\sqrt{8(3\rho^3 - 2\rho)\sin \theta}$</td>
<td>Primary $y$ coma</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>1</td>
<td>$\sqrt{8(3\rho^3 - 2\rho)\cos \theta}$</td>
<td>Primary $x$ coma</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>3</td>
<td>$\sqrt{8\rho^3 \sin 3\theta}$</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>3</td>
<td>$\sqrt{8\rho^3 \cos 3\theta}$</td>
<td></td>
</tr>
</tbody>
</table>

*Mahajan 1993: Zernike Circle Polynomials and Optical Aberrations of Systems with Circular Pupils*
This is the evolution of the scattering transform
Let us look at a parameterized family $q_\lambda$ of radial potentials

Assume that $q_0 \in C_0^\infty(\mathbb{R}^2)$ is of conductivity type (there is a smooth, strictly positive function $\psi$ with $\lim_{|z| \to \infty} \psi(z) = 1$ so that $q = \psi^{-1}(\Delta \psi)$) and rotationally symmetric: $q_0(z) = q_0(|z|)$. Set

$$q_\lambda = q_0 + \lambda w,$$

where $w \in C_0^\infty(\mathbb{R}^2)$ is a non-negative rotationally symmetric test function.

Then the scattering transform $t_\lambda : \mathbb{C} \to \mathbb{C}$ corresponding to $q_\lambda$ is real-valued and rotationally symmetric: $t_\lambda(k) = t_\lambda(|k|)$.

For any $\lambda < 0$, the potential $q_\lambda$ is not of conductivity type [Murata 1987].
Consider the Dirichlet problem

$$(-\Delta + q_\lambda) u = 0 \text{ in } B_1$$
$$u|_{S^1} = f. \quad (1)$$

Assume that zero is not a Dirichlet eigenvalue of $$(-\Delta + q_\lambda)$$ in $$B_1$$. If $$u$$ denotes the unique solution to (1), we set

$$\Lambda_{q_\lambda} f = \left. \frac{\partial u}{\partial \nu} \right|_{S^1}.$$  

Rotational symmetry implies $$\Lambda_{q_\lambda} \varphi_n = \mu_n(q_\lambda) \varphi_n$$ for $$\varphi_n(\theta) = \frac{e^{i n \theta}}{\sqrt{2\pi}}$$. We denote $$\mu(q_\lambda) := \mu_0(q_\lambda).$$
Simple example potential

Let’s take $q_\lambda = \lambda w$ with this radial, nonnegative test function $w$:
Here is the zeroth eigenvalue of $\Lambda_{q\lambda}$
Recall how to determine the scattering transform from the DN map $\Lambda_{q\lambda}$

First solve for the traces of the CGO solutions from the boundary integral equation

$$\psi|_{S^1} = e^{ikz} - S_k (\Lambda_q - \Lambda_0) \psi|_{S^1},$$

where $S_k$ is the integral operator

$$(S_k \psi)(z) = \int_{S^1} G_k(z - w)\psi(w) \, d\sigma(w).$$

Now compute

$$t(k) = \int_{S^1} e^{ikz} [(\Lambda_q - \Lambda_0) \psi](z, k) \, d\sigma(z).$$
Profile of scattering transform at $\lambda = -5$
Profile of scattering transform at $\lambda = -15$
Profile of scattering transform at $\lambda = -30$
Theorem (Music, Perry & S 2012)

(1) For $\lambda > 0$ sufficiently small there are no exceptional points, and the scattering transform $t_\lambda$ is $C^\infty$ away from $k = 0$.

(2) For $\lambda < 0$ sufficiently small and a unique $r(\lambda) > 0$, the exceptional set $\mathcal{E}$ is a circle $C_\lambda$ of radius $r(\lambda)$ about the origin, and the function $t_\lambda$ is $C^\infty$ on $\mathbb{R}^2 \setminus [C_\lambda \cup \{0\}]$, while

$$\lim_{|k| \to r(\lambda)} |t_\lambda(k)| = \infty.$$ 

The radius $r(\lambda)$ obeys the formula

$$r(\lambda) \sim_{\lambda \uparrow 0} \exp \left[ -2\pi \left( h - \frac{(1 + O(\lambda))}{2\pi \mu(\lambda)} \right) \right]$$

where $h = -\gamma/(2\pi)$ with Euler’s constant $\gamma$, and $\mu(\lambda)$ is the eigenvalue of the DN map $\Lambda_{q\lambda}$ corresponding to constant functions.
Knowledge about zero-energy exceptional points in dimension two
The asymptotic behaviour predicted by theory matches the numerical results remarkably well.

\[ \exp(-\gamma + \frac{1}{\mu(\lambda)}) \]
Dynamics of NV solutions for $\lambda < 0$ (A. Stahel)
Dynamics of NV solutions for $\lambda > 0$ (A. Stahel)