Edge-Preserving Electrical Impedance Tomography

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This talk provides a connection between three traditions of inverse problems research

(a) Analytic-geometric theory of inverse boundary-value problems for partial differential equations (PDE),
(b) Functional-analytic regularization theory,
(c) Imaging science.

Take-home messages:

PDE techniques developed in (a) provide regularization strategies of type (b) for highly nonlinear inverse problems.

Image enhancement methods from (c) can be stably driven using nonlinear transforms arising from (a).
Applications of Electrical Impedance Tomography (EIT)

EIT is an ill-posed and nonlinear inverse problem

Regularization and EIT

Regularized D-bar method for EIT

The CGO sinogram

Edge-preserving EIT
This talk concentrates on applications of EIT to chest imaging

Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos \theta$

Measure the resulting voltages at the 32 electrodes.
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 2\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos 3\theta$

Measure the resulting voltages at the 32 electrodes.
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern \( \cos 4\theta \)

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 5\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos 16\theta$

Measure the resulting voltages at the 32 electrodes.
The D-bar method works for real EIT data, such as laboratory phantoms and *in vivo* human data.

Saline and agar phantom

Reconstruction ($R = 4$)

[Isaacson, Mueller, Newell & S 2004]
[Montoya 2012]
The most promising use of EIT is detection of breast cancer in combination with mammography. ACT4 and mammography devices 

Radiolucent electrodes

Cancerous tissue is up to four times more conductive than healthy breast tissue [Jossinet 1998]. The above experiment by David Isaacson’s team measures 3D X-ray mammograms and EIT data at the same time.
Which of these three breasts have cancer?
Spectral EIT can detect cancerous tissue

[Kim, Isaacson, Xia, Kao, Newell & Saulnier 2007]
EIT can be used for heart imaging

Lab: Rensselaer Polytechnic Institute 1998

Article: Mueller Isaacson Newell 1999
EIT can potentially be used for imaging changes in vocal folds due to excessive voice use.
EIT can be used for nondestructive testing: here for crack detection in concrete structures

[Karhunen, Seppänen, Lehikoinen, Monteiro & Kaipio 2010]
[Karhunen, Seppänen, Lehikoinen, Monteiro, Kaipio, Blunt, Hyvönen]
Outline

Applications of Electrical Impedance Tomography (EIT)

EIT is an ill-posed and nonlinear inverse problem

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Edge-preserving EIT
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón.

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\left\{ \begin{array}{l}
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = f.
\end{array} \right.$$ 

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_\sigma : f \mapsto \sigma \left( \frac{\partial u}{\partial \vec{n}} \right)|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_\sigma$. It is a nonlinear and ill-posed inverse problem.
Why is the forward map $F : \sigma \mapsto \Lambda_\sigma$ nonlinear?

Define a quadratic form $\mathcal{P}_\sigma$ for functions $f : \partial \Omega \rightarrow \mathbb{R}$ by

$$\mathcal{P}_\sigma(f) = \int_\Omega \sigma |\nabla u|^2 \, dz,$$

where $u$ is the solution of the Dirichlet problem

$$\begin{cases} 
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
 u|_{\partial \Omega} &= f.
\end{cases}$$

Now the map $\sigma \mapsto \mathcal{P}_\sigma$ is nonlinear because $u$ depends on $\sigma$ in (1). Physically, $\mathcal{P}_\sigma(f)$ is the power needed for maintaining the voltage potential $f$ on the boundary $\partial \Omega$. Integrate by parts in (1):

$$\mathcal{P}_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \mathbf{n}} \right) \, ds = \int_{\partial \Omega} f (\Lambda_\sigma f) \, ds.$$

Thus the map $\sigma \mapsto \Lambda_\sigma$ cannot be linear in $\sigma$. 
We illustrate the ill-posedness of EIT using a simulated example
We apply the voltage distribution $f(\theta) = \cos \theta$ at the boundary of the two different phantoms.
The measurement is the distribution of current through the boundary

\[ \sigma_1 \quad \sigma_2 \]

\[ \theta \]

\[ \frac{\partial u_1}{\partial \vec{n}} \]

\[ \frac{\partial u_2}{\partial \vec{n}} \]
The current data are very similar, although the conductivities are quite different.
Let us apply the more oscillatory distribution \( f(\theta) = \cos 2\theta \) of voltage at the boundary.
The measurement is again the distribution of current through the boundary

\[
\sigma_1 \frac{\partial u_1}{\partial \vec{n}} \quad \sigma_2 \frac{\partial u_2}{\partial \vec{n}}
\]
The current distribution measurements are almost the same

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \quad \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data

\[ \sigma_1, \sigma_2 \]

\[
\cos \theta, \cos 2\theta, \cos 3\theta, \cos 4\theta, \cos 5\theta, \cos 6\theta
\]
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data.
There are many EIT reconstruction methods:

**Linearization:** Barber, Bikowski, Brown, Calderón, Cheney, Isaacson, Mueller, Newell

**Iterative regularization:** Dobson, Gehre, Harbrecht, Hohage, Hua, Jin, Kaipio, Kindermann, Kluth, Leitão, Lechleiter, Lipponen, Maass, Neubauer, Rieder, Rondi, Santosa, Seppänen, Tompkins, Webster, Woo

**Bayesian inversion:** Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkainen, Seppänen, Somersalo, Vauhkonen, Voutilainen

**Resistor network methods:** Borcea, Druskin, Mamonov, Vasquez

**Layer stripping:** Cheney, Isaacson, Isaacson, Somersalo

**D-bar methods:** Astala, Bikowski, Bowerman, Delbary, Hamilton, Hansen, Herrera, Isaacson, Kao, Knudsen, Lassas, Montoya, Mueller, Murphy, Nachman, Newell, Päivärinta, Perämäki, Saulnier, S, Tamasan, Tamminen

**Teichmüller space methods:** Kolehmainen, Lassas, Ola, S

**Methods for partial information:** Alessandrini, Ammari, Bilotta, Brühl, Eckel, Erhard, Gebauer, Hanke, Harrach, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kress, Kwon, Lechleiter, Lim, Maass, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, ...
Applications of Electrical Impedance Tomography (EIT)

EIT is an ill-posed and nonlinear inverse problem

Regularization and EIT

Regularized D-bar method for EIT

The CGO sinogram

Edge-preserving EIT
The forward map $F : X \supset D(F) \rightarrow Y$ does not have a continuous inverse!

Model space $X = L^\infty(\Omega)$

Data space $Y = L(H^{1/2}(\partial \Omega), H^{-1/2}(\partial \Omega))$

Furthermore, the noisy data $\Lambda_\delta^\sigma$ does not belong to the range $F(D(F))$. 
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.

Regularization must be based on combining the incomplete measurement data with a priori information about the conductivity.
A **regularization strategy** needs to be constructed so that the assumptions below are satisfied

A family $\Gamma_\alpha : Y \to X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a **regularization strategy** for $F$ if

$$\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\sigma) - \sigma \|_X = 0$$

for each fixed $\sigma \in D(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called **admissible** if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $\sigma \in D(F)$ the following holds:

$$\sup_{\Lambda_\delta} \left\{ \| \Gamma_\alpha(\delta)(\Lambda_\sigma) - \sigma \|_X : \| \Lambda_\delta - \Lambda_\sigma \|_Y \leq \delta \right\} \to 0 \text{ as } \delta \to 0.$$
1. **Tikhonov regularization:** write a penalty functional

\[
\Phi(x) = \|A\tilde{\sigma} - A\delta\|^2_Y + \alpha\|\tilde{\sigma}\|^2_X,
\]

and define \(\Gamma_\alpha(A\delta)\) by \(\Phi(\Gamma_\alpha(A\delta)) = \min_{\tilde{\sigma} \in X} \{\Phi(\tilde{\sigma})\}\).

**Pro:** The same code applies to many problems.

**Con:** Repeated solution of direct problem needed.

**Con:** Prone to get stuck in local minima.

Current theory of iterative regularization does not cover full EIT because of high degree of nonlinearity.

2. **Problem-specific regularization**

**Pro:** Can deal efficiently with a specific nonlinearity.

**Con:** Each code applies to only one problem.

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Edge-preserving EIT
This part of the talk is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA

Kim Knudsen, Technical University of Denmark

Matti Lassas, University of Helsinki, Finland

Jon Newell, Rensselaer Polytechnic Institute, USA

Jennifer Mueller, Colorado State University, USA
There exists a nonlinear Fourier transform adapted to electrical impedance tomography.
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements.
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
There is currently only one regularized method for reconstructing the full conductivity distribution - BIE followed by nonlinear IFFT. Practical measurement techniques include [S, Mueller & Isaacson 2000] and [Knudsen, Lassas, Mueller & S 2009].
This is a brief history of the two-dimensional regularized D-bar method for EIT

1966 Faddeev: Complex geometric optics (CGO) solutions

1987 Sylvester and Uhlmann: CGO solutions for inverse boundary-value problems; uniqueness for 3D EIT with smooth conductivities and infinite-precision data

1988 R. G. Novikov: Outline of the core ideas of the D-bar method; no rigorous proof

1996 Nachman: Uniqueness and reconstruction for 2D EIT with $C^2$ conductivities and infinite-precision data

2000 S, Mueller and Isaacson: Numerical implementation of Nachman’s proof using a Born approximation

2006 Isaacson, Mueller, Newell and S: Application of the D-bar method to EIT data measured from a human subject

2009 Knudsen, Lassas, Mueller and S: Regularization proof
Nonlinear low-pass filtering yields a regularization strategy with convergence speed

Theorem (Knudsen, Lassas, Mueller & S 2009)

Fix a conductivity $\sigma \in D(F)$. Assume given noisy data $\Lambda^\delta_\sigma$ satisfying

$$\|\Lambda^\delta_\sigma - \Lambda_\sigma\|_Y \leq \delta.$$ 

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_\alpha(\delta)(\Lambda^\delta_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \delta)^{-1/14}.$$
Regularized reconstructions from simulated data with noise amplitude $\delta = \|\Lambda^\delta - \Lambda_\sigma\|_Y$

\[
\delta \approx 10^{-6} \quad \delta \approx 10^{-5} \quad \delta \approx 10^{-4} \quad \delta \approx 10^{-3} \quad \delta \approx 10^{-2}
\]

The percentages are the relative square norm errors in the reconstructions.
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Nachman’s 1996 uniqueness proof in 2D relies on complex geometric optics solutions

Define a potential \( q \) by setting \( q(z) \equiv 0 \) for \( z \) outside \( \Omega \) and

\[
q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for} \quad z \in \Omega.
\]

Then \( q \in C_0(\Omega) \). We look for solutions of the Schrödinger equation

\[
(-\Delta + q)\psi(\cdot, k) = 0 \quad \text{in} \quad \mathbb{R}^2
\]

parametrized by \( k \in \mathbb{C} \setminus 0 \) and satisfying the asymptotic condition

\[
e^{-ikz}\psi(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2),
\]

where \( \tilde{p} > 2 \) and \( ikz = i(k_1 + ik_2)(x + iy) \).
Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

\[ \varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}. \]

We invert the linear operator appearing in the equation

\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = [I + S_k(\Lambda_\sigma^\delta - \Lambda_1)]^{-1} e^{ikz}|_{\partial\Omega} \]

as a matrix in \( \text{span}(\{\varphi_n\}_{n=-N}^N) \).

The single-layer operator

\[ (S_k \phi)(z) = \int_{\partial\Omega} G_k(z-w) \phi(w) \, ds(w) \]

uses Faddeev’s Green’s function.
What is this so-called CGO sinogram?

The CGO sinogram is a collection of traces of the modified exponential functions used in the nonlinear Fourier transform.

Define the CGO sinogram by setting $\mu(z, k) = e^{-ikz}\psi(z, k)$ and

$$S_\sigma(\theta, \varphi, R) := \mu(e^{i\theta}, Re^{i\varphi}),$$

where both $\theta$ and $\varphi$ range in the interval $[0, 2\pi)$.

The radius $R > 0$ should be small enough for the solution of the boundary integral equation to be stable.
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi$ and set $\mu(z, k) = e^{-ikz}\psi(z, k)$.

Conductivity (z-plane)

$k$-plane

CGO sinogram

$z = e^{i\theta}$

$k = \text{Re} e^{i\varphi}$

$\mu(e^{i\theta}, \text{Re} e^{i\varphi})$
CGO sinogram in the context of EIT

Solve \( \psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi \) and set \( \mu(z, k) = e^{-ikz}\psi(z, k) \).

Conductivity (\( z \)-plane)

\[ z = e^{i\theta} \]

\( k \)-plane

\[ k = \text{Re}^{i\varphi} \]

CGO sinogram

\[ \mu(e^{i\theta}, \text{Re}^{i\varphi}) \]
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi$ and set $\mu(z, k) = e^{-ikz}\psi(z, k)$.

Conductivity ($z$-plane)

$k$-plane

CGO sinogram

$z = e^{i\theta}$

$k = \text{Re}^{i\varphi}$

$\varphi$

$\theta$

$\mu(e^{i\theta}, \text{Re}^{i\varphi})$
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi$ and set $\mu(z, k) = e^{-ikz}\psi(z, k)$.
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Conductivity (\( z \)-plane)

\[ z = e^{i\theta} \]

\( k \)-plane

\[ k = \text{Re}^i\varphi \]

CGO sinogram

\[ \mu(e^{i\theta}, \text{Re}^i\varphi) \]
CGO sinogram in the context of EIT

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Conductivity (z-plane)

\[ z = e^{i\theta} \]

k-plane

\[ k = Re^{i\varphi} \]

CGO sinogram

\[ \mu(e^{i\theta}, Re^{i\varphi}) \]
CGO sinogram in the context of EIT

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Conductivity (z-plane)
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z = e^{i\theta}
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k-plane
\[
k = \text{Re}e^{i\varphi}
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CGO sinogram
\[
\mu(e^{i\theta}, \text{Re}e^{i\varphi})
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CGO sinogram in the context of EIT

Solve \( \psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \) and set \( \mu(z, k) = e^{-ikz}\psi(z, k) \).

Conductivity (\( z \)-plane)

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CGO sinogram in the context of EIT

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CGO sinogram in the context of EIT

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CGO sinogram in the context of EIT

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Conductivity (\( z \)-plane)

\( z = e^{i\theta} \)

\( k = \text{Re}^{i\varphi} \)

CGO sinogram

\( \mu(e^{i\theta}, \text{Re}^{i\varphi}) \)
CGO sinogram in the context of EIT

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Conductivity ($z$-plane)

$z = e^{i\theta}$

$k$-plane

$k = \text{Re}^{i\varphi}$

CGO sinogram

$\varphi$

$\theta$

$\mu(e^{i\theta}, \text{Re}^{i\varphi})$
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_{\sigma} - \Lambda_1)\psi$ and set $\mu(z, k) = e^{-ikz}\psi(z, k)$. 

Conductivity ($z$-plane)

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$\mu(e^{i\theta}, \text{Re}^{i\varphi})$
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$k = \text{Re} e^{i\varphi}$

$\mu(e^{i\theta}, \text{Re} e^{i\varphi})$
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial \Omega} = e^{i k z} - S_k (\Lambda^\delta - \Lambda_1) \psi$ and set $\mu(z, k) = e^{-i k z} \psi(z, k)$.

Conductivity ($z$-plane)

$k$-plane

CGO sinogram

$z = e^{i \theta}$

$k = \text{Re} e^{i \varphi}$

$\varphi$

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$\mu(e^{i \theta}, \text{Re} e^{i \varphi})$
CGO sinogram in the context of EIT

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$$\mu(e^{i\theta}, \text{Re}^{i\varphi})$$
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Conductivity ($z$-plane) $\quad k$-plane $\quad$ CGO sinogram

$z = e^{i\theta} \quad k = \text{Re}^{i\varphi} \quad \mu(e^{i\theta}, \text{Re}^{i\varphi})$
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi$ and set $\mu(z, k) = e^{-ikz}\psi(z, k)$.

Conductivity ($z$-plane) $k$-plane

$z = e^{i\theta}$ $k = \text{Re}^{i\varphi}$

CGO sinogram

$\mu(e^{i\theta}, \text{Re}^{i\varphi})$
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi$ and set $\mu(z, k) = e^{-ikz}\psi(z, k)$.
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CGO sinogram in the context of EIT

Solve \( \psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\delta - \Lambda_1)\psi \) and set \( \mu(z, k) = e^{-ikz}\psi(z, k) \).

Conductivity (z-plane) \( z = e^{i\theta} \)

\( k \)-plane \( k = \text{Re} e^{i\varphi} \)

CGO sinogram \( \mu(e^{i\theta}, \text{Re} e^{i\varphi}) \)
CGO sinogram in the context of EIT

Solve \( \psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda^\delta - \Lambda_1)\psi \) and set \( \mu(z, k) = e^{-ikz}\psi(z, k) \).

 Conductivity (\( z \)-plane)

\[ z = e^{i\theta} \]

\( k \)-plane

\[ k = \text{Re}^{i\varphi} \]

CGO sinogram

\[ \mu(e^{i\theta}, \text{Re}^{i\varphi}) \]
CGO sinogram in the context of EIT

Solve $\psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda^\delta_\sigma - \Lambda_1)\psi$ and set $\mu(z, k) = e^{-ikz}\psi(z, k)$.

Conductivity ($z$-plane)

$k$-plane

CGO sinogram

$z = e^{i\theta}$

$k = \text{Re}e^{i\varphi}$

$\mu(e^{i\theta}, \text{Re}e^{i\varphi})$
The CGO sinogram is more intuitive geometrically than the DN matrix: here a simple example

Conductivity      DN matrix      CGO sinogram
The CGO sinogram is more intuitive geometrically than the DN matrix: a more complicated case.
Conductivity 1

Conductivity 2

Difference

CGO sinogram 1

CGO sinogram 2

Difference

$\frac{5\pi}{4}$

$4\frac{5\pi}{4}$

$2\pi$
Outline

Applications of Electrical Impedance Tomography (EIT)

EIT is an ill-posed and nonlinear inverse problem

Regularization and EIT

Regularized D-bar method for EIT

The CGO sinogram

Edge-preserving EIT
This part of the talk is a joint work with

Sarah Hamilton, University of Helsinki, Finland

Andreas Hauptmann, University of Helsinki, Finland

Also shown are results with Xiaoqun Zhang and Juan Manuel Reyes
We choose the best (contrast-enhanced) image from the Ambrosio-Tortorelli segmentation flow. This provides a link between imaging methods and PDE-based inversion.

<table>
<thead>
<tr>
<th>Iteration number</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Segmentation by AT flow</strong></td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
<tr>
<td><strong>Enhanced contrast</strong></td>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
<td><img src="image10.png" alt="Image" /></td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
</tr>
<tr>
<td>Error in CGO sinogram (%)</td>
<td>20.3</td>
<td>17.76</td>
<td>17.41</td>
<td>17.28</td>
<td>17.36</td>
<td>17.56</td>
</tr>
</tbody>
</table>
Let us use the CGO sinogram for correcting the contrast in a D-bar reconstruction.
We modify the contrast in the reconstruction using a parameter $0 \leq s \leq 1$
The CGO-controlled Ambrosio-Tortorelli flow gives a nonlinear, edge-preserving EIT method.

Conductivity  
D-bar reconstruction  
AT and D-bar

[Hamilton, Hauptmann & S 2014]
A quick note on some new unpublished results

Hamilton, Reyes, S and Zhang, work in progress
A quick note on some new unpublished results

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A quick note on some new unpublished results

Hamilton, Reyes, S and Zhang, work in progress
A quick note on some new unpublished results

Hamilton, Reyes, S and Zhang, work in progress
D-bar method is the only proven regularization method for EIT, based on low-pass filtering a nonlinear Fourier transform.

Practical noise levels require so severe truncation that high frequencies are lost in the D-bar reconstruction: all the missing high-frequency information is (incorrectly) replaced by zero.

The CGO sinogram approach allows combining stable low-frequency information with *a priori* knowledge about the conductivity. The present examples of using segmentation flows to promote piecewise constantness is just one example of a general strategy.

This is a new bridge between PDE-inversion and imaging science.
Part I: Linear Inverse Problems
1 Introduction
2 Naïve reconstructions and inverse crimes
3 Ill-Posedness in Inverse Problems
4 Truncated singular value decomposition
5 Tikhonov regularization
6 Total variation regularization
7 Besov space regularization using wavelets
8 Discretization-invariance
9 Practical X-ray tomography with limited data
10 Projects

Part II: Nonlinear Inverse Problems
11 Nonlinear inversion
12 Electrical impedance tomography
13 Simulation of noisy EIT data
14 Complex geometrical optics solutions
15 A regularized D-bar method for direct EIT
16 Other direct solution methods for EIT
17 Projects
We define spaces for our regularization strategy

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \rightarrow \mathbb{R}$ with

- $\|\sigma\|_{C^2(\overline{\Omega})} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators $A : H^{1/2}(\partial \Omega) \rightarrow H^{-1/2}(\partial \Omega)$ satisfying

- $A(1) = 0$,
- $\int_{\partial \Omega} A(f) \, ds = 0$. 
The observed radii are better (=larger) than those given by the theoretical formula \( R(\delta) = -\frac{1}{10} \log \delta \)
The complex geometric optics solutions of Astala and Päivärinta enable accurate study of the nonlinear Fourier transform.

Nonlinear low-pass filtering with cutoff frequency $R = 40$.

[Astala, Päivärinta, Reyes & S 2014]