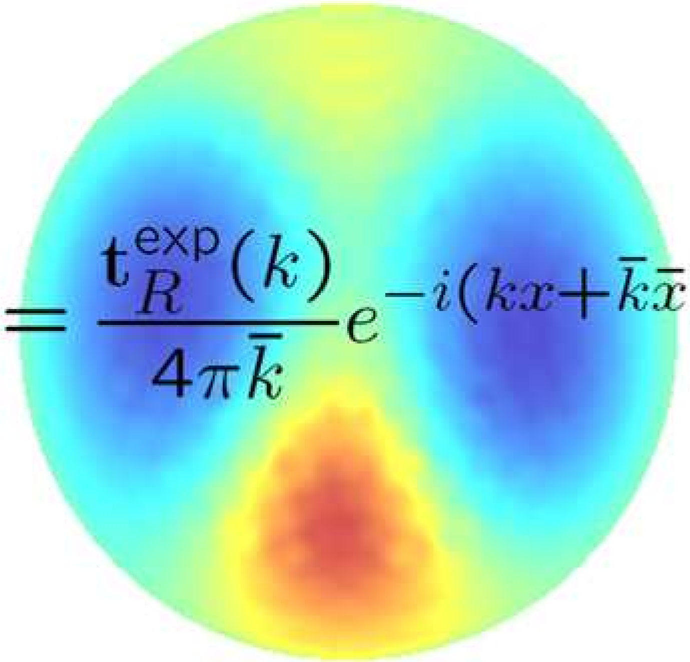
Regularized D-bar method for electrical impedance tomography

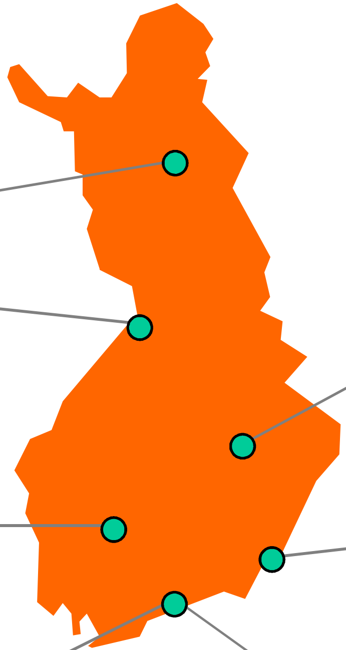


$$\frac{\partial}{\partial \bar{k}}\mu_R(x,k) = \frac{\mathbf{t}_R^{\mathrm{exp}}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu_R(x,k)}$$

Samuli.Siltanen@iki.fi
Tampere University of Technology
Finland

Control and Inverse Problems in PDE
C.I.R.M., February 19, 2009

Finnish Centre of Excellence in Inverse Problems Research



OULUN YLIOPISTO / UNIVERSITY of OULU

UNIVERSITY OF KUOPIO

TAMPERE UNIVERSITY OF TECHNOLOGY

LAPPEENRANTA UNIVERSITY OF TECHNOLOGY

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY

UNIVERSITY OF HELSINKI
Rolf Nevanlinna Institute
Research institute of mathematics, computer science and statistics

http://math.tkk.fi/inverse-coe/

This is a joint work with

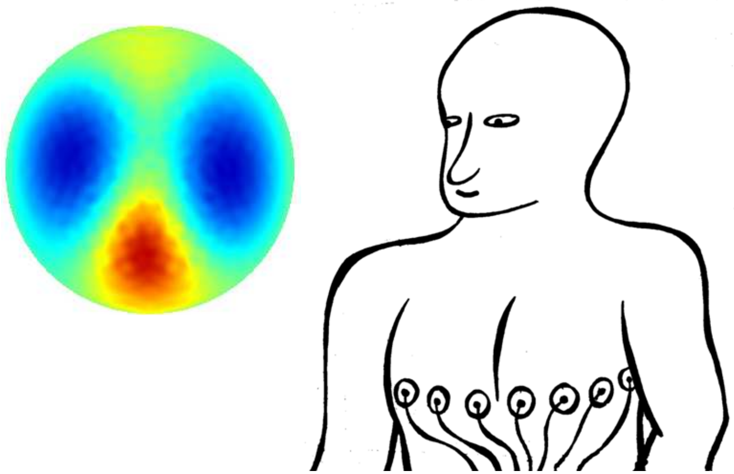
David Isaacson
Rensselaer Polytechnic Institute, USA

Kim Knudsen
Technical University of Denmark

Matti Lassas
University of Helsinki, Finland

Jennifer Mueller
Colorado State University, USA

Jon Newell
Rensselaer Polytechnic Institute, USA



1. Electrical impedance tomography

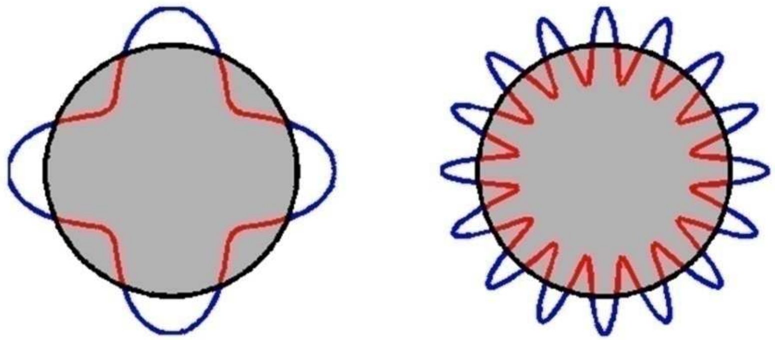
$$\frac{\partial}{\partial \bar{k}} \mu(x,k) = \frac{\mathbf{t}(k)}{4\pi \bar{k}} e^{-i(kx+\bar{k}\bar{x})} \overline{\mu(x,k)}.$$

2. Theory of the D-bar method

3. Regularization results

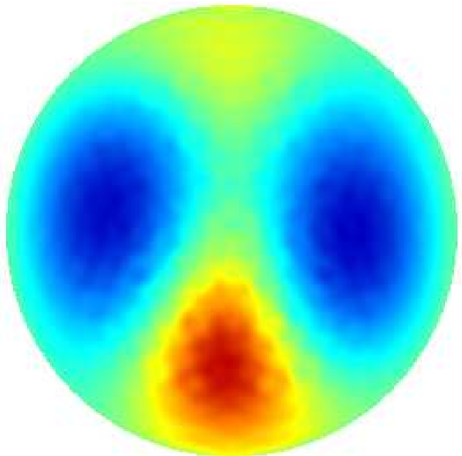
$$\sup_{\Lambda_\gamma^\varepsilon} \left\{ \|\Gamma_{\alpha(\varepsilon)}\Lambda_\gamma^\varepsilon - \gamma\|_{L^\infty(\Omega)} : \|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \le \varepsilon \right\}$$

4. Practical computation

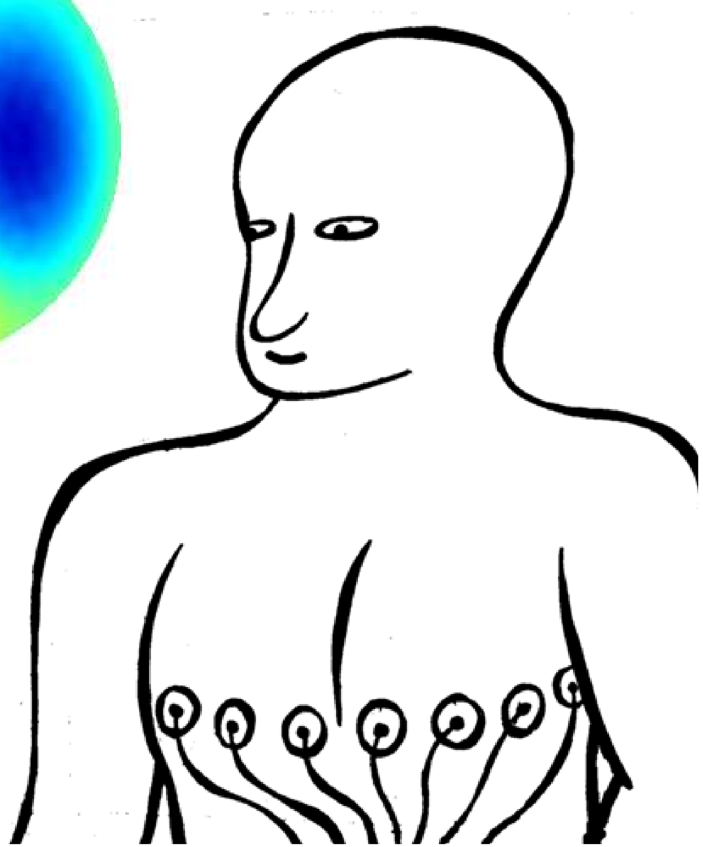


Electrical impedance tomography (EIT) is an emerging medical imaging method

Feed electric currents through electrodes, measure voltages



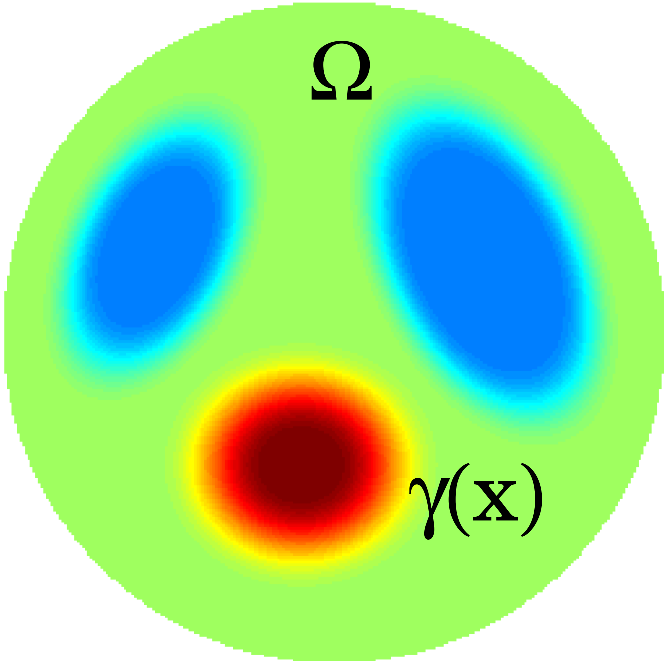
Reconstruct the image of electric conductivity in a two-dimensional slice



Applications include:

monitoring heart and lungs of unconscious patients, detecting pulmonary edema, enhancing ECG and EEG

The inverse conductivity problem of Calderón is the mathematical model of EIT



$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

$$\begin{aligned} \nabla \cdot \gamma \nabla u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \partial\Omega. \end{aligned}$$

We assume that $0 < c \leq \gamma(x) \leq C$ for all $x \in \Omega$.

Problem: given the Dirichlet-to-Neumann map, how to reconstruct the conductivity?

The reconstruction problem is nonlinear and ill-posed.

EIT reconstruction algorithms can be divided roughly into the following classes:

Linearization

Iterative **regularization methods**

Statistical inversion

Resistor network method

Convexification method

Layer stripping

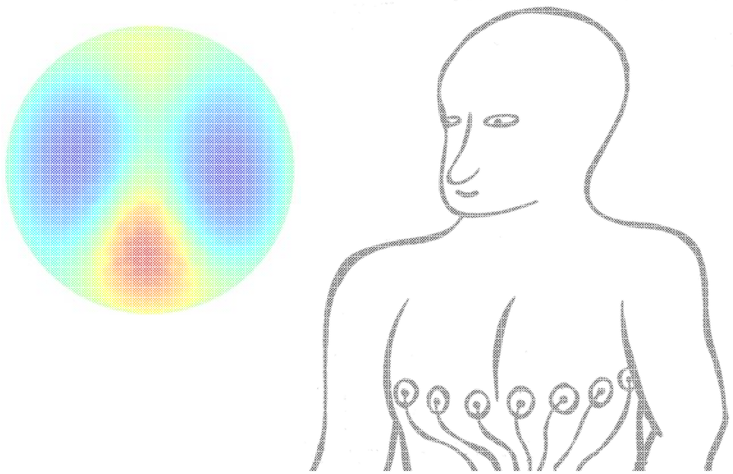
The inverse scattering approach, or the **d-bar method**

This is a brief history of regularization methods for electrical impedance tomography

1991 **Hua, Woo, Webster** and **Tompkins** (Tikhonov and smoothness)
1992 **Goble, Cheney** and **Isaacson** (truncated Newton method)
1994 **Dobson** and **Santosa** (Total variation)
1999 **Vauhkonen** et al. (Tikhonov in 3D)
2001 **Kindermann** and **Neubauer** (surface representation)
2001 **Rondi** and **Santosa** (Mumford-Shah-functional)
2003 **Lukaschewitsch, Maass** and **Pidcock** (Tikhonov regularization)
2005 **Chung, Chan** and **Tai** (level set, total variation)
2005 **Eppler** and **Harbrecht** (Newton regularization)
2006 **Lechleiter** and **Rieder** (numerical Newton regularization)
2008 **Rondi** (theory for regularized recovery of discontinuities)
2008 **Lechleiter** and **Rieder** (local convergence of Newton regularization)

This is a brief history of the development of the d-bar method in dimension two

Theory	Practice
1980 Calderón	2008 Bikowski and Mueller
1987 Sylvester and Uhlmann 1987 R G Novikov 1988 Nachman	
1996 Nachman 1997 Liu	2000 S, Mueller and Isaacson 2003 Mueller and S 2004 Isaacson, Mueller, Newell and S 2006 Isaacson, Mueller, Newell and S 2007 Murphy 2008 Knudsen, Lassas, Mueller and S
1997 Brown and Uhlmann 2001 Barceló, Barceló and Ruiz	2001 Knudsen and Tamasan 2003 Knudsen
2000 Francini	
2003 Astala and Päivärinta 2007 Barceló, Faraco and Ruiz 2008 Clop, Faraco and Ruiz	2008 Astala, Mueller, Päivärinta and S



1. Electrical impedance tomography

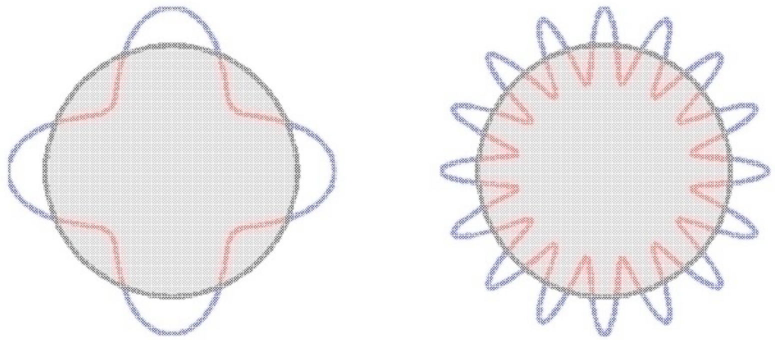
$$\frac{\partial}{\partial \bar{k}}\mu(x,k) = \frac{\mathbf{t}(k)}{4\pi \bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

2. Theory of the D-bar method

3. Regularization results

$$\sup_{\Lambda_\gamma^\varepsilon} \left\{ \|\Gamma_{\alpha(\varepsilon)}\Lambda_\gamma^\varepsilon - \gamma\|_{L^\infty(\Omega)} \,:\, \|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \leq \varepsilon \right\}$$

4. Practical computation



Nachman's 1996 proof consists of two steps:

$$\Lambda_\gamma \longrightarrow \mathbf{t} \longrightarrow \gamma$$

The intermediate object **t** is a complex-valued function called *scattering transform* and defined as follows:

$$\mathbf{t}(k) := \int_{\mathbb{R}^2} e^{i\bar{k}\bar{x}} q(x)\psi(x,k)dx$$

$$q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$$

$$(-\Delta + q)\psi(\cdot, k) = 0$$

$$\psi(x,k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)}$$

Step 1: from DN map to scattering transform

Solve traces of ψ from the boundary integral equation

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot,k),$$

where the single-layer operator has
Faddeev Green's function as kernel.

Compute the scattering transform as

$$\mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma - \Lambda_1)\psi(x,k)\,d\sigma(x).$$

Let us take a closer look at Faddeev Green's function and the related single layer operator

The operator

$$(S_k\phi)(x) := \int_{\partial\Omega} G_k(x-y)\phi(y)d\sigma(y)$$

involves the Faddeev Green's function G_k for the Laplacian:

$$-\Delta G_k(x) = \delta_0(x).$$

The function G_k can be written in the form

$$G_k(x) := e^{ikx} g_k(x),$$

where

$$g_k(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} d\xi.$$

Step 2: from scattering transform to γ

Define $\mu(x,k) = e^{-ikx}\psi(x,k)$

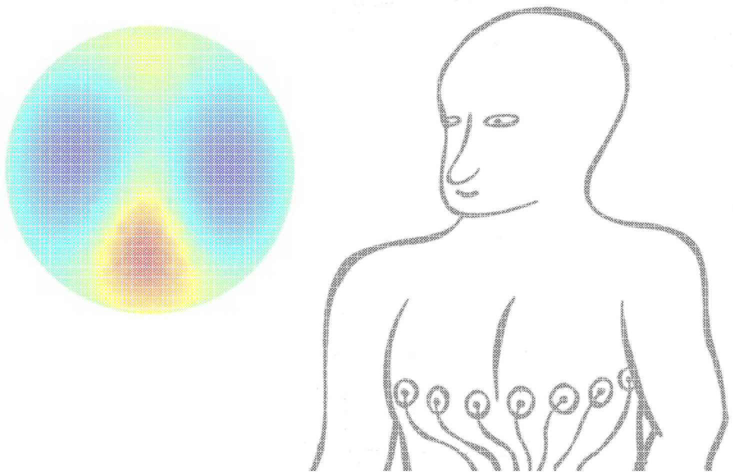
Then the following d-bar equation holds:

$$\frac{\partial}{\partial \bar{k}}\mu(x,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}}e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

Here $\frac{\partial}{\partial \bar{k}} = \frac{1}{2}(\frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_2})$.

The d-bar equation has a unique solution for all x.
The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k\to 0} \mu(x,k).$$



1. Electrical impedance tomography

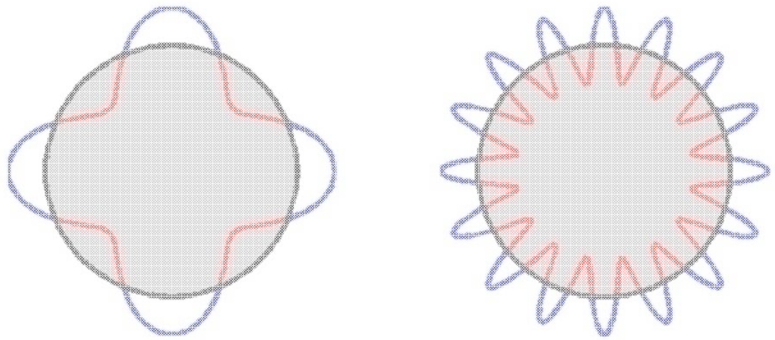
$$\frac{\partial}{\partial \bar{k}}\mu(x,k) = \frac{\mathrm{t}(k)}{4\pi \bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

2. Theory of the D-bar method

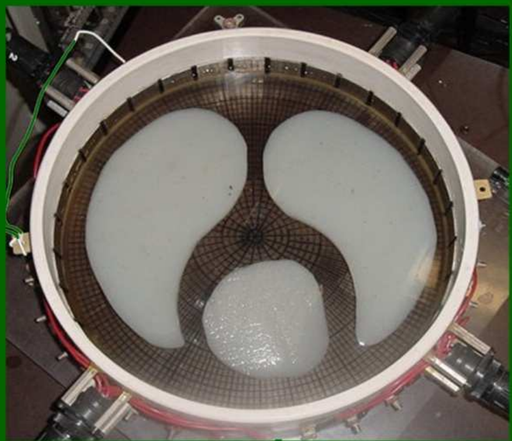
3. Regularization results

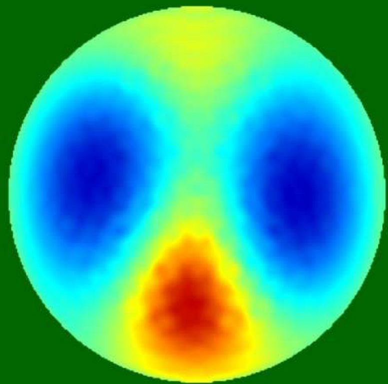
$$\sup_{\Lambda_\gamma^\varepsilon}\left\{\|\Gamma_{\alpha(\varepsilon)}\Lambda_\gamma^\varepsilon-\gamma\|_{L^\infty(\Omega)} \,:\, \|\Lambda_\gamma^\varepsilon-\Lambda_\gamma\|_Y\le\varepsilon\right\}$$

4. Practical computation



We have achieved excellent EIT reconstructions from practical data (phantom and *in vivo* human) using Born approximation to linearize Step 1.





However, that approach does not allow regularization analysis.

We work within the following assumptions:

Let $\Omega \subset \mathbb{R}^2$ be the open unit disc.

Define the forward map F between the spaces

$F : \mathcal{D}(F) \subset L^\infty(\Omega) \to Y$.

Domain $\mathcal{D}(F)$ is defined as follows.

Let $M > 0$ and $0 < \rho < 1$. The set $\mathcal{D}(F)$ contains functions $\gamma : \Omega \to \mathbb{R}$ satisfying

(a) $\|\gamma\|_{C^2(\overline{\Omega})} \leq M$,

(b) $\gamma(x) \geq M^{-1}$ for all $x \in \Omega$,

(c) $\gamma(x) \equiv 1$ for $\rho < |x| < 1$.

Space Y of data is defined as follows.

Y consists of bounded linear operators

$$\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

satisfying $\int_{\partial\Omega} \Lambda(f) d\sigma = 0$ and $\Lambda(1) = 0$.

Let us emphasize one of the strengths of our new results

Conditional stability results have the form

$$\|\gamma_1 - \gamma_2\|_Z \leq f(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_Y),$$

where γ_1, γ_2 belong to some function space Z and f is a continuous function with $f(0) = 0$.

The above estimate is not practically relevant. The noisy measurement $\Lambda_\gamma^\varepsilon$ is in general not the DN map of some conductivity.

In contrast, we prove regularization properties for the D-bar method under the practically feasible assumption $\|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \leq \varepsilon$.

Let us define nonlinear regularization strategy
(following Engl, Hanke & Neubauer and Kirsch)

Recall direct problem: $\gamma \in X$ maps to $\Lambda_\gamma \in Y$.

A family of continuous mappings $\Gamma_\alpha : Y \to X$ with $0 < \alpha < \infty$ is a regularization strategy if

$$\lim_{\alpha \to 0} \|\Gamma_\alpha \Lambda_\gamma - \gamma\|_{L^\infty(\Omega)} = 0$$

for each fixed $\gamma \in X$. A regularization strategy with a choice $\alpha = \alpha(\varepsilon)$ is called admissible if

$$\alpha(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

and for any fixed $\gamma \in X$ the following holds:

$$\sup_{\Lambda_\gamma^\varepsilon} \left\{ \|\Gamma_{\alpha(\varepsilon)} \Lambda_\gamma^\varepsilon - \gamma\|_{L^\infty(\Omega)} \;:\; \|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \le \varepsilon \right\} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

This is our regularized D-bar method for EIT

Given the noise level $\varepsilon > 0$, solve the equation

$$\psi^{\varepsilon}(\,\cdot\,,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma^{\varepsilon} - \Lambda_1)\psi^{\varepsilon}(\,\cdot\,,k)|_{\partial\Omega}$$

for $|k| < R(\varepsilon) := -\frac{1}{10}\log(\varepsilon)$.

Introduce nonlinear low-pass filtering

$$\mathbf{t}_R^{\varepsilon}(k) := \begin{cases} \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\gamma^{\varepsilon} - \Lambda_1)\psi^{\varepsilon}(\cdot,k)d\sigma & \text{for } |k| < R(\varepsilon), \\ \text{zero for } |k| \geq R(\varepsilon). \end{cases}$$

For each $x \in \Omega$, solve the integral equation

$$\mu_R(x,k) = 1 + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\mathbf{t}_R^{\varepsilon}(s)}{(k-s)\bar{s}} e_{-x}(s)\overline{\mu_R(x,s)}ds_1 ds_2,$$

and define $\alpha(\varepsilon) = \frac{1}{R(\varepsilon)}$ and $(\Gamma_\alpha \Lambda_\gamma^{\varepsilon})(x) := (\mu_R(x,0))^2$.

Theorem [Knudsen, Lassas, Mueller & S 2008]

The family Γ_α is well-defined for small $\alpha > 0$. It is an admissible regularization strategy with

$$\alpha(\varepsilon) = (-\frac{1}{10}\log(\varepsilon))^{-1}.$$

Furthermore, we have the explicit estimate

$$\sup_{\Lambda_\gamma^\varepsilon}\left\{\|\Gamma_{\alpha(\varepsilon)}\Lambda_\gamma^\varepsilon - \gamma\|_{L^\infty(\Omega)} : \|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \leq \varepsilon\right\}$$

$$\leq \quad C(-\log\varepsilon)^{-1/14}$$

$$\to \quad 0 \text{ as } \varepsilon \to 0.$$

Proof of the main theorem is divided into several lemmata. First a D-bar estimate:

Lemma 1. Let $4/3 < r_0 < 2$ and suppose that $\phi_1, \phi_2 \in L^r(\mathbb{R}^2)$ for all $r \geq r_0$. Let μ_1, μ_2, be the solutions of

$$\mu_j(x,k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\phi_j(k')}{(k-k')} \overline{\mu_j(x,k')} dk'_1 dk'_2,$$

$j = 1, 2$. Then for fixed $x \in \overline{\Omega}$ we have

$$\|\mu_1(x,\cdot) - \mu_2(x,\cdot)\|_{C^\alpha(\mathbb{R}^2)} \leq C\|\phi_1 - \phi_2\|_{L^{r_0} \cap L^{r_0'}(\mathbb{R}^2)},$$

where $\alpha < 2/r_0 - 1$ and $1/r_0' = 1 - 1/r_0$.

Proof. Combination of well-known results.

These results follow from careful analysis of Faddeev's Green function

Lemma 2. Let $\phi_0 \in H^{-1/2}(\partial\Omega)$ with $\int \phi_0 = 0$. Then we have the estimate

$$\|S_k \phi_0\|_{H^{1/2}(\partial\Omega)} \leq C e^{2|k|}(1 + |k|)\|\phi_0\|_{H^{-1/2}(\partial\Omega)}.$$

Lemma 3. For $k \in \mathbb{C}$ we have the estimate

$$\left\|[I + S_k(\Lambda_\gamma - \Lambda_1)]^{-1}\right\|_{L(H^s(\partial\Omega))} \leq C_2 e^{2|k|}(1 + |k|),$$

where C_2 depends only on M and ρ.

Combining previous results, a perturbation argument, and delicate L^p analysis shows

Lemma 4. There exists $\varepsilon_0 > 0$, depending only on M and ρ, such that equation

$$\psi^{\varepsilon}(\cdot,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma^{\varepsilon} - \Lambda_1)\psi^{\varepsilon}(\cdot,k)|_{\partial\Omega}$$

is solvable in $H^{1/2}(\partial\Omega)$ for all $0 < \varepsilon \leq \varepsilon_0$ and $|k| < R$ with

$$R = R(\varepsilon) = -\frac{1}{10}\log\varepsilon.$$

Furthermore, for $p > 1$ we have the estimate

$$\left\|\frac{\mathbf{t}(k) - \mathbf{t}_R^{\varepsilon}(k)}{\bar{k}}\right\|_{L^p(|k|\leq R)} \leq C\varepsilon^{1/10}\left(-\frac{1}{10}\log\varepsilon\right)^{2/p},$$

where C is independent of p and R and ε.

Sketch of proof of main theorem

(i) $\lim_{\alpha\to 0} \|\Gamma_\alpha \Lambda_\gamma - \gamma\|_{L^\infty(\Omega)} = 0$ for $\gamma \in X$.

(ii) $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$,

(iii) $\sup_{\Lambda_\gamma^\varepsilon} \left\{ \|\Gamma_{\alpha(\varepsilon)} \Lambda_\gamma^\varepsilon - \gamma\|_{L^\infty(\Omega)} \; : \; \|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \leq \varepsilon \right\}$
tends to zero as $\varepsilon \to 0$.

Claim (i) follows from [Nachman 1996] (with delicate choices of L^p spaces) and Lemma 1.
Claim (ii) is OK: $\alpha(\varepsilon) = \frac{1}{R(\varepsilon)} = -10(\log \varepsilon)^{-1}$.

Sketch of proof of main theorem

To prove that

$$\sup_{\Lambda_\gamma^\varepsilon} \left\{ \|\Gamma_{\alpha(\varepsilon)}\Lambda_\gamma^\varepsilon - \gamma\|_{L^\infty(\Omega)} \ : \ \|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \leq \varepsilon \right\}$$

tends to zero as $\varepsilon \to 0$ we combine [Nachman 1996] with Lemmata 1 and 4 to estimate

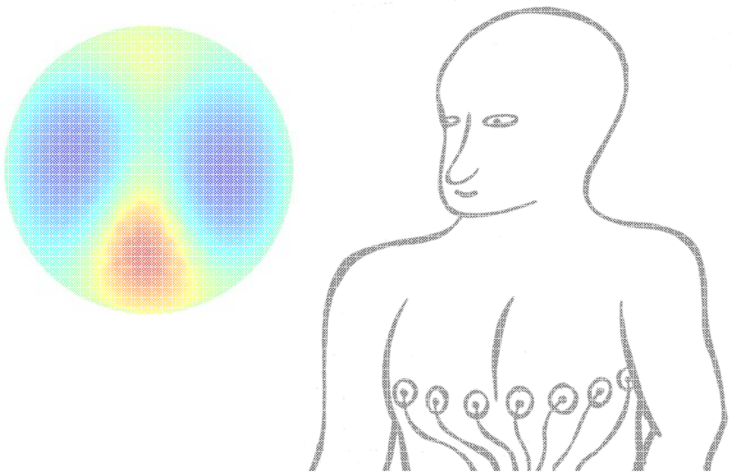
$$\begin{aligned}
&\|\mu(x,\cdot) - \mu_R(x,\cdot)\|_{C^\alpha(\mathbb{R}^2)} \\
&\leq C \left\| \frac{\mathbf{t}(k) - \mathbf{t}_R^\varepsilon(k)}{\bar{k}} \right\|_{L^p \cap L^{p'}(\mathbb{R}^2)} \\
&\leq C \left\| \frac{\mathbf{t}(k) - \mathbf{t}_R^\varepsilon(k)}{\bar{k}} \right\|_{L^p \cap L^{p'}(|k|<R)} + C \left\| \frac{\mathbf{t}(k)}{\bar{k}} \right\|_{L^p(|k|>R)} \\
\leq \quad & C(-\frac{1}{10}\log\varepsilon)^{10/7}\,\varepsilon^{1/10} + CR(\varepsilon)^{-\frac{1}{7}} + CR(\varepsilon)^{-\frac{1}{14}} \\
= \quad & C(-\log\varepsilon)^{10/7}\,\varepsilon^{1/10} + C(-\log\varepsilon)^{-\frac{1}{7}} + C(-\log\varepsilon)^{-\frac{1}{14}}.
\end{aligned}$$

One more thing: the regularization strategy is not yet defined on all of data space Y

The range $F(\mathcal{D}(F)) \subset Y$ is not known, and its structure may be complicated.
(This is related to the open and notoriously difficult *characterization problem*.)

The previous results show the claim only for operators ε_0-close to the range $F(\mathcal{D}(F))$.

The problem can be overcome using spectral theoretical arguments.



1. Electrical impedance tomography

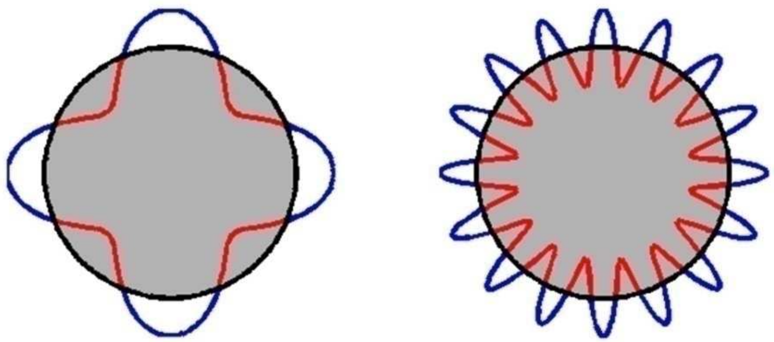
$$\frac{\partial}{\partial \bar{k}}\mu(x,k) = \frac{\mathbf{t}(k)}{4\pi\bar{k}} e^{-i(kx+\bar{k}\bar{x})}\overline{\mu(x,k)}.$$

2. Theory of the D-bar method

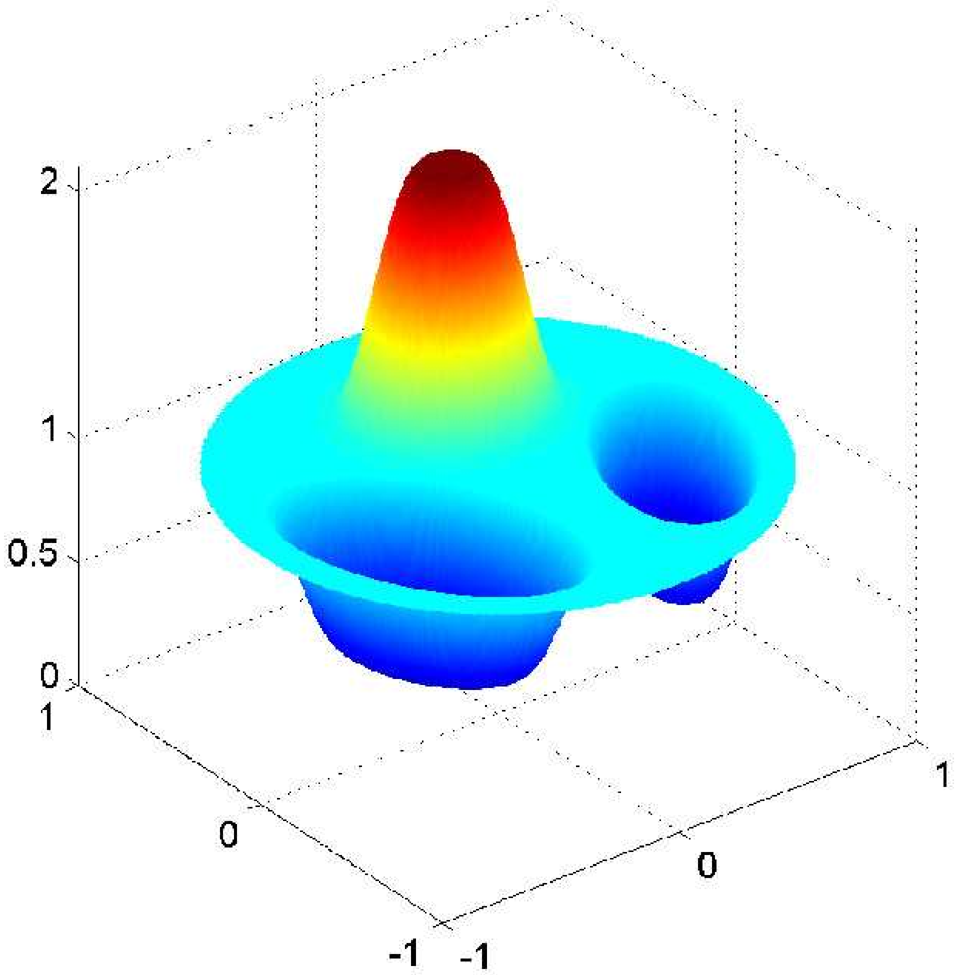
3. Regularization results

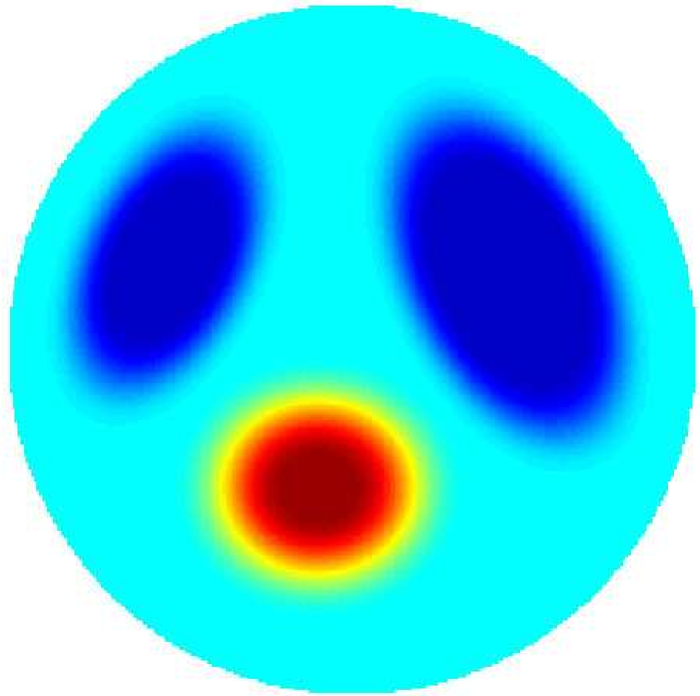
$$\sup_{\Lambda_\gamma^\varepsilon}\left\{\|\Gamma_{\alpha(\varepsilon)}\Lambda_\gamma^\varepsilon - \gamma\|_{L^\infty(\Omega)} \,:\, \|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \leq \varepsilon\right\}$$

4. Practical computation

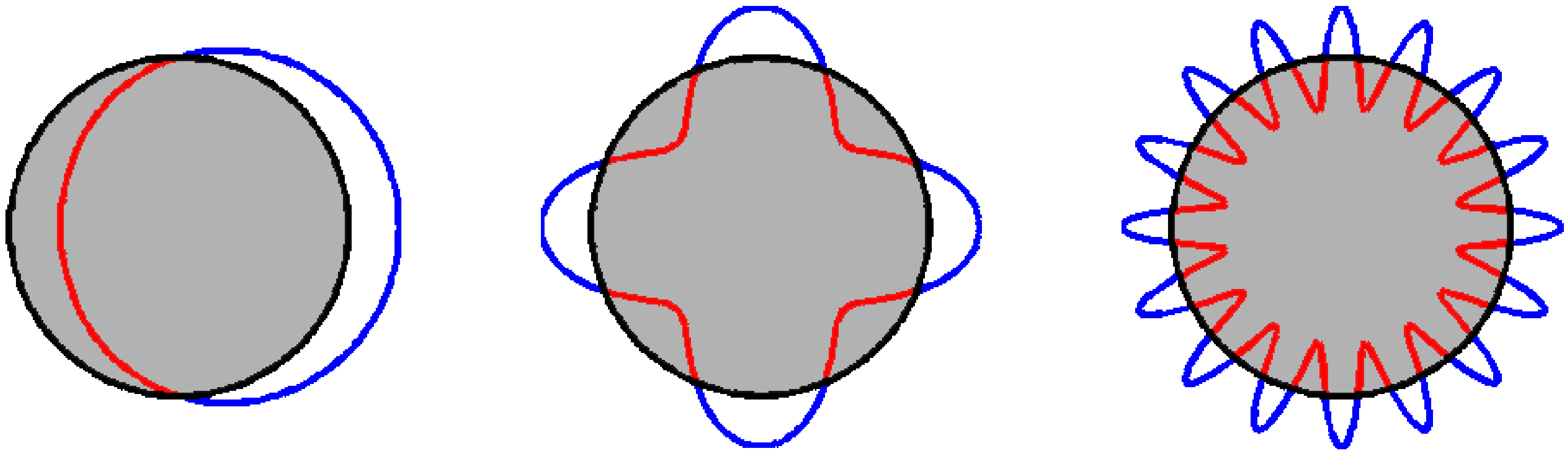


We construct a simulated human chest phantom for numerical testing





We use the truncated Fourier basis $\{e^{in\theta}\}_{n=-N}^{N}$ to express functions defined on the unit circle.



Integral operators $A : H^s(\partial\Omega) \to H^r(\partial\Omega)$ are represented as finite matrices $[\langle Ae^{in\theta}, e^{im\theta}\rangle]$.

- Λ_1 we know analytically,
- Λ_γ we compute using Finite Element Method,
- S_k we evaluate by numerical integration.

This is our practical two-step regularized D-bar method for EIT

1. We solve for $|k| < R$ the matrix version of

$$\psi^{\varepsilon}(\,\cdot\,,k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma^{\varepsilon} - \Lambda_1)\psi^{\varepsilon}(\,\cdot\,,k)|_{\partial\Omega}$$

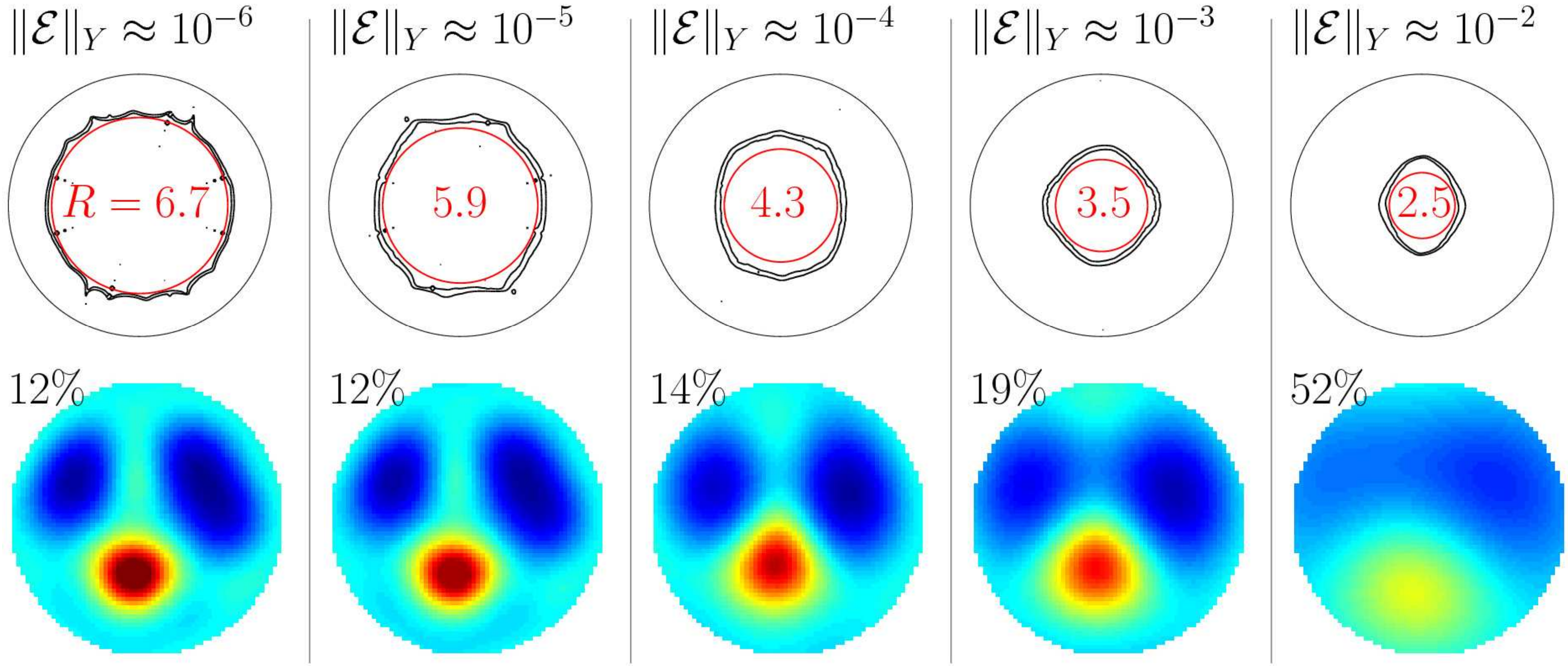
with R as large as numerically stable.

2. The integral equation

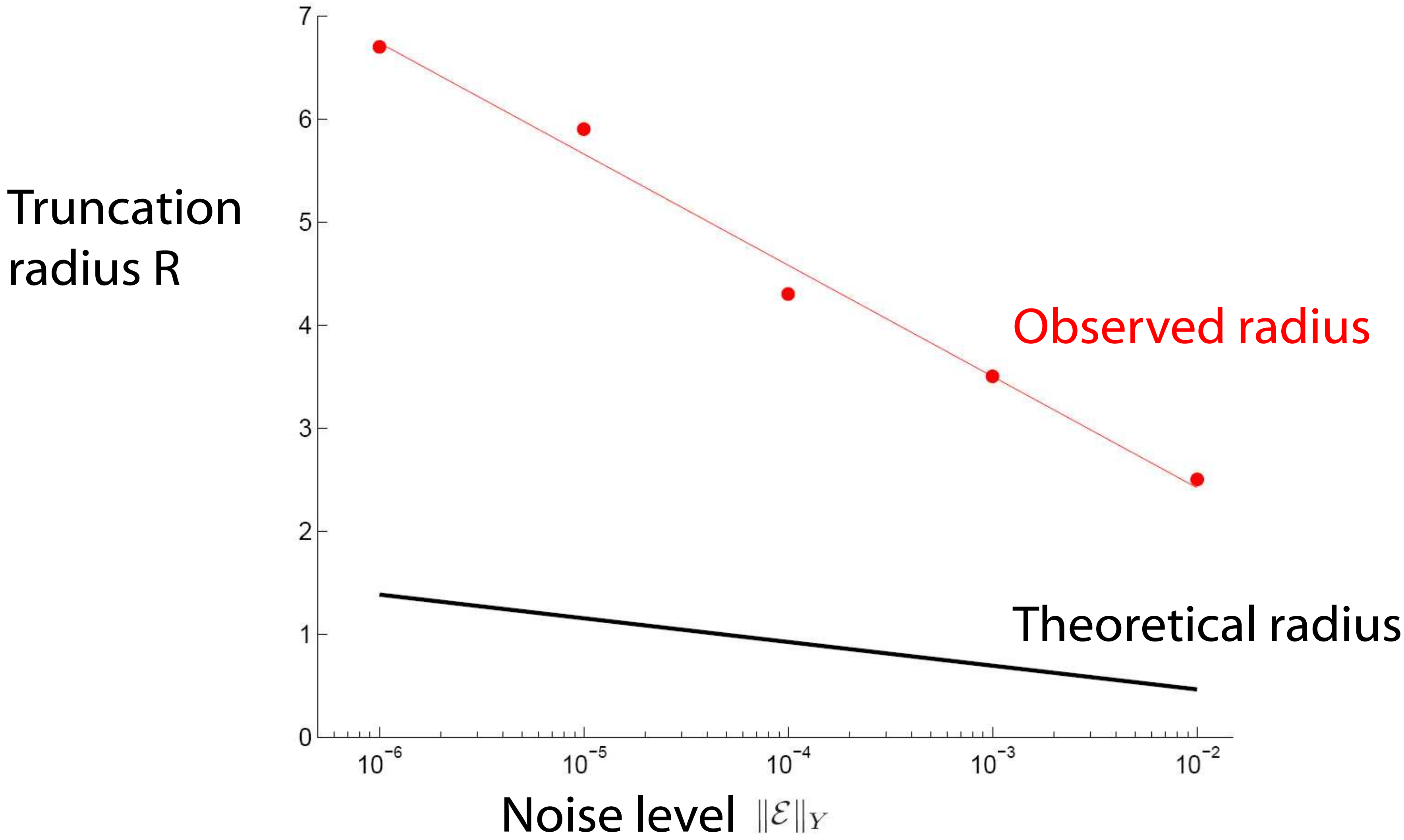
$$\mu_R(x,k) = 1 + \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{\mathbf{t}_R^{\varepsilon}(s)}{(k-s)\bar{s}} e_{-x}(s)\overline{\mu_R(x,s)}\,ds_1 ds_2$$

can be solved by our established D-bar solver. The reconstructed conductivity is $\mu_R(x,0)$.

Here we see the reconstructions corresponding to various levels of measurement noise



The numerical results actually improve the exponential behaviour predicted by theory



Thank you!

Preprints available at
www.siltanen-research.net

Knudsen K, Lassas M, Mueller J L and Siltanen S
Reconstructions of Piecewise Constant Conductivities by the D-bar Method for Electrical Impedance Tomography
To appear in proceedings of Applied Inverse Problems 2007, Vancouver.

Knudsen K, Lassas M, Mueller J L and Siltanen S 2007
D-bar method for electrical impedance tomography with discontinuous conductivities
SIAM Journal of Applied Mathematics **67**(3), pp. 893-913

Lassas M, Mueller J L and Siltanen S 2007
Mapping properties of the nonlinear Fourier transform in dimension two
Communications in Partial Differential Equations **32**(4), pp. 591-610

Isaacson D, Mueller J L, Newell J and Siltanen S 2006
Imaging Cardiac Activity by the D-bar Method for Electrical Impedance Tomography
Physiological Measurement **27**, pp. S43-S50

Isaacson D, Mueller J L, Newell J and Siltanen S 2004
Reconstructions of chest phantoms by the d-bar method for electrical impedance tomography
IEEE Transactions on Medical Imaging **23**(7), pp. 821-828

Knudsen K, Mueller J L and Siltanen S 2004
Numerical solution method for the dbar-equation in the plane
Journal of Computational Physics **198**(2), pp. 500-517

Mueller J L and Siltanen S 2003
Direct reconstructions of conductivities from boundary measurements
SIAM Journal of Scientific Computation **24**(4), pp. 1232-1266

Siltanen S, Mueller J L and Isaacson D 2000
An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem
Inverse Problems 16, pp. 681-699; *Erratum* Inverse problems **17** , pp. 1561-1563