New computational techniques for static and dynamic sparse-data X-ray tomography

Samuli Siltanen

Department of Mathematics and Statistics
University of Helsinki, Finland
samuli.siltanen@helsinki.fi
http://www.siltanen-research.net

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http://wiki.helsinki.fi/display/inverse/Home
Outline

Motivation: dental X-ray imaging

Part I: New parameter choice rules for TV regularization
   Total variation regularized tomography
   The S-curve method
   A multiresolution choice rule

Part II: Dynamic sparse-data tomography
   Modified level set method
   Computational examples with simulated data
   Computational example with measured data
Application: dental implant planning, where a missing tooth is replaced with an implant
Nowadays, a digital panoramic imaging device is standard equipment at dental clinics.

A panoramic dental image offers a general overview showing all teeth and other dento-maxillofacial structures simultaneously.

Panoramic images are not suitable for dental implant planning because of unavoidable geometric distortion.
We reprogram the panoramic X-ray device so that it collects projection data by scanning.
We reprogram the panoramic X-ray device so that it collects projection data by scanning.

Number of projection images: 11

Angle of view: 40 degrees

Image size: $1000 \times 1000$ pixels

The detector of a panoramic device is very narrow, so images are formed using a scanning movement analogously to a xerox machine.
Here are example images of an actual patient: navigation image (left) and desired slice (right).


The radiation dose of the VT device is lowest among 3D dental imaging modalities

<table>
<thead>
<tr>
<th>Modality</th>
<th>$\mu$Sv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head CT</td>
<td>2100</td>
</tr>
<tr>
<td>CB Mercuray</td>
<td>558</td>
</tr>
<tr>
<td>i-Cat</td>
<td>193</td>
</tr>
<tr>
<td>NewTom 3G</td>
<td>59</td>
</tr>
<tr>
<td>VT device</td>
<td>13</td>
</tr>
</tbody>
</table>

Ludlow, Davies-Ludlow, Brooks & Howerton 2006
The VT device was developed in 2001–2012 by

Nuutti Hyvönen
Seppo Järvenpää
Jari Kaipio
Martti Kalke
Petri Koistinen
Ville Kolehmainen
Matti Lassas
Jan Moberg
Kati Niinimäki
Juha Pirttilä
Maaria Rantala
Eero Saksman
Henri Setälä
Erkki Somersalo
Antti Vanne
Simopekka Vänskä
Richard L. Webber
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Keijo Hämäläinen, University of Helsinki, Finland

Aki Kallonen, University of Helsinki, Finland

Ville Kolehmainen, University of Eastern Finland

Matti Lassas, University of Helsinki, Finland

Esa Niemi, University of Helsinki, Finland

Kati Niinimäki, University of Eastern Finland
We write the reconstruction problem in matrix form

\[
\begin{bmatrix}
  f_1 & f_4 & f_7 \\
  f_2 & f_5 & f_8 \\
  f_3 & f_6 & f_9 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  m_1 \\
  m_2 \\
  m_3 \\
\end{bmatrix}
\]

\[
f = \begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
  f_4 \\
  f_5 \\
  f_6 \\
  f_7 \\
  f_8 \\
  f_9 \\
\end{bmatrix}, \quad m = \begin{bmatrix}
  m_1 \\
  m_2 \\
  m_3 \\
  m_4 \\
  m_5 \\
  m_6 \\
\end{bmatrix}
\]

Measurement model: \( m = Af + \varepsilon \)
This is the matrix equation related to the above measurement

\[
\begin{bmatrix}
    m_1 \\
    m_2 \\
    m_3 \\
    m_4 \\
    m_5 \\
    m_6
\end{bmatrix} =
\begin{bmatrix}
    0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\
    \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} \\
    0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 \\
    1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
    0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
    0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3 \\
    f_4 \\
    f_5 \\
    f_6 \\
    f_7 \\
    f_8 \\
    f_9
\end{bmatrix} +
\begin{bmatrix}
    \varepsilon_1 \\
    \varepsilon_2 \\
    \varepsilon_3 \\
    \varepsilon_4 \\
    \varepsilon_5 \\
    \varepsilon_6
\end{bmatrix}
\]
This is the matrix equation related to the above measurement

\[
\begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5 \\
m_6 
\end{bmatrix} = \begin{bmatrix}
0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\
\sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} \\
0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
f_7 \\
f_8 \\
f_9 
\end{bmatrix} + \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix}
\]

Original image by G. Hounsfield from the 1970’s
Brief history of TV-regularized tomography

Rudin, Osher & Fatemi 1992: denoise images by minimizing
\[ \Phi(f) = \|Af - m\|_2^2 + \alpha\|\nabla f\|_1, \] where \( A = I \).

Later, \( A \) was taken to be the tomography matrix:

1998 Delaney & Bresler (simulated data)
2003 Kolehmainen, S, Järvenpää, Kaipio, Koistinen, Lassas, Pirttilä & Somersalo (measured data)
2006 Kolehmainen, Vanne, S, Järvenpää, Kaipio, Lassas & Kalke
2006 Sidky, Kao & Pan
2008 Liao & Sapiro
2008 Sidky & Pan
2008 Herman & Davidi
2009 Tang, Nett & Chen
2009 Duan, Zhang, Xing, Chen & Cheng
2010 Bian, Han, Sidky, Cao, Lu, Zhou & Pan
2011 Jensen, Jørgensen, Hansen & Jensen
2011 Tian, Jia, Yuan, Pan & Jiang
2012–dozens of articles indicated by Google Scholar
This is Professor Keijo Hämäläinen’s X-ray lab
We collected X-ray projection data of a walnut from 1200 directions.

The data was collected by Keijo Hämäläinen and Aki Kallonen at University of Helsinki.
This is the reconstruction using all 1200 projections and filtered back-projection
When only few projection angles are available, TV regularization performs better than FBP.

These images were computed by Kati Niinimäki.
How to choose the regularization parameter in the total variation (TV) functional?

These are the currently available methods in the literature:

1. The classical **L-curve method**.

2. **Discrepancy principle**, introduced for TV regularization by Wen and Chan in 2012 in the context of image restoration.

3. **S-curve method**, introduced by Kolehmainen, Lassas, Niinimäki & S in 2012 can be extended to total variation regularization.

4. Quasi-optimality principle and Hanke-Raus rules were described for TV by Kindermann, Mutimbu and Resmerita in 2013.

5. **Multiresolution choice rule**, introduced now. No *a priori* information needed.

Practical experience suggests that no single choice rule works perfectly for all applications. Therefore it is good to have a collection of rules.
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We took photographs of walnuts cut in half

These photos are used for estimating the expected total variation norm (number of nonzero gradient elements) in a two-dimensional tomographic reconstruction. We compute the discrete TV norm for these images downsampled to $256 \times 256$ pixel resolution.

[Hämäläinen, Kallonen, Kolehmainen, Lassas, Niinimäki & S 2013]

Special thanks to Esa Niemi for his careful job in sawing the walnuts.
The S-curve method applied to the real data case with no added noise. Vertical axis: TV norm

\[ S\text{-curve} \]

\[ \alpha = 3 \]

\[ 256 \times 256 \]

[Niinimäki, Lassas, Hämäläinen, Kallonen, Kolehmainen, Niemi & S, submitted]
The S-curve method applied to the real data case with added 5% noise. Vertical axis: TV norm

S-curve

$\alpha = 40$

[Niinimäki, Lassas, Hämäläinen, Kallonen, Kolehmainen, Niemi & S, submitted]
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The continuous tomographic model needs to be approximated using a discrete model

Continuous model:

In this schematic setup we have 5 projection directions and a 10-pixel detector. Therefore, the number of data points is \(50\).
The resolution of the discrete model can be freely chosen according to computational resources.

**Continuous model:**

In this schematic setup we have 5 projection directions and a 10-pixel detector. Therefore, the number of data points is 50.

The number of degrees of freedom in the three discrete models below are 16, 64 and 256, respectively.

**Discrete models:**
Intuition: discrete TV reconstructions at different resolutions should converge to a continuous limit.
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**Intuition:** discrete TV reconstructions at different resolutions should converge to a continuous limit.
We define the total variation (TV) norm consistently for continuous and discrete cases

Continuous anisotropic TV norm for attenuation coefficient \( f : \Omega \rightarrow \mathbb{R} \):

\[
\int_{\Omega} \left( \left| \frac{\partial f}{\partial x_1} \right| + \left| \frac{\partial f}{\partial x_2} \right| \right) dx.
\]

Discrete anisotropic TV norm for an image matrix of size \( n \times n \):

\[
\frac{1}{n} \sum |f_{\kappa} - f_{\kappa'}| ,
\]

where the sum is over horizontally and vertically neighboring pixel values \( f_\kappa \) and \( f_{\kappa'} \).

The above is based on this approximate two-dimensional computation:

\[
\int_{\Omega} \left| \frac{f(x_1 + \frac{1}{n}, x_2) - f(x_1, x_2)}{1/n} \right| dx \approx (1/n)^2 \sum \left| \frac{f_\kappa - f_{\kappa'}}{1/n} \right| ,
\]

where the sum is over horizontally neighboring pixel values \( f_\kappa \) and \( f_{\kappa'} \).
Low-noise TV reconstructions of a walnut using several regularization parameters

\( \alpha = 0.001 \) \hspace{1cm} \( \alpha = 1 \) \hspace{1cm} \( \alpha = 1000 \)

Too small \( \alpha \) \hspace{1cm} Just right \( \alpha \) \hspace{1cm} Too large \( \alpha \)
What happens when we compare reconstructions at different resolutions?
Low-noise TV reconstructions of a walnut at many resolutions using $\alpha = 0.001$
Low-noise TV reconstructions of a walnut at many resolutions using $\alpha = 1$

128 × 128

192 × 192

256 × 256
Low-noise TV reconstructions of a walnut at many resolutions using $\alpha = 1000$
## TV norms of low-noise reconstructions with various resolutions and parameters $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>128 × 128</th>
<th>192 × 192</th>
<th>256 × 256</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>1.51</td>
<td>2.29</td>
<td>3.64</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>1.51</td>
<td>2.29</td>
<td>3.46</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>1.50</td>
<td>2.23</td>
<td>2.97</td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>1.43</td>
<td>1.85</td>
<td>1.93</td>
</tr>
<tr>
<td>$10^{0}$</td>
<td>1.08</td>
<td>1.11</td>
<td>1.11</td>
</tr>
<tr>
<td>$10^{1}$</td>
<td>0.78</td>
<td>0.78</td>
<td>0.77</td>
</tr>
<tr>
<td>$10^{2}$</td>
<td>0.48</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>$10^{3}$</td>
<td>0.12</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>$10^{4}$</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>$10^{5}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$10^{6}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
What happens when we add noise to the data?
5% noise TV reconstructions of a walnut at many resolutions using $\alpha = 0.001$
5% noise TV reconstructions of a walnut at many resolutions using $\alpha = 10$

$128 \times 128$

$192 \times 192$

$256 \times 256$
5% noise TV reconstructions of a walnut at many resolutions using $\alpha = 10000$
TV norms of reconstructions using various noise levels, resolutions and parameters $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Low noise</th>
<th>5% noise</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$128^2$</td>
<td>$192^2$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>1.51</td>
<td>2.29</td>
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<td>1.08</td>
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<tr>
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<td>0.78</td>
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</tr>
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<td>$10^2$</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>$10^3$</td>
<td>0.12</td>
<td>0.12</td>
</tr>
<tr>
<td>$10^4$</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>$10^5$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$10^6$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

[Niinimäki, Lassas, Hämäläinen, Kallonen, Kolehmainen, Niemi & S, submitted]
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Ville Kolehmainen, University of Eastern Finland

Matti Lassas, University of Helsinki, Finland

Esa Niemi, University of Helsinki, Finland
We propose a new tomographic imaging modality based on multiple source-detector pairs

We place several X-ray sources and detectors in fixed positions. There are no moving parts.

Digital flat-panel X-ray detectors are available with 400Hz frame-rate, giving projection videos.

Reconstructing the 3D structure at all times leads to 4D tomography.

Applications include cardiac imaging, angiography, veterinary medicine, nondestructive testing.
The greatest benefit of the new imaging modality is probably taking angiography from 2D to 3D.

Video by Dr. Magda Bayoumi, downloaded from Dailymotion
For simplicity, let us consider a two-dimensional sparse-angle measurement configuration.

We place several sources and detectors in fixed positions. There are no moving parts.

We simulate a simple 2D target changing in time and collect X-ray data from all directions at multiple times.

This lets us study different options for temporal regularization without too heavy computations.
Dynamic Spatial Reconstructor

[Robb, Hoffman, Sinak, Harris & Ritman 1983]
Very brief overview of multi-source tomographic studies, all based on FBP-type algorithms


Static multi-source arrangements have received very little attention in the literature. Filtered back-projection type methods are not well-suited for resulting sparse datasets.
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The level set method [Osher, Santosa, Sethian] parametrizes curves and surfaces in a flexible way.
A generalization of the classical level set method was introduced in [Kolehmainen, Lassas & S 2008].

We model the X-ray attenuation function as $g(\Phi(x, y))$, where

$$g(\tau) = \begin{cases} \tau, & \text{if } \tau \geq 0 \\ 0, & \text{if } \tau < 0. \end{cases}$$

The smooth level set function $\Phi(x, y) := \lim_{s \to \infty} \phi(x, y, s)$ is the large-time limit of the solution of the evolution equation

$$\begin{cases} \phi_s = -A^*(A(g(\phi)) - m) + \beta \Delta \phi, \\ (\nu \cdot \nabla - r)\phi|_{\partial \Omega} = 0, \end{cases}$$

with a suitable initial condition.

Here $\beta > 0$, $r \geq 0$, $A^*$ denotes the transpose of $A$, and $\Delta \phi = \phi_{xx} + \phi_{yy}$.

We can prove the existence of the limit $\Phi(x, y)$. Such a result is not currently available for the original level set method.
The generalized level set method works nicely for stationary limited-angle tomography.

Ground truth

Full angle

Back-projection

Limited angle

Level set method

Limited angle

[Kolehmainen, Lassas & S 2008]
We write our new level set in (2+1)D spacetime to extend it to the dynamic case

**The 2D static case:**

We model the X-ray attenuation function as \( g(\Phi(x, y)) \), where the level set function

\[
\Phi(x, y) := \lim_{s \to \infty} \phi(x, y, s)
\]

comes from the evolution equation

\[
\begin{cases}
\phi_s = -A^{*}(A(g(\phi)) - m) + \beta \Delta \phi, \\
(\nu \cdot \nabla - r)\phi|_{\partial \Omega} = 0,
\end{cases}
\]

with an appropriate initial condition. Here \( \Delta \phi = \phi_{xx} + \phi_{yy} \).

[Kolehmainen, Lassas & S 2008]

**The (2+1)D dynamic case:**

We model the X-ray attenuation function as \( g(\Phi(x, y, t)) \), where the level set function

\[
\Phi(x, y, t) := \lim_{s \to \infty} \phi(x, y, t, s)
\]

comes from the evolution equation

\[
\begin{cases}
\phi_s = -A^{*}(A(g(\phi)) - m) + \beta \Delta \phi, \\
(\nu \cdot \nabla - r)\phi|_{\partial \Omega} = 0,
\end{cases}
\]

with an appropriate initial condition. Here \( \Delta \phi = \phi_{xx} + \phi_{yy} + \phi_{tt} \).

[Niemi, Lassas & S 2013]
Using the function $g$ has both computational and theoretical advantages over the classical method

**Theorem:** The functional

$$F_n(u) := \frac{1}{2} \| A g(u) - m \|_{L^2(D)}^2 + \frac{\alpha}{2} \sum_{1 \leq |\beta| \leq n} c_\beta \| D^\beta u \|_{L^2(\Omega)}^2,$$

where $A$ is an operator modeling 2D Radon transforms measured at several times, has a **global minimizer**. Further, the minimizer is unique for $n = 1$.

**Observation:** We can present an argument suggesting that the minimizer of $F_1$ can be looked for by following the descent flow

$$\partial_s \phi(x, y, t, s) = -H(\phi(x, y, t, s)) A^* (A f(\phi(x, y, t, s)) - m) + \alpha \Delta \phi(x, y, t, s).$$

[Niemi, Lassas, Kallonen, Harhanen, Hämäläinen & S, submitted]
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Simulated example in spacetime dimension (2+1)

Left: Slice-by-slice view of the simulated phantom evolving in time.
Below: Spacetime view of the same phantom.
Spacetime reconstructions

Original			Level set		Tikhonov

[Images of reconstructions with percentages: 28%, 47%, 27%, 39%, 25%, 42%]

[Lassas, Niemi & S]
Two more simulated examples, based on only seven (7) projection directions:

Imaging geometry:

Spacetime phantoms:
<table>
<thead>
<tr>
<th>Original</th>
<th>FBP</th>
<th>Tikhonov</th>
<th>Level set</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Original Image" /></td>
<td><img src="image2" alt="FBP Image" /></td>
<td><img src="image3" alt="Tikhonov Image" /></td>
<td><img src="image4" alt="Level set Image" /></td>
</tr>
<tr>
<td><img src="image5" alt="Original Image" /></td>
<td><img src="image6" alt="FBP Image" /></td>
<td><img src="image7" alt="Tikhonov Image" /></td>
<td><img src="image8" alt="Level set Image" /></td>
</tr>
<tr>
<td><img src="image9" alt="Original Image" /></td>
<td><img src="image10" alt="FBP Image" /></td>
<td><img src="image11" alt="Tikhonov Image" /></td>
<td><img src="image12" alt="Level set Image" /></td>
</tr>
</tbody>
</table>
Level set reconstruction in spacetime

Original phantom
Level set reconstruction in spacetime

Original phantom
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Our first dynamic phantom was a stop motion animation made of modelling clay (Play-Doh)
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Our first dynamic phantom was a stop motion animation made of modelling clay (Play-Doh)
Our first dynamic phantom was a stop motion animation made of modelling clay (Play-Doh)
However, the Play-Doh data turned out to be useless because of beam hardening
Our second dynamic phantom was a stop motion animation made of regular sand.
Our second dynamic phantom was a stop motion animation made of regular sand.
Our second dynamic phantom was a stop motion animation made of regular sand.
Our second dynamic phantom was a stop motion animation made of regular sand.
Our second dynamic phantom was a stop motion animation made of regular sand.
Here is the tomographic data collected from frame 2 of the sand phantom
The sand data suffered from beam hardening, too. Solution: use sugar instead!
Level set reconstruction from 10 projections

FBP reconstruction from 120 projections

$t = 1$

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
Level set reconstruction from 10 projections

FBP reconstruction from 120 projections

\[ t = 2 \]

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
Level set reconstruction from 10 projections

FBP reconstruction from 120 projections

\[ t = 3 \]

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
Level set reconstruction from 10 projections

FBP reconstruction from 120 projections

\[ t = 4 \]

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
FBP reconstruction from 120 projections

\[ t = 5 \]

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
FBP reconstruction from 120 projections

\[ t = 6 \]

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
FBP reconstruction from 120 projections

$t = 7$

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
FBP reconstruction from 120 projections

\[ t = 8 \]

Tomographic data:
Keijo Hämäläinen
Aki Kallonen
Reconstruction:
Esa Niemi

Level set reconstruction from 10 projections
FBP reconstruction from 120 projections

\[ t = 9 \]

Tomographic data: Keijo Hämäläinen
Aki Kallonen
Reconstruction: Esa Niemi

Level set reconstruction from 10 projections
FBP reconstruction from **120 projections**

\[ t = 10 \]

**Tomographic data:**
Keijo Hämäläinen
Aki Kallonen

**Reconstruction:**
Esa Niemi

Level set reconstruction from **10 projections**
This is a movie showing the recovered level set in (2+1) dimensional spacetime

Computation and visualization by Esa Niemi
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http://aip2015.fips.fi
Many numerical methods have been suggested for total variation regularization

**Primal-dual algorithms** Chambolle, Chan, Chen, Esser, Golub, Mulet, Nesterov, Zhang

**Thresholding** Candès, Chambolle, Chaux, Combettes, Daubechies, Defrise, DeMol, Donoho, Pesquet, Starck, Teschke, Vese, Wajs

**Bregman iteration** Cai, Burger, Darbon, Dong, Goldfarb, Mao, Osher, Shen, Xu, Yin, Zhang

**Splitting approaches** Chan, Esser, Fornasier, Goldstein, Langer, Osher, Schönlieb, Setzer, Wajs

**Nonlocal TV** Bertozzi, Bresson, Burger, Chan, Lou, Osher, Zhang

We found that **quadratic programming** works well for us. Kati Niinimäki wrote a primal-dual path following interior-point method for our purposes.
We solve the total variation regularization problem using quadratic programming (QP)

The minimizer of the functional

$$\arg\min_{f \in \mathbb{R}_+^n} \left\{ \|Af - m\|_2^2 + \alpha \|L_H f\|_1 + \alpha \|L_V f\|_1 \right\}$$

can be transformed into the standard form

$$\arg\min_{z \in \mathbb{R}^{5n}} \left\{ \frac{1}{2} z^T Q z + c^T z \right\}, \quad z \geq 0, \quad Ez = b,$$

where $Q$ is symmetric and $E$ implements equality constraints.

Large-scale primal-dual interior point QP method was developed in Kolehmainen, Lassas, Niinimäki & S (2012) and Hämäläinen, Kallonen, Kolehmainen, Lassas, Niinimäki & S (2013).
Reduction to \( \arg\min_{z \in \mathbb{R}^{5n}} \left\{ \frac{1}{2} z^T Q z + c^T z \right\} \)

Denote horizontal and vertical differences by

\[
L_H f = u_H^+ - u_H^- \quad \text{and} \quad L_V f = u_V^+ - u_V^- ,
\]

where \( u_H^\pm, u_V^\pm \geq 0 \). TV minimization is now

\[
\arg\min_{f \in \mathbb{R}_+^n} \left\{ f^T A^T A f - 2 f^T A^T m + \alpha \mathbf{1}^T ( u_H^+ + u_H^- + u_V^+ + u_V^- ) \right\} ,
\]

where \( \mathbf{1} \in \mathbb{R}^n \) is vector of all ones. Further, we denote

\[
z = \begin{bmatrix} f \\ u_H^+ \\ u_H^- \\ u_V^+ \\ u_V^- \end{bmatrix} , \quad Q = \begin{bmatrix} \frac{1}{\sigma^2} A^T A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \quad c = \begin{bmatrix} -2 A^T m \\ \alpha \mathbf{1} \\ \alpha \mathbf{1} \\ \alpha \mathbf{1} \end{bmatrix} .
\]
Can we offer any theoretical justification for the multi-resolution parameter choice?

In a one-dimensional deconvolution context one can show that the discrete TV regularized reconstructions converge towards the continuous TV regularized reconstruction. In such a case the multi-resolution choice rule can be intuitively explained. See [Lassas & S 2004], [Lassas, Saksman and S 2009] and [Lucka 2012].

In the above two-dimensional tomographic situation we do not have an analogous argument readily available. So the method is currently just a (promising) numerical experiment.
The L-curve method applied to the real data case with no added noise

Reconstruction with $\alpha = 1358$

[Niinimäki, Lassas, Hämäläinen, Kallonen, Kolehmainen, Niemi & S, submitted]
The L-curve method applied to the real data case with added 5% noise

Reconstruction with $\alpha = 10.4$

[Niinimäki, Lassas, Hämäläinen, Kallonen, Kolehmainen, Niemi & S, submitted]