Regularized D-bar method for electrical impedance tomography

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LATP Seminar, Marseille, April 12, 2011
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1. Electrical impedance tomography

2. Nonlinear regularization for EIT

3. Computational results

4. The Astala-Päivärinta approach
Electrical impedance tomography (EIT) is an emerging medical imaging method

Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

Applications include: monitoring heart and lungs of unconscious patients, detecting pulmonary edema, enhancing ECG and EEG
Early detection of breast cancer is effective using combined X-ray mammography and EIT.

Cancerous tissue is up to four times more conductive than healthy tissue. [Jossinet 1998]

X-ray attenuation is almost the same in cancerous and healthy tissue.

**David Isaacson** and his team have achieved good results in early detection of breast cancer using EIT.
Application of EIT to non-destructive testing: imaging cracks in concrete structures

Karhunen, Seppänen, Lehikoinen, Monteiro, Kaipio, Blunt, Hyvönen
The inverse conductivity problem of Calderón is the mathematical model of EIT

\[ \Lambda \gamma f = \gamma \frac{\partial u}{\partial \nu} |_{\partial \Omega}, \]

\[ \nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \]

\[ u = f \text{ on } \partial \Omega. \]

We assume that \( 0 < c \leq \gamma(x) \leq C \) for all \( x \in \Omega \).

**Problem:** given the Dirichlet-to-Neumann map, how to reconstruct the conductivity? The reconstruction problem is nonlinear and ill-posed.
Nonlinearity of Calderón’s problem

The weak formulation of the Dirichlet-to-Neumann map

$$\wedge_\gamma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)$$

is given by

$$\langle \wedge_\gamma f, g \rangle = \int_{\Omega} \gamma \nabla u \cdot \nabla v,$$

where \(v\) is any \(H^1(\Omega)\) function with trace \(g\), and \(u\) satisfies the Dirichlet problem

$$\begin{cases}
\nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}$$

Now the map \(\gamma \mapsto \wedge_\gamma\) is nonlinear because \(u\) depends on \(\gamma\).
Let us choose two simple conductivities to demonstrate the ill-posedness of EIT:
Here we show the voltage potentials resulting from the same boundary data.
Current measurements corresponding to the two conductivities are almost the same.
We can try another voltage pattern as well:
We can try yet another voltage pattern:
Distinguishing two very different targets from small differences in data is ill-posedness.
EIT reconstruction algorithms can be divided roughly into the following classes:

**Linearization** (Barber, Bikowski, Brown, Cheney, Isaacson, Mueller, Newell)

**Iterative regularization** (Dobson, Hua, Kindermann, Leitão, Lechleiter, Neubauer, Rieder, Rondi, Santosa, Tompkins, Webster, Woo)

**Bayesian inversion** (Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen)

**Resistor network methods** (Borcea, Druskin, Mamonov, Vasquez)

**Convexification** (Beilina, Klibanov)

**Layer stripping** (Cheney, Isaacson, Isaacson, Somersalo)

**D-bar methods** (Astala, Bikowski, Bowerman, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Saulnier, S, Tamasan)

**Teichmüller space methods** (Kolehmainen, Lassas, Ola)

**Methods for partial information** (Alessandrini, Ammari, Bilotta, Brühl, Erhard, Gebauer, Hanke, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kwon, Lechleiter, Lim, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others)
## History of global CGO-based methods for 2D EIT

<table>
<thead>
<tr>
<th>Infinite-precision data</th>
<th>Practical data</th>
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<td><strong>1980 Calderón</strong></td>
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<td>2008 Boverman, Isaacson, Kao, Saulnier &amp; Newell</td>
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<td>1987 Sylvester and Uhlmann</td>
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<td>(C^2(Ω))</td>
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<td>1997 Brown and Uhlmann</td>
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<td>(L^∞(Ω))</td>
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<td>2007 Barceló, Faraco and Ruiz</td>
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1. Electrical impedance tomography

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4. The Astala-Päivärinta approach
EIT is an ill-posed inverse problem: the forward map $F$ does not have a continuous inverse. The structure of the range $F(D(F))$ is not known.

Model space $X$  

Data space $Y$

The structure of the range $F(D(F))$ is not known.
Approximate and noise-robust inversion requires regularization

We can only measure noisy data $\Lambda_\gamma^\varepsilon$ instead of the ideal data $\Lambda_\gamma$
Conditional stability estimates are not satisfactory for practical purposes.

Conditional stability results for EIT are typically of the form

$$\|\gamma_1 - \gamma_2\|_Z \leq f(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_Y),$$

where $\gamma_1, \gamma_2 \in Z$ and $f$ is a continuous function with $f(0) = 0$.

This is not enough in practice. Namely, the noisy measurement $\Lambda_{\gamma}^\varepsilon$ is in general not the DN map corresponding to any conductivity.
This is the official definition of regularization

A family of continuous mappings $\Gamma_\alpha : Y \to X$ is called a regularization strategy parametrized by $0 < \alpha < \infty$ if $\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\gamma) - \gamma \|_X = 0$ for each fixed conductivity $\gamma \in X$.

A choice of $\alpha = \alpha(\varepsilon)$ as function of the noise level $\varepsilon > 0$ is admissible if $\lim_{\varepsilon \to 0} \alpha(\varepsilon) = 0$ and for any fixed $\gamma \in X$ the following holds:

$$\sup_{\Lambda_\gamma} \left\{ \| \Gamma_\alpha(\Lambda_\varepsilon) - \gamma \|_X : \| \Lambda_\varepsilon - \Lambda_\gamma \|_Y \leq \varepsilon \right\} \to 0 \quad \varepsilon \to 0$$
Inverse problems so far provided with a regularization analysis are marked with red.
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All inverse problems

Close to linear problems

Linear inverse problems

- Bissantz, Hanke, Hofmann, Hohage, Kaltenbacher, Kindermann, Lu, Mathé, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Resmerita, Scherzer, Teschke (*1997-present*)

- Tikhonov’s work, see also Engl, Hanke & Neubauer (*1996*)
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- **Close to linear problems**
  - Blind deconvolution: Justin & Ramlau (2006)
  - Bissantz, Hanke, Hofmann, Hohage, Kaltenbacher, Kindermann, Lu, Mathé, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Resmerita, Scherzer, Teschke (1997-present)
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Inverse problems so far provided with a regularization analysis are marked with red

**All inverse problems**

- **IS**
- **BD**
- **OT**
- **EIT**

**Close to linear problems**

**Linear inverse problems**

- Blind deconvolution: Justin & Ramlau (2006)
- Bissantz, Hanke, Hofmann, Hohage, Kaltenbacher, Kindermann, Lu, Mathé, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Resmerita, Scherzer, Teschke (1997-present)
- Tikhonov’s work, see also Engl, Hanke & Neubauer (1996)
We describe a nonlinear regularization strategy for EIT under these assumptions:

Let \( \Omega \subset \mathbb{R}^2 \) be the open unit disc and

\[
F : \mathcal{D}(F) \subset L^\infty(\Omega) \to Y.
\]

Let \( M > 0 \) and \( 0 < \rho < 1 \). The domain \( \mathcal{D}(F) \) is the set of functions \( \gamma : \Omega \to \mathbb{R} \) satisfying

\[
\|\gamma\|_{\mathcal{C}^2(\overline{\Omega})} \leq M,
\]

\[
\gamma(x) \geq 1/M,
\]

\[
\gamma(x) \equiv 1 \text{ for } \rho < |x| < 1.
\]

The data space \( Y \) consists of bounded linear operators \( \Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega) \) satisfying \( \Lambda(1) = 0 \) and \( \int_{\partial\Omega} \Lambda(f) \, d\sigma = 0 \).
These are the two main steps in Nachman’s 2D reconstruction method for EIT:

\[ \Lambda \gamma \rightarrow t \rightarrow \gamma \]

The intermediate object \( t \) is a complex-valued function called \textit{scattering transform} and defined as follows:

\[ t(k) := \int_{\mathbb{R}^2} e^{i\bar{k}x} q(x) \psi(x, k) \, dx \]

The function \( t \) is also a nonlinear Fourier transform.

\[ q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \]

\[ (-\Delta + q)\psi(\cdot, k) = 0 \]

\[ \psi(x, k) \sim e^{ikx} = e^{i(k_1+ik_2)(x_1+ix_2)} \]
The main tool in the method is the construction of complex geometrical optics solutions

We look for \( \psi(x, k) = e^{ikx} \mu(x, k) \) satisfying
\[
(-\Delta + q)\psi(x, k) = 0 \text{ and } \mu(\cdot, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2).
\]

Then \( \mu \) must satisfy \( (-\Delta - 4ik\bar{\partial} + q)\mu = 0, \) which can be written in the form

\[
\mu = 1 - g_k \ast (q\mu).
\]

Here \( (-\Delta - 4ik\bar{\partial})g_k(x) = \delta(x). \) The solution is given by \( \mu - 1 = [I + g_k \ast (q \cdot)]^{-1}(g_k \ast q) \) for all \( k \in \mathbb{C} \setminus 0 \) whenever \( q = \frac{\Delta \gamma^{1/2}}{\gamma_{1/2}} \) [Nachman -96].
Define $G_k(x) = e^{ikx}g_k(x)$; then $-\Delta G_k = \delta$.

We call $G_k$ Faddeev Green’s function. Now

$$\psi(x, k) = e^{ikx}\mu(x, k)$$

$$= e^{ikx} - \int_{\mathbb{R}^2} e^{ikx} g_k(x - y)q(y)\mu(y, k)\,dy$$

$$= e^{ikx} - \int_{\mathbb{R}^2} G_k(x - y)q(y)\psi(y, k)\,dy,$$

and Alessandrini’s identity gives

$$\psi(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)(\psi(\cdot, k)|_{\partial\Omega}),$$

where $(S_k\varphi)(x) = \int_{\partial\Omega} G_k(x - y)\varphi(y)\,dS(y)$. 
Our regularized D-bar reconstruction method is based on nonlinear low-pass filtering

Solve the integral equation

\[ \psi^\varepsilon(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial \Omega} \]

for \(|k| < R\). Set

\[ t^\varepsilon_R(k) = \begin{cases} \int_{\partial \Omega} e^{ikx}(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)dS & \text{for } |k| < R, \\ 0 & \text{otherwise.} \end{cases} \]

Solve the D-bar equation

\[ \frac{\partial}{\partial k} \mu_R(x, k) = \frac{t^\varepsilon_R(k)}{4\pi |k|} e^{-i(kx + \bar{k}\bar{x})} \frac{1}{\mu_R(x, k)} \]

with large \(|k|\) asymptotics \(\mu_R(x, k) \sim 1\).
Regularization strategy:

\[ \Gamma_{\alpha \Lambda^\varepsilon} = \mu_R(x, 0)^2 \]

Truncation radius:

\[ R(\varepsilon) = -\frac{1}{10} \log \varepsilon \]

Regularization parameter:

\[ \alpha(\varepsilon) = \frac{1}{R(\varepsilon)} \]
Starting point: ideal data $\Lambda_\gamma$

Scattering transform $t(k)$

$$\frac{\partial}{\partial k} \mu(x, k) = \frac{t(k)}{4\pi k} e^{-i(kx+k\bar{x})} \mu(x, k)$$

Solve $\overline{\partial}$ equation

Perfect reconstruction

$$\gamma(x) = \mu(x, 0)^2$$

Starting point: noisy data $\Lambda_\varepsilon$

Noisy scattering transform $t^\varepsilon_R(k)$ truncated at $R(\varepsilon)$

$$\frac{\partial}{\partial k} \mu_R(x, k) = \frac{t^\varepsilon_R(k)}{4\pi k} e^{-i(kx+k\bar{x})} \mu_R(x, k)$$

Solve $\overline{\partial}$ equation

Approximate reconstruction

$$\gamma(x) \approx \mu_R(x, 0)^2$$
Why is $\sqrt{\gamma(x)} = \lim_{k \to 0} \mu(x, k)$?

Substituting $\psi(x, k) = e^{ikx} \mu(x, k)$ into equation $(-\Delta + q)\psi(\cdot, k) = 0$ yields

$$(-\Delta - 4ik\bar{\partial} + q(x))\mu(x, k) = 0. \quad (1)$$

Now (1) has a unique solution with large $|x|$ asymptotics $\mu(x, k) \sim 1$.

Recall $q = \gamma^{-1/2} \Delta \gamma^{1/2}$. Now $k = 0$ in (1) gives

$$(-\Delta + \frac{\Delta \sqrt{\gamma(x)}}{\sqrt{\gamma(x)}})\mu(x, 0) = 0,$$

whose unique solution must be $\mu(x, 0) = \sqrt{\gamma(x)}$. 
**Theorem** [Knudsen, Lassas, Mueller & S 2008]

The family $\Gamma_\alpha$ is well-defined for small $\alpha > 0$. It is an admissible regularization strategy with

$$\alpha(\varepsilon) = \left( -\frac{1}{10} \log(\varepsilon) \right)^{-1}.$$  

Furthermore, we have the explicit estimate

$$\sup_{\Lambda_\gamma^\varepsilon} \left\{ \| \Gamma_{\alpha(\varepsilon)} \Lambda_\gamma^\varepsilon - \gamma \|_{L^\infty(\Omega)} : \| \Lambda_\gamma^\varepsilon - \Lambda_\gamma \|_Y \leq \varepsilon \right\}$$

$$\leq C(-\log\varepsilon)^{-1/14}$$

$$\rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$
1. Electrical impedance tomography

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4. The Astala-Päivärinta approach
This is a typical configuration for electrode measurements in EIT.

Here we have N=32 electrodes (ACT3).
The machine is in Rensselaer Polytechnic Institute, USA.
We approximate discrete current patterns by Fourier basis functions

\[
\cos(\theta) \quad \cos(4\theta) \quad \cos(16\theta)
\]
We construct a simulated human chest phantom and compute DN map using FEM.
This is our practical two-step regularized D-bar method for EIT

1. We solve for $|k| < R$ the matrix version of

$$
\psi^\varepsilon(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k (\Lambda_\gamma^\varepsilon - \Lambda_1) \psi^\varepsilon(\cdot, k)|_{\partial\Omega}
$$

with $R$ as large as numerically stable.

2. The integral equation

$$
\mu_R(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t_R^\varepsilon(s)}{(k - s)^3} e^{-x(s)\mu_R(x, s)} ds_1 ds_2
$$

can be solved by our established D-bar solver. The reconstructed conductivity is $\mu_R(x, 0)^2$. 

We solve the boundary integral equation using matrices in truncated Fourier basis

Given linear map $A$ and $N > 0$, define matrix $A : \mathbb{C}^{2N+1} \to \mathbb{C}^{2N+1}$ by $A := [A_{mn}]$ with

$$A_{mn} := \frac{1}{2\pi} \int_0^{2\pi} (Ae^{in\theta}) e^{-im\theta} d\theta.$$

We write all operators in the equation

$$\psi(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)(\psi(\cdot, k)|_{\partial \Omega})$$

in matrix form, and solve

$$[I + S_k(\Lambda_\gamma - \Lambda_1)]\psi(\cdot, k)|_{\partial \Omega} = e^{ikx}.$$
This is how the scattering transform $t(k)$ looks like

Real part of $t(k)$   Imaginary part of $t(k)$

Here $|k|<10$. 
The effect of measurement noise is clearly visible in $t(k)$

Real part of $t(k)$  Imaginary part of $t(k)$

Here $|k|<10$. 
The d-bar equation is written in integral form for numerical solution by Vainikko’s method.

Write the d-bar equation

\[
\frac{\partial}{\partial k} \mu_R(x, k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + \overline{k}x)} \mu_R(x, k)
\]

in integral form using the appropriate Green function:

\[
\mu_R(x, k) = 1 + \frac{1}{\pi k} \ast \left( \frac{t_R(k)}{4\pi k} e^{-i(kx + \overline{k}x)} \mu_R(x, k) \right).
\]

This equation of the Lippmann-Schwinger form can be solved numerically as explained below. Then

\[
\gamma_R^{1/2}(x) = \mu_R(x, 0).
\]
Periodic convolution is conveniently implemented using the FFT

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\rightarrow \text{FFT} \rightarrow \text{Multiplication} \rightarrow \text{IFFT} \rightarrow \begin{bmatrix}
T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} \\
T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} \\
T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} \\
T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi} & T_R\bar{\varphi}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\varphi & \varphi & \varphi & \varphi \\
\end{bmatrix}
\rightarrow - \rightarrow + \rightarrow \varphi - \tilde{g} \ast (\tilde{T}_R\bar{\varphi})
Here we see the reconstructions corresponding to various levels of measurement noise.

\[ \| \Lambda_\gamma - \Lambda_\gamma \|_Y \approx 10^{-6} \]

\[ \| \Lambda_\gamma - \Lambda_\gamma \|_Y \approx 10^{-5} \]

\[ \| \Lambda_\gamma - \Lambda_\gamma \|_Y \approx 10^{-4} \]

\[ \| \Lambda_\gamma - \Lambda_\gamma \|_Y \approx 10^{-3} \]

\[ \| \Lambda_\gamma - \Lambda_\gamma \|_Y \approx 10^{-2} \]

The percentages shown are relative square norm errors.
The numerical results actually improve the exponential behaviour predicted by theory.

\[ R(\varepsilon) = -\frac{1}{10} \log \varepsilon \]

![Graph showing the relationship between noise level and observed radius](image)
Reconstruction from measured phantom data

Relative error 23% (lung) and 12% (heart). Dynamical range is 94% of the true range.
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4. The Astala-Päivärinta approach
The reconstruction method is based on the use of complex geometrical optics (CGO) solutions

Set \( \mu := (1 - \sigma)(1 + \sigma)^{-1} \)
in the Beltrami equation

\[
\partial_z f_\mu = \mu \partial_z f_\mu.
\]

We look for solutions

\[
f_\mu(z, k) = e^{ikz} M_\mu(z, k) = e^{ikz} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right)
\]
as \( |z| \to \infty \). The complex \( k \) is a Fourier variable.

Also, we use \( u_j \) as in

\[
\nabla \cdot \sigma \nabla u_1(\cdot, k) = 0,
\]
\[
\nabla \cdot \sigma^{-1} \nabla u_2(\cdot, k) = 0.
\]

Define \( h_+ = \frac{1}{2}(f_\mu + f_{-\mu}) \)
and \( h_- = \frac{1}{2}(f_\mu - f_{-\mu}) \); then the solutions \( u_1(z, k) \) and \( u_2(z, k) \) are given by

\[
u_1 = h_+ - ih_-,
\]
\[
u_2 = i(h_+ + ih_-).
\]

The Astala-Päivärinta approach can recover discontinuous conductivities quite accurately.

Thank you!

Preprints available at www.siltanen-research.net

Forthcoming book: Mueller & S, Linear and nonlinear inverse problems with practical applications, SIAM
Regularization of nonlinear inverse problems is an active and challenging area of research.

There are two main approaches for regularizing nonlinear inverse problems:

1. Iterative regularization
   + Generic: numerically applicable to any inverse problem,
   + Quick to develop optimization-based solution software,
     o Rigorously defined only for almost linear inverse problems due to problems with local minima.

2. Tailored nonlinear regularization strategies:
   + Rigorous mathematical analysis available for algorithms,
   + Provides a link between several schools of research,
     o One method applies only to one inverse problem.
This is an ideal process for creating rigorous and useful computational inversion methods

1. Uniqueness analysis
2. Constructive reconstruction proof (ideal data)
3. Conditional stability (ideal data)
4. Characterization of the range of the forward map
5. Regularization strategy with a practical algorithm

This talk is all about showing how 1 and 2 and 3 can be used for achieving 5 in the case of electrical impedance tomography, a fundamental nonlinear ill-posed inverse problem.