Regularized D-bar method for the inverse conductivity problem

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http://math.tkk.fi/inverse-coe/
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1. Inverse problems and regularization

2. The inverse conductivity problem of Calderón

3. Theory of d-bar imaging: infinite precision data

4. Regularized d-bar imaging for noisy data

5. Numerical results
Hadamard’s definition of a “well-posed problem” has three parts

(H1) A solution exists

(H2) The solution is unique

(H3) The output depends continuously on the input

A problem is called “ill-posed”, or inverse problem, if (H1), (H2) or (H3) fails.

Jacques Salomon Hadamard (1865-1963)
Ill-posed problems typically arise from indirect measurements

Example: Radon transform in X-ray tomography

X-ray attenuation coefficient belongs to model space $X$.

Radon transform (sinogram) belongs to data space $Y$.

Forward map $F$ is the function mapping a coefficient to its data.
In ill-posed problems, the forward map

\[ F : \mathcal{D}(F) \subset X \rightarrow Y \]

does not have a continuous inverse.

Moreover, we cannot measure in practice the ideal data \( m_x = F(x) \) but only a noisy approximation \( m_x^\varepsilon = P(F(x)) + \varepsilon \).

How can we recover \( x \) approximately from the knowledge of \( m_x^\varepsilon \) in a noise-robust way? The answer is *regularization*. 
A family of continuous mappings $\Gamma_\alpha : Y \to X$ is called a regularization strategy parametrized by $0 < \alpha < \infty$ if $\lim_{\alpha \to 0} \| \Gamma_\alpha(m_x) - x \|_X = 0$ for each fixed $x \in X$.

Furthermore, a choice of $\alpha = \alpha(\varepsilon)$ as function of the noise level $\varepsilon > 0$ is called admissible if $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0$ and for any fixed $x \in X$ the following holds:

$$\sup_{m_x} \{ \| \Gamma_\alpha(m_x^\varepsilon) - x \|_X : \| m_x^\varepsilon - m_x \|_Y \leq \varepsilon \} \to 0$$
as $\varepsilon \to 0$. 
Tikhonov’s pioneering work in the 1930’s led to regularization of linear inverse problems

\[ \Gamma_\alpha(m^\varepsilon_x) = \underset{x \in \mathcal{D}(F)}{\text{arg\,min}} \left\{ \|Fx - m^\varepsilon_x\|_Y^2 + \alpha \|x\|_X^2 \right\} \]

Andrei Nikolaevich Tikhonov (1906-1993) contributed to several fields of mathematics, including topology and functional analysis. His studies of compact embeddings are the basis of the most popular solution methods for inverse problems, called **Tikhonov regularization**.
Regularization of nonlinear inverse problems is an active and challenging area of research

There are two main approaches for regularizing nonlinear inverse problems:

1. **Iterative regularization**
   + Generic: applicable in principle to any inverse problem,
   + Quick to develop optimization-based solution software,
   o Possible problems with local minima.

2. **Tailored nonlinear regularization strategies:**
   + Rigorous mathematical analysis available for algorithms,
   + Provides a link between two schools of research,
   o Difficult to develop,
   o One method applies only to one inverse problem.
Iterative regularization theory is discussed in the following references:

**Hilbert space:**
- 2008 Mathé & Hofmann
- 2008 Hohage & Pricop
- 2007 Lu, Pereverzev & Ramlau
- 2004 Bissantz, Hohage & Munk
- 1997 Hanke

**Banach space:**
- 2008 Ramlau
- 2008 Kaltenbacher, Neubauer & Scherzer
- 2008 Kindermann & Neubauer
- 2007 Hofmann, Kaltenbacher, Pöschl & Scherzer
- 2006 Kaltenbacher & Neubauer
- 2006 Ramlau & Teschke
- 2005 Resmerita
2009 Knudsen, Lassas, Mueller & S
Regularized D-bar method for EIT.

2006 Justin and Ramlau
Regularized blind deconvolution.

2006 Lechleiter (based on Hyvönen 2004)
Regularized inclusion detection for EIT using the factorization method.

2004 Ikehata & S
Regularized inclusion detection for EIT based on probing with cones.

2004 Arens
Regularized inverse obstacle scattering by linear sampling.

Practical research tradition
Given noisy measurement $m_x^e$, compute robust approximation of $x$.

Theoretical research tradition
Prove that the ideal measurement $m_x$ uniquely determines $x$. 
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Electrical impedance tomography (EIT) is an emerging medical imaging method

Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

Applications include: monitoring heart and lungs of unconscious patients, detecting pulmonary edema, enhancing ECG and EEG
EIT can be used as well in industrial process monitoring
EIT can be used for finding defects in materials

The inverse conductivity problem of Calderón is the mathematical model of EIT

Problem: given the Dirichlet-to-Neumann map, how to reconstruct the conductivity?

The reconstruction problem is nonlinear and ill-posed.

\[ \Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}|_{\partial \Omega}, \]

\[ \nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \]

\[ u = f \text{ on } \partial \Omega. \]

We assume that \( 0 < c \leq \gamma(x) \leq C \) for all \( x \in \Omega \).
Nonlinearity of Calderón’s problem

The weak formulation of the Dirichlet-to-Neumann map

$$\Lambda_{\gamma} : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$$

is given by

$$\langle \Lambda_{\gamma} f, g \rangle = \int_{\Omega} \gamma \nabla u \cdot \nabla v,$$

where $v$ is any $H^1(\Omega)$ function with trace $g$, and $u$ satisfies the Dirichlet problem

$$\begin{cases} 
\nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega, \\
\n\quad u = f & \text{on } \partial\Omega.
\end{cases}$$

Now the map $\gamma \mapsto \Lambda_{\gamma}$ is nonlinear because $u$ depends on $\gamma$. 

The weak formulation of the Dirichlet-to-Neumann map
EIT reconstruction algorithms can be divided roughly into the following classes:

**Linearization** (Barber, Bikowski, Brown, Cheney, Isaacson, Mueller, Newell)

**Iterative regularization** (Dobson, Hua, Kindermann, Leitão, Lechleiter, Neubauer, Rieder, Rondi, Santosa, Tompkins, Webster, Woo)

**Bayesian inversion** (Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen)

**Resistor network methods** (Borcea, Druskin, Vasquez)

**Convexification** (Beilina, Klibanov)

**Layer stripping** (Cheney, Isaacson, Isaacson, Somersalo)

**D-bar methods** (Astala, Bikowski, Bowerman, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Saulnier, S, Tamasan)

**Teichmüller space methods** (Kolehmainen, Lassas, Ola)

**Methods for partial information** (Alessandrini, Ammari, Bilotta, Brühl, Erhard, Gebauer, Hanke, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kwon, Lechleiter, Lim, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others)
This is a brief history of D-bar methods in 2D

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Nachman’s 1996 proof consists of two steps:

\[ \wedge \gamma \rightarrow t \rightarrow \gamma \]

The intermediate object \( t \) is a complex-valued function called \textit{scattering transform} and defined as follows:

\[ t(k) := \int_{\mathbb{R}^2} e^{ik\overline{x}} q(x) \psi(x, k) \, dx \]

The function \( t \) is also a nonlinear Fourier transform.

\[
q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \quad \begin{align*}
(-\Delta + q)\psi(\cdot, k) &= 0 \\
\psi(x, k) &\sim e^{ikx} = e^{i(k_1 + ik_2)(x_1 + ix_2)}
\end{align*}
\]
Let’s look at Nachman’s construction of the complex geometrical optics (CGO) solutions

We need a $\psi$ satisfying $(-\Delta + q)\psi(x, k) = 0$
and $\psi(x, k) \sim e^{ikx}$ as $x \to \infty$. We try to write
$\psi(x, k) = e^{ikx} \mu(x, k)$ where $\mu(\cdot, k) - 1 \in W^{1,p}(\mathbb{R}^2)$.

Now $(-\Delta + q)e^{ikx} \mu(x, k) = 0$ is equivalent to

$$(-\Delta - 4ik\bar{\partial} + q(x))\mu(x, k) = 0.$$

This, together with the asymptotic condition, can be written in the form of the Lippmann-Schwinger type equation

$$\mu = 1 - g_k \ast (q\mu),$$

where the fundamental solution $g_k$ satisfies
$(-\Delta - 4ik\bar{\partial})g_k(x) = \delta(x)$ and the symbol $\ast$
denotes convolution over $\mathbb{R}^2$. The solution is
given by $\mu - 1 = [I + g_k \ast (q\cdot)]^{-1}(g_k \ast q)$. 
Now we can derive a boundary integral equation for the CGO solutions

Define \( G_k(x) = e^{ikx} g_k(x) \) and recall the identity \((-\Delta - 4ik\bar{\partial})g_k(x) = \delta(x)\). It follows that

\[-\Delta G_k = \delta.\]

We call \( G_k \) Faddeev Green’s function. Now

\[
\psi(x, k) = e^{ikx} \mu(x, k) \\
= e^{ikx} - e^{ikx}(g_k * (q\mu)) \\
= e^{ikx} - \int_{\mathbb{R}^2} e^{ik(x-y)} g_k(x-y) q(y) \mu(y, k) \, dy \\
= e^{ikx} - \int_{\mathbb{R}^2} G_k(x-y) q(y) \psi(y, k) \, dy,
\]

and Alessandrini’s identity gives

\[
\psi(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)(\psi(\cdot, k)|_{\partial\Omega}),
\]

where the Faddeev single layer operator \( S_k \) is defined by \((S_k\varphi)(x) = \int_{\partial\Omega} G_k(x-y) \varphi(y) \, dS(y)\).
Step 1: from DN map to scattering transform

Solve traces of \( \psi \) from the boundary integral equation

\[
\psi(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k),
\]

where the single-layer operator has Faddeev Green’s function as kernel.

Compute the scattering transform as

\[
t(k) = \int_{\partial \Omega} e^{ik\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(x, k) d\sigma(x).
\]
Step 2: from scattering transform to $g$

Define $\mu(x, k) = e^{-ikx} \psi(x, k)$

Then the following D-bar equation holds:

$$\frac{\partial}{\partial k} \mu(x, k) = \frac{t(k)}{4\pi k} e^{-i(kx + k\bar{x})} \mu(x, k).$$

Here $\frac{\partial}{\partial k} = \frac{1}{2} \left( \frac{\partial}{\partial k_1} + i \frac{\partial}{\partial k_2} \right)$.

The D-bar equation has a unique solution for all $x$. The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k \to 0} \mu(x, k).$$
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We work within the following assumptions:

Let \( \Omega \subset \mathbb{R}^2 \) be the open unit disc.

Define the forward map \( F \) between the spaces \( F : \mathcal{D}(F) \subset L^\infty(\Omega) \to Y \).

Domain \( \mathcal{D}(F) \) is defined as follows.

Let \( M > 0 \) and \( 0 < \rho < 1 \). The set \( \mathcal{D}(F) \) contains functions \( \gamma : \Omega \to \mathbb{R} \) satisfying

(a) \( \|\gamma\|_{C^2(\bar{\Omega})} \leq M \),
(b) \( \gamma(x) \geq M^{-1} \) for all \( x \in \Omega \),
(c) \( \gamma(x) \equiv 1 \) for \( \rho < |x| < 1 \).

Space \( Y \) of data is defined as follows.

\( Y \) consists of bounded linear operators

\[ \Lambda : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \]

satisfying \( \int_{\partial \Omega} \Lambda(f) d\sigma = 0 \) and \( \Lambda(1) = 0 \).
Let us define nonlinear regularization strategy
(following Engl, Hanke & Neubauer and Kirsch)

Recall direct problem: $\gamma \in X$ maps to $\Lambda_\gamma \in Y$.

A family of continuous mappings $\Gamma_\alpha : Y \rightarrow X$ with $0 < \alpha < \infty$ is a regularization strategy if

$$\lim_{\alpha \rightarrow 0} \| \Gamma_\alpha \Lambda_\gamma - \gamma \|_{L^\infty(\Omega)} = 0$$

for each fixed $\gamma \in X$. A regularization strategy with a choice $\alpha = \alpha(\varepsilon)$ is called admissible if

$$\alpha(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

and for any fixed $\gamma \in X$ the following holds:

$$\sup_{\Lambda_\gamma^\varepsilon} \left\{ \| \Gamma_{\alpha(\varepsilon)} \Lambda_\gamma^\varepsilon - \gamma \|_{L^\infty(\Omega)} : \| \Lambda_\gamma^\varepsilon - \Lambda_\gamma \|_Y \leq \varepsilon \right\} \rightarrow 0 \quad \varepsilon \rightarrow 0.$$
This is our regularized D-bar method for EIT

Given the noise level $\varepsilon > 0$, solve the equation

$$\psi^\varepsilon(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda_1^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial \Omega}$$

for $|k| < R(\varepsilon) := -\frac{1}{10} \log(\varepsilon)$.

Introduce nonlinear low-pass filtering

$$t^\varepsilon_R(k) := \begin{cases} \int_{\partial \Omega} e^{ikx}(\Lambda_1^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k) d\sigma & \text{for } |k| < R(\varepsilon), \\ \text{zero for } |k| \geq R(\varepsilon). \end{cases}$$

For each $x \in \Omega$, solve the integral equation

$$\mu_R(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t^\varepsilon_R(s)}{(k - s)\bar{s}} e^{-x(s)} \mu_R(x, s) ds_1 ds_2,$$

and define $\alpha(\varepsilon) = \frac{1}{R(\varepsilon)}$ and $(\Gamma \alpha \Lambda_1^\varepsilon)(x) := (\mu_R(x, 0))^2$. 
Starting point: ideal data $\Lambda_{\gamma}$

Scattering transform $t(k)$

Solve $\bar{\partial}$ equation

$$\frac{\partial}{\partial k} \mu(x, k) = \frac{t(k)}{4\pi k} e^{-i(kx + \bar{k}\bar{x})} \mu(x, k)$$

Perfect reconstruction

$\gamma(x) = \mu(x, 0)^2$

Starting point: noisy data $\Lambda_{\gamma}^{\varepsilon}$

Noisy scattering transform $t_{R}^{\varepsilon}(k)$ truncated at $R(\varepsilon)$

Solve $\bar{\partial}$ equation

$$\frac{\partial}{\partial k} \mu_R(x, k) = \frac{t_{R}^{\varepsilon}(k)}{4\pi k} e^{-i(kx + \bar{k}\bar{x})} \mu_R(x, k)$$

Approximate reconstruction

$\gamma(x) \approx \mu_R(x, 0)^2$
Truncation radius is $R(\varepsilon) = -\frac{1}{10} \log(\varepsilon)$,

regularization parameter is $\alpha = 1/R$. 
**Theorem** [Knudsen, Lassas, Mueller & S 2008]

The family $\Gamma_\alpha$ is well-defined for small $\alpha > 0$. It is an admissible regularization strategy with

$$\alpha(\varepsilon) = \left(-\frac{1}{10} \log(\varepsilon)\right)^{-1}.$$ 

Furthermore, we have the explicit estimate

$$\sup_{\Lambda^\varepsilon_\gamma} \left\{ \|\Gamma_\alpha(\varepsilon) \Lambda^\varepsilon_\gamma - \gamma\|_{L^\infty(\Omega)} : \|\Lambda^\varepsilon_\gamma - \Lambda_\gamma\|_Y \leq \varepsilon \right\}$$

$$\leq C(-\log \varepsilon)^{-1/14}$$

$$\to 0 \text{ as } \varepsilon \to 0.$$
Let us emphasize one of the strengths of our new results

Conditional stability results have the form

$$\|\gamma_1 - \gamma_2\|_Z \leq f(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_Y),$$

where $\gamma_1, \gamma_2$ belong to some function space $Z$ and $f$ is a continuous function with $f(0) = 0$.

The above estimate is not practically relevant. The noisy measurement $\Lambda^\varepsilon_\gamma$ is in general not the DN map of some conductivity.

In contrast, we prove regularization properties for the D-bar method under the practically feasible assumption $\|\Lambda^\varepsilon_\gamma - \Lambda_\gamma\|_Y \leq \varepsilon$. 
One more thing: the regularization strategy is not yet defined on all of data space $Y$

The range $F(D(F)) \subset Y$ is not known, and its structure may be complicated. (This is related to the open and notoriously difficult characterization problem.)

The previous results show the claim only for operators $\varepsilon_0$-close to the range $F(D(F))$.

The problem can be overcome using spectral theoretical arguments.
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This is a typical configuration for electrode measurements in EIT

Here we have $N=32$ electrodes. The machine is in Rensselaer Polytechnic Institute, USA.
There is a finite number of linearly independent current patterns.

Here are three examples with \( N=32 \):

Altogether, there are \( N-1 \) linearly independent current patterns due to conservation of charge.
We construct a simulated human chest phantom and compute DN map using FEM
We approximate discrete current patterns by Fourier basis functions

\[ \cos(\theta) \quad \cos(4\theta) \quad \cos(16\theta) \]
This is our practical two-step regularized D-bar method for EIT

1. We solve for $|k| < R$ the matrix version of

$$\psi^\varepsilon(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial\Omega}$$

with $R$ as large as numerically stable.

2. The integral equation

$$\mu_R(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t_R^\varepsilon(s)}{(k - s)^2} e^{-x(s)} \mu_R(x, s) ds_1 ds_2$$

can be solved by our established D-bar solver. The reconstructed conductivity is $\mu_R(x, 0)^2$. 
We solve the boundary integral equation using matrices in truncated Fourier basis

Given linear map $A$ and $N > 0$, define matrix $A : \mathbb{C}^{2N+1} \to \mathbb{C}^{2N+1}$ by

$$A_{mn} := \frac{1}{2\pi} \int_0^{2\pi} (Ae^{in\theta}) e^{-im\theta} d\theta.$$ 

We write all operators in the equation

$$\psi(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k)|_{\partial\Omega}$$

in matrix form, and solve

$$[I + S_k(\Lambda_\gamma - \Lambda_1)]\psi(\cdot, k)|_{\partial\Omega} = e^{ikx}.$$
For $x$ in the unit disc we compute $g_1(x)$ with a formula by [Boiti et al 1987]

$$g_1(x) = -\frac{e^{-ix}}{4\pi} (2\gamma + \log |x|^2 + \sum_{n=1}^{\infty} \frac{(ix)^n + (-i\bar{x})^n}{nn!})$$

Here $g$ is Euler’s constant
We write $g_1(x)$ as a formula containing a rapidly converging one-dimensional integral

$$g_1(x) = \frac{e^{-ix_1}}{2\pi} \text{Re} \left[ - e^{ix_1} \sum_{j=0}^{N} \frac{j!}{(ix)^{j+1}} + (N + 1)! e^{ix_1} \frac{1}{(-x)^{N+1}} \int_{0}^{\infty} \frac{e^{-t(x_1 + ix_2)}}{(t - i)^{N+2}} dt \right]$$

For other domains we use residue calculus and reflectional symmetry
The d-bar equation is written in integral form for numerical solution by Vainikko’s method.

Write the d-bar equation

\[ \frac{\partial}{\partial k} \mu_R(x, k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + \bar{k}x)} \mu_R(x, k) \]

in integral form using the appropriate Green function:

\[ \mu_R(x, k) = 1 + \frac{1}{\pi k} \ast \left( \frac{t_R(k)}{4\pi k} e^{-i(kx + \bar{k}x)} \mu_R(x, k) \right). \]

This equation of the Lippmann-Schwinger form can be solved numerically as explained below. Then

\[ \gamma_R^{1/2}(x) = \mu_R(x, 0). \]
The d-bar equation is defined on the whole k-plane.

However, numerical solution requires a finite computational domain.

To this end, we consider periodic functions. The plane is tiled by the square

\[ S = [-2R - \epsilon, 2R + \epsilon]^2. \]
We introduce an $S$-periodic version of the d-bar equation.

Green's functions $g : \mathbb{C} \to \mathbb{C}$ and $\tilde{g} : S \to \mathbb{C}$:

$$g(k) = \frac{1}{\pi k}, \quad \tilde{g}(k) = \frac{1}{\pi k} \bigg|_S$$

Denote the multiplier function as follows:

$$T_R(k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + \bar{k}x)}$$

and set $\tilde{T}_R(k) = T_R(k)|_S$. The d-bar equations:

$$\mu_R(x, k) = 1 + \int_{\mathbb{C}} g(k - \lambda) T_R(\lambda) \overline{\mu_R(x, \lambda)} d\lambda$$

$$\tilde{\mu}_R(x, k) = 1 + \int_{S} \tilde{g}(k - \lambda) \tilde{T}_R(\lambda) \overline{\tilde{\mu}_R(x, \lambda)} d\lambda$$
Now the d-bar equation can be essentially solved in the square $S$ instead of the $k$-plane.

Green’s functions $g : \mathbb{C} \rightarrow \mathbb{C}$ and $\tilde{g} : S \rightarrow \mathbb{C}$:

$$g(k) = \frac{1}{\pi k}, \quad \tilde{g}(k) = \frac{1}{\pi k}\big|_S$$

Denote the multiplier function as follows:

$$T_R(k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + \bar{k}\bar{x})},$$

and set $\tilde{T}_R(k) = T_R(k)|_S$. The d-bar equations:

$$\mu_R(x, k) = 1 + \int_{D(0,R)} g(k - \lambda)T_R(\lambda)\overline{\mu_R(x, \lambda)}d\lambda$$

$$\tilde{\mu}_R(x, k) = 1 + \int_{D(0,R)} \tilde{g}(k - \lambda)\tilde{T}_R(\lambda)\overline{\tilde{\mu}_R(x, \lambda)}d\lambda$$

It can be shown that

$$\mu_R(x, k) = \tilde{\mu}_R(x, k) \quad \text{for } |k| < R.$$
We form a grid suitable for FFT (fast Fourier transform).

Here 8x8 grid is shown; in practice we typically use 512x512 points.

Periodic functions are represented by their values at the grid points.
Periodic functions are represented by their values at the grid points, real and imaginary parts separately:

\[
\begin{bmatrix}
\text{Re } \varphi(k_1) \\
\text{Re } \varphi(k_2) \\
\vdots \\
\text{Re } \varphi(k_{64}) \\
\text{Im } \varphi(k_1) \\
\text{Im } \varphi(k_2) \\
\vdots \\
\text{Im } \varphi(k_{64})
\end{bmatrix} \in \mathbb{R}^{128}
\]
The solution of the periodic equation is reduced to iterative solution of a linear system

Write the periodic integral equation

\[ \tilde{\mu}_R(x, k) = 1 + \int_S \tilde{g}(k - \lambda) \tilde{T}_R(\lambda) \overline{\tilde{\mu}_R(x, \lambda)} d\lambda \]

in the form

\[ [I - \tilde{g} \ast (\tilde{T}_R \cdot -)] \tilde{\mu}_R = 1. \]  \hspace{1cm} (1)

We represent the solution \( \tilde{\mu}_R \) as a vector of point values as shown above.

Then we can use an iterative solver, such as GMRES, to solve (1) provided we have a computational routine for the real-linear operation

\[ \varphi \mapsto \varphi - \tilde{g} \ast (\tilde{T}_R \varphi). \]

See [Vainikko 2000] and [Knudsen, Mueller and S 2004].
Periodic convolution is conveniently implemented using the FFT

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\xrightarrow{\text{FFT}}
\begin{bmatrix}
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi}
\end{bmatrix}
\xrightarrow{\text{FFT}}
\begin{bmatrix}
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\end{bmatrix}
\xrightarrow{\text{Multiplication}}
\begin{bmatrix}
\varphi - \tilde{g} \ast (\tilde{T}_R\tilde{\varphi})
\end{bmatrix}
\xrightarrow{\text{IFFT}}
\begin{bmatrix}
\varphi - \tilde{g} \ast (\tilde{T}_R\tilde{\varphi})
\end{bmatrix}
Here we see the reconstructions corresponding to various levels of measurement noise.

\[ \| \mathcal{E} \|_Y \approx 10^{-6} \]

\[ R = 6.7 \]

\[ 12\% \]

\[ \| \mathcal{E} \|_Y \approx 10^{-5} \]

\[ 5.9 \]

\[ 12\% \]

\[ \| \mathcal{E} \|_Y \approx 10^{-4} \]

\[ 4.3 \]

\[ 14\% \]

\[ \| \mathcal{E} \|_Y \approx 10^{-3} \]

\[ 3.5 \]

\[ 19\% \]

\[ \| \mathcal{E} \|_Y \approx 10^{-2} \]

\[ 2.5 \]

\[ 52\% \]
The numerical results actually improve the exponential behaviour predicted by theory.

![Graph showing the relationship between noise level and truncation radius. The graph compares observed and theoretical radii.](image)
Thank you!

Preprints available at

www.siltanen-research.net