Regularization for electrical impedance tomography using nonlinear Fourier transform

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Finnish Centre of Excellence in Inverse Problems Research

http://wiki.helsinki.fi/display/inverse/Home
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1. Electrical impedance tomography

2. Nonlinear regularization for EIT

3. Convergence theorem

4. Computational results
Electrical impedance tomography (EIT) is an emerging medical imaging method

Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

Applications include: monitoring heart and lungs of unconscious patients, detecting pulmonary edema, enhancing ECG and EEG
Early detection of breast cancer is effective using combined X-ray mammography and EIT.

Cancerous tissue is up to four times more conductive than healthy tissue. [Jossinet -98]

X-ray attenuation is almost the same in cancerous and healthy tissue.

David Isaacson and his team have achieved good results in early detection of breast cancer using EIT.
Application of EIT to non-destructive testing: imaging cracks in concrete structures

Karhunen, Seppänen, Lehikoinen, Monteiro, Kaipio, Blunt, Hyvönen
The inverse conductivity problem of Calderón is the mathematical model of EIT.

\[ \Lambda \gamma f = \gamma \frac{\partial u}{\partial n} |_{\partial \Omega}, \]

\[ \nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \]
\[ u = f \text{ on } \partial \Omega. \]

We assume that \( 0 < c \leq \gamma(x) \leq C \) for all \( x \in \Omega \).

**Problem:** given the Dirichlet-to-Neumann map, how to reconstruct the conductivity?
The reconstruction problem is nonlinear and ill-posed.
Nonlinearity of Calderón’s problem

The weak formulation of the Dirichlet-to-Neumann map

\[ \Lambda_\gamma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \]

is given by

\[ \langle \Lambda_\gamma f, g \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v, \]

where \( v \) is any \( H^1(\Omega) \) function with trace \( g \), and \( u \) satisfies the Dirichlet problem

\[
\begin{align*}
\nabla \cdot \gamma \nabla u &= 0 & \text{in } \Omega, \\
\n\n\gamma \gamma &= f & \text{on } \partial \Omega.
\end{align*}
\]

Now the map \( \gamma \mapsto \Lambda_\gamma \) is nonlinear because \( u \) depends on \( \gamma \).
Let us choose two simple conductivities to demonstrate the ill-posedness of EIT:
Here we show the voltage potentials resulting from the same boundary data.
Current measurements corresponding to the two conductivities are almost the same.
We can try another voltage pattern as well:

\[ \gamma_1 \frac{\partial u_1}{\partial n} |_{\partial \Omega} \]

\[ \gamma_2 \frac{\partial u_2}{\partial n} |_{\partial \Omega} \]
We can try yet another voltage pattern:

\[ \gamma_1 \]

\[ \gamma_2 \]

\[ u_1 \]

\[ u_2 \]

\[ \gamma_1 \frac{\partial u_1}{\partial n} \big|_{\partial \Omega} \]

\[ \gamma_2 \frac{\partial u_2}{\partial n} \big|_{\partial \Omega} \]
Distinguishing two very different targets from small differences in data is ill-posedness.
EIT reconstruction algorithms can be divided roughly into the following classes:

**Linearization** (Barber, Bikowski, Brown, Cheney, Isaacson, Mueller, Newell)

**Iterative regularization** (Dobson, Hua, Kindermann, Leitão, Lechleiter, Neubauer, Rieder, Rondi, Santosa, Tompkins, Webster, Woo)

**Bayesian inversion** (Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen)

**Resistor network methods** (Borcea, Druskin, Mamonov, Vasquez)

**Convexification** (Beilina, Klibanov)

**Layer stripping** (Cheney, Isaacson, Isaacson, Somersalo)

**D-bar methods** (Astala, Bikowski, Bowerman, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Saulnier, S, Tamasan)

**Teichmüller space methods** (Kolehmainen, Lassas, Ola)

**Methods for partial information** (Alessandrini, Ammari, Bilotta, Brühl, Erhard, Gebauer, Hanke, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kwon, Lechleiter, Lim, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others)
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<th>Infinite-precision data</th>
<th>Practical data</th>
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1. Electrical impedance tomography

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4. Computational results
EIT is an ill-posed inverse problem: the forward map $F$ does not have a continuous inverse.

The structure of the range $F(D(F))$ is not known.
Approximate and noise-robust inversion requires regularization.

We can only measure noisy data $\Lambda_\gamma^\epsilon$ instead of the ideal data $\Lambda_\gamma$. 

Diagram showing the mapping from model space $X$ to data space $Y$ via $F$, with $D(F)$ in model space and $F(D(F))$ in data space.
Conditional stability estimates are not satisfactory for practical purposes.

Conditional stability results for EIT are typically of the form

$$\|\gamma_1 - \gamma_2\|_Z \leq f(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_Y),$$

where $\gamma_1, \gamma_2 \in Z$ and $f$ is a continuous function with $f(0) = 0$.

This is not enough in practice. Namely, the noisy measurement $\Lambda_{\gamma}^\varepsilon$ is in general not the DN map corresponding to any conductivity.
This is the definition of regularization

A family of continuous mappings $\Gamma_\alpha : Y \to X$ is called a *regularization strategy* parametrized by $0 < \alpha < \infty$ if $\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\gamma) - \gamma \|_X = 0$ for each fixed conductivity $\gamma \in X$.

A choice of $\alpha = \alpha(\varepsilon)$ as function of the noise level $\varepsilon > 0$ is *admissible* if $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0$ and for any fixed $\gamma \in X$ the following holds:

$$\sup_{\Lambda_\varepsilon \gamma} \left\{ \| \Gamma_\alpha(\Lambda_\varepsilon) - \gamma \|_X : \| \Lambda_\varepsilon - \Lambda_\gamma \|_Y \leq \varepsilon \right\} \xrightarrow{\varepsilon \to 0} 0.$$
Regularization of nonlinear inverse problems is an active and challenging area of research

There are two main approaches for regularizing nonlinear inverse problems:

1. Iterative regularization
   + Generic: numerically applicable to any inverse problem,
   + Quick to develop optimization-based solution software,
     o Rigorously defined only for almost linear inverse problems due to problems with local minima.

2. Tailored nonlinear regularization strategies:
   + Rigorous mathematical analysis available for algorithms,
   + Provides a link between several schools of research,
     o One method applies only to one inverse problem.
Inverse problems so far provided with a regularization analysis are marked with red.
Inverse problems so far provided with a regularization analysis are marked with red.

All inverse problems

Close to linear problems

Linear inverse problems

Bissantz, Hanke, Hofmann, Hohage, Kaltenbacher, Kindermann, Lu, Mathé, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Resmerita, Scherzer, Teschke (1997-present)

Tikhonov’s work, see also Engl, Hanke & Neubauer (1996)
All inverse problems

Inverse problems so far provided with a regularization analysis are marked with red

Linear inverse problems

Close to linear problems

EIT

Electrical inclusion detection:

Bissantz, Hanke, Hofmann, Hohage, Kaltenbacher, Kindermann, Lu, Mathé, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Resmerita, Scherzer, Teschke (1997-present)

Tikhonov’s work, see also
Engl, Hanke & Neubauer (1996)
Inverse problems so far provided with a regularization analysis are marked with red.

All inverse problems:

- Blind deconvolution: Justin & Ramlau (2006)
- Bissantz, Hanke, Hofmann, Hohage, Kaltenbacher, Kindermann, Lu, Mathé, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Resmerita, Scherzer, Teschke (1997-present)
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Close to linear problems

Linear inverse problems
Inverse problems so far provided with a regularization analysis are marked with red.

- **Electrical impedance tomography**: Knudsen, Lassas, Mueller & S (2009)
- **Blind deconvolution**: Justin & Ramlau (2006)
- **Electrical inclusion detection**: Ikehata & S (2004), Lechleiter (2006)
- **Close to linear problems**
  - Bissantz, Hanke, Hofmann, Hohage, Kaltenbacher, Kindermann, Lu, Mathé, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Resmerita, Scherzer, Teschke (1997-present)
- **Linear inverse problems**
  - Tikhonov’s work, see also Engl, Hanke & Neubauer (1996)
1. Electrical impedance tomography

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4. Computational results
We describe a nonlinear regularization strategy for EIT under these assumptions:

Let $\Omega \subset \mathbb{R}^2$ be the open unit disc and

$$F : \mathcal{D}(F) \subset L^\infty(\Omega) \to Y.$$ 

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ is the set of functions $\gamma : \Omega \to \mathbb{R}$ satisfying

$$\|\gamma\|_{C^2(\overline{\Omega})} \leq M,$$

$$\gamma(x) \geq 1/M,$$

$$\gamma(x) \equiv 1 \text{ for } \rho < |x| < 1.$$ 

The data space $Y$ consists of bounded linear operators $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying $\Lambda(1) = 0$ and $\int_{\partial\Omega} \Lambda(f) \, d\sigma = 0.$
These are the two main steps in Nachman’s 2D reconstruction method for EIT:

\[ \Lambda \gamma \rightarrow t \rightarrow \gamma \]

The intermediate object \( t \) is a complex-valued function called *scattering transform* and defined as follows:

\[ t(k) := \int_{\mathbb{R}^2} e^{i\bar{k} \cdot x} q(x) \psi(x, k) \, dx \]

The function \( t \) is also a nonlinear Fourier transform.

\[ q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \]

\[ (-\Delta + q) \psi(\cdot, k) = 0 \]

\[ \psi(x, k) \sim e^{ik \cdot x} = e^{i(k_1 + ik_2)(x_1 + ix_2)} \]
The main tool in the method is the construction of complex geometrical optics solutions

We look for \( \psi(x, k) = e^{ikx} \mu(x, k) \) satisfying 
\((-\Delta + q)\psi(x, k) = 0 \) and \( \mu(\cdot, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2) \).

Then \( \mu \) must satisfy 
\((-\Delta - 4ik\bar{\partial} + q)\mu = 0 \),
which can be written in the form

\[ \mu = 1 - g_k \ast (q\mu). \]

Here \((-\Delta - 4ik\bar{\partial})g_k(x) = \delta(x) \). The solution is given by \( \mu - 1 = [I + g_k \ast (q \cdot)]^{-1}(g_k \ast q) \) for all \( k \in \mathbb{C} \setminus 0 \) whenever \( q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \) [Nachman -96].
Derivation of a boundary integral equation for traces of complex geometrical optics solutions

Define \( G_k(x) = e^{ikx} g_k(x) \); then \(-\Delta G_k = \delta\).

We call \( G_k \) Faddeev Green’s function. Now

\[
\psi(x, k) = e^{ikx} \mu(x, k) \\
= e^{ikx} - \int_{\mathbb{R}^2} e^{ikx} g_k(x - y) q(y) \mu(y, k) \, dy \\
= e^{ikx} - \int_{\mathbb{R}^2} G_k(x - y) q(y) \psi(y, k) \, dy,
\]

and Alessandrini’s identity gives

\[
\psi(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)(\psi(\cdot, k)|_{\partial\Omega}),
\]

where \((S_k \varphi)(x) = \int_{\partial\Omega} G_k(x - y) \varphi(y) \, dS(y)\).
Our regularized D-bar reconstruction method is based on nonlinear low-pass filtering

Solve the integral equation

$$\psi^\varepsilon(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial\Omega}$$

for $|k| < R$. Set

$$t^\varepsilon_R(k) = \begin{cases} \int_{\partial\Omega} e^{ikx}(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k) dS & \text{for } |k| < R, \\ 0 & \text{otherwise.} \end{cases}$$

Solve the D-bar equation

$$\frac{\partial}{\partial k} \mu_R(x, k) = \frac{t^\varepsilon_R(k)}{4\pi k} e^{-i(kx + \bar{k}x)} \mu_R(x, k)$$

with large $|k|$ asymptotics $\mu_R(x, k) \sim 1$. 
Regularization strategy:

$$\Gamma_{\alpha} \Lambda_{\gamma}^\varepsilon = \mu_R(x, 0)^2$$

Truncation radius:

$$R(\varepsilon) = -\frac{1}{10} \log \varepsilon$$

Regularization parameter:

$$\alpha(\varepsilon) = \frac{1}{R(\varepsilon)}$$
Starting point: ideal data $\Lambda_\gamma$

- Scattering transform $t(k)$
  
  \[ \frac{\partial}{\partial k} \mu(x, k) = \frac{t(k)}{4\pi k} e^{-i(kx+k|x)} \mu(x, k) \]

- Solve $\overline{\nabla}$ equation

- Perfect reconstruction $\gamma(x) = \mu(x, 0)^2$

Starting point: noisy data $\Lambda_\varepsilon$

- Noisy scattering transform $t_R^\varepsilon(k)$ truncated at $R(\varepsilon)$

- Solve $\overline{\nabla}$ equation

- Approximate reconstruction $\gamma(x) \approx \mu_R(x, 0)^2$
Why is $\sqrt{\gamma(x)} = \lim_{k \to 0} \mu(x, k)$?

Substituting $\psi(x, k) = e^{ikx} \mu(x, k)$ into equation $(-\Delta + q)\psi(\cdot, k) = 0$ yields

$$(-\Delta - 4ik\delta + q(x))\mu(x, k) = 0. \quad (1)$$

Now (1) has a unique solution with large $|x|$ asymptotics $\mu(x, k) \sim 1$.

Recall $q = \gamma^{-1/2} \Delta \gamma^{1/2}$. Now $k = 0$ in (1) gives

$$(-\Delta + \frac{\Delta \sqrt{\gamma(x)}}{\sqrt{\gamma(x)}})\mu(x, 0) = 0,$$

whose unique solution must be $\mu(x, 0) = \sqrt{\gamma(x)}$. 
Theorem [Knudsen, Lassas, Mueller & S 2008]

The family $\Gamma_\alpha$ is well-defined for small $\alpha > 0$. It is an admissible regularization strategy with

$$\alpha(\varepsilon) = \left( \frac{1}{10} \log(\varepsilon) \right)^{-1}.$$

Furthermore, we have the explicit estimate

$$\sup_{\Lambda_\gamma^\varepsilon} \left\{ \| \Gamma_{\alpha(\varepsilon)} \Lambda_\gamma^\varepsilon - \gamma \|_{L^\infty(\Omega)} : \| \Lambda_\gamma^\varepsilon - \Lambda_\gamma \|_Y \leq \varepsilon \right\} \leq C (- \log \varepsilon)^{-1/14}

\rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$
Proof of the main theorem is divided into several lemmata. First a D-bar estimate:

**Lemma 1.** Let $4/3 < r_0 < 2$ and suppose that $\phi_1, \phi_2 \in L^r(\mathbb{R}^2)$ for all $r \geq r_0$. Let $\mu_1, \mu_2$, be the solutions of

$$\mu_j(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\phi_j(k')}{(k - k')} \mu_j(x, k') dk'_1 dk'_2,$$

$j = 1, 2$. Then for fixed $x \in \overline{\Omega}$ we have

$$\|\mu_1(x, \cdot) - \mu_2(x, \cdot)\|_{C^\alpha(\mathbb{R}^2)} \leq C \|\phi_1 - \phi_2\|_{L^{r_0} \cap L^{r_0'}(\mathbb{R}^2)},$$

where $\alpha < 2/r_0 - 1$ and $1/r_0' = 1 - 1/r_0$.

**Proof.** Combination of well-known results.
These results follow from careful analysis of Faddeev’s Green function.

**Lemma 2.** Let \( \phi_0 \in H^{-1/2}(\partial \Omega) \) with \( \int \phi_0 = 0 \). Then we have the estimate

\[
\|S_k \phi_0\|_{H^{1/2}(\partial \Omega)} \leq C e^{2|k|}(1 + |k|)\|\phi_0\|_{H^{-1/2}(\partial \Omega)}.
\]

**Lemma 3.** For \( k \in \mathbb{C} \) we have the estimate

\[
\|[I + S_k(\Lambda_\gamma - \Lambda_1)]^{-1}\|_{L(H^s(\partial \Omega))} \leq C_2 e^{2|k|}(1 + |k|),
\]

where \( C_2 \) depends only on \( M \) and \( \rho \).
Combining previous results, a perturbation argument, and delicate $L^p$ analysis shows

**Lemma 4.** There exists $\varepsilon_0 > 0$, depending only on $M$ and $\rho$, such that the equation

$$\psi^\varepsilon(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial \Omega}$$

is solvable in $H^{1/2}(\partial \Omega)$ for all $0 < \varepsilon \leq \varepsilon_0$ and $|k| < R$ with

$$R = R(\varepsilon) = -\frac{1}{10} \log \varepsilon.$$ 

Furthermore, for $p > 1$ we have the estimate

$$\left\| \frac{t(k) - t_R^\varepsilon(k)}{\overline{k}} \right\|_{L^p(|k| \leq R)} \leq C \varepsilon^{1/10} \left( -\frac{1}{10} \log \varepsilon \right)^{2/p},$$

where $C$ is independent of $p$ and $R$ and $\varepsilon$. 
Sketch of proof of main theorem

(i) $\lim_{\alpha \to 0} \| \Gamma_{\alpha} \Lambda_{\gamma} - \gamma \|_{L^{\infty}(\Omega)} = 0$ for $\gamma \in X$.

(ii) $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$,

(iii) $\sup_{\Lambda_{\gamma}^{\varepsilon}} \left\{ \| \Gamma_{\alpha(\varepsilon)} \Lambda_{\gamma}^{\varepsilon} - \gamma \|_{L^{\infty}(\Omega)} : \| \Lambda_{\gamma}^{\varepsilon} - \Lambda_{\gamma} \|_{Y} \leq \varepsilon \right\}$
    tends to zero as $\varepsilon \to 0$.

Claim (i) follows from [Nachman 1996] (with delicate choices of $L^p$ spaces) and Lemma 1. Claim (ii) is OK: $\alpha(\varepsilon) = \frac{1}{R(\varepsilon)} = -10(\log \varepsilon)^{-1}$.
Sketch of proof of main theorem

To prove that
\[ \sup_{\Lambda_\gamma \in \Lambda_\gamma^\varepsilon} \left\{ \| \Gamma_\alpha(\varepsilon) \Lambda_\gamma^\varepsilon - \gamma \|_{L^\infty(\Omega)} : \| \Lambda_\gamma^\varepsilon - \Lambda_\gamma \|_Y \leq \varepsilon \right\} \]
tends to zero as \( \varepsilon \to 0 \) we combine [Nachman 1996] with Lemmata 1 and 4 to estimate

\[ \| \mu(x, \cdot) - \mu_R(x, \cdot) \|_{C^\alpha(\mathbb{R}^2)} \]
\[ \leq C \left\| \frac{t(k) - t_{R}^\varepsilon(k)}{\overline{k}} \right\|_{L^p \cap L^{p'}(\mathbb{R}^2)} \]
\[ \leq C \left\| \frac{t(k) - t_{R}^\varepsilon(k)}{\overline{k}} \right\|_{L^p \cap L^{p'}(|k|<R)} + C \left\| \frac{t(k)}{\overline{k}} \right\|_{L^p(|k|>R)} \]
\[ \leq C(-\frac{1}{10} \log \varepsilon)^{10/7} \varepsilon^{1/10} + CR(\varepsilon)^{-\frac{1}{7}} + CR(\varepsilon)^{-\frac{1}{14}} \]
\[ = C(- \log \varepsilon)^{10/7} \varepsilon^{1/10} + C(- \log \varepsilon)^{-\frac{1}{7}} + C(- \log \varepsilon)^{-\frac{1}{14}}. \]
One more thing: the regularization strategy is not yet defined on all of data space $Y$.

The range $F(D(F)) \subset Y$ is not known, and its structure may be complicated. (This is related to the open and notoriously difficult characterization problem.)

The previous results show the claim only for operators $\varepsilon_0$-close to the range $F(D(F))$.

The problem can be overcome using spectral theoretical arguments.
1. Electrical impedance tomography

2. Nonlinear regularization for EIT

3. Convergence theorem

4. Computational results
This is a typical configuration for electrode measurements in EIT. Here we have $N=32$ electrodes (ACT3). The machine is in Rensselaer Polytechnic Institute, USA.
We approximate discrete current patterns by Fourier basis functions:

- $\cos(\theta)$
- $\cos(4\theta)$
- $\cos(16\theta)$
We construct a simulated human chest phantom and compute DN map using FEM
This is our practical two-step regularized D-bar method for EIT

1. We solve for $|k| < R$ the matrix version of

$$\psi^\varepsilon(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda^\varepsilon_\gamma - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial\Omega}$$

with $R$ as large as numerically stable.

2. The integral equation

$$\mu_R(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t_R^\varepsilon(s)}{(k - s)^2} e^{-\rho(x(s) + \mu_R(x, s))} ds_1 ds_2$$

can be solved by our established D-bar solver. The reconstructed conductivity is $\mu_R(x, 0)$.2
We solve the boundary integral equation using matrices in truncated Fourier basis

Given linear map $A$ and $N > 0$, define matrix $A : \mathbb{C}^{2N+1} \to \mathbb{C}^{2N+1}$ by $A := [A_{mn}]$ with

$$A_{mn} := \frac{1}{2\pi} \int_0^{2\pi} (A e^{in\theta}) e^{-im\theta} d\theta.$$

We write all operators in the equation

$$\psi(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_{\gamma} - \Lambda_1)(\psi(\cdot, k)|_{\partial\Omega})$$

in matrix form, and solve

$$[I + S_k(\Lambda_{\gamma} - \Lambda_1)]\psi(\cdot, k)|_{\partial\Omega} = e^{ikx}.$$
This is how the scattering transform $t(k)$ looks like

Real part of $t(k)$  Imaginary part of $t(k)$

Here $|k|<10$. 
The effect of measurement noise is clearly visible in $t(k)$

Real part of $t(k)$  Imaginary part of $t(k)$

Here $|k|<10$. 
The d-bar equation is written in integral form for numerical solution by Vainikko’s method.

Write the d-bar equation

\[
\frac{\partial}{\partial k} \mu_R(x, k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + \bar{k}x)} \mu_R(x, k)
\]

in integral form using the appropriate Green function:

\[
\mu_R(x, k) = 1 + \frac{1}{\pi k} \left( \frac{t_R(k)}{4\pi k} e^{-i(kx + \bar{k}x)} \mu_R(x, k) \right).
\]

This equation of the Lippmann-Schwinger form can be solved numerically as explained below. Then

\[
\gamma_R^{1/2}(x) = \mu_R(x, 0).
\]
Periodic convolution is conveniently implemented using the FFT

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\xrightarrow{\text{FFT}}
\begin{bmatrix}
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi}
\end{bmatrix}
\xrightarrow{\text{Multiplication}}
\begin{bmatrix}
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi
\end{bmatrix}
\xrightarrow{-}
\begin{bmatrix}
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi
\end{bmatrix}
\xrightarrow{\text{IFFT}}
\begin{bmatrix}
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} \\
T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi} & T_R\tilde{\varphi}
\end{bmatrix}
\xrightarrow{+}
\varphi - \tilde{g} \ast (T_R\tilde{\varphi})
Here we see the reconstructions corresponding to various levels of measurement noise.

The percentages shown are relative square norm errors.
The numerical results actually improve the exponential behaviour predicted by theory.

\[ R(\varepsilon) = -\frac{1}{10} \log \varepsilon \]
Reconstruction from measured phantom data

Relative error 23% (lung) and 12% (heart).
Dynamical range is 94% of the true range.
Thank you!

Preprints available at www.siltanen-research.net

Forthcoming book: Mueller & S, Linear and nonlinear inverse problems with practical applications, SIAM
The reconstruction method is based on the use of complex geometrical optics (CGO) solutions.

Set \( \mu := (1 - \sigma)(1 + \sigma)^{-1} \) in the Beltrami equation

\[
\partial_z f_\mu = \mu \bar{\partial}_z f_\mu.
\]

We look for solutions

\[
f_\mu(z, k) = e^{ikz} M_\mu(z, k) = e^{ikz} \left( 1 + O \left( \frac{1}{z} \right) \right)
\]

as \( |z| \to \infty \). The complex \( k \) is a Fourier variable.

Also, we use \( u_j \) as in

\[
\nabla \cdot \sigma \nabla u_1(\cdot, k) = 0,
\]

\[
\nabla \cdot \sigma^{-1} \nabla u_2(\cdot, k) = 0.
\]

Define \( h_+ = \frac{1}{2}(f_\mu + f_{-\mu}) \) and \( h_- = \frac{1}{2}(f_\mu - f_{-\mu}) \); then the solutions \( u_1(z, k) \) and \( u_2(z, k) \) are given by

\[
u_1 = h_+ - ih_-,
\]

\[
u_2 = i(h_+ + ih_-).
\]

The Astala-Päivärinta approach can recover discontinuous conductivities quite accurately.

Conductivity $\sigma_1$  
Reconstruction from ideal data, $R = 6$  
Reconstruction with 0.01% noise, $R = 5.5$

Astala K, Mueller J L, Päivärinta L, Perämäki A, and Siltanen S,  
*Direct electrical impedance tomography for nonsmooth conductivities.* To appear.