Electrical impedance tomography, virtual X-rays, and machine learning

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Finland

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Outline

Electrical impedance tomography (EIT) and stroke

Complex geometric optics solutions for EIT

Virtual Hybrid Edge Detection (VHED)

Filtered back-projection theorem

Why machine learning?
Motivation of this study: imaging stroke with EIT

Ischemic stroke: low conductivity.
CT image from Jansen 2008

Hemorrhagic stroke: high conductivity.
CT image from Nakano et al. 2001

Same symptoms in both cases!

Difficulties: resistive skull layer and unknown background. However, see
- Romsauerova, McEwan, Horesh, Yerworth, Bayford & Holder 2006
- Shi, You, Xu, Wang, Fu, Liu & Dong 2009
- Bayford & Tizzard 2012
- Malone, Jehl, Arridge, Betcke & Holder 2014
- Boverman, Kao, Wang, Ashe, Davenport & Amm 2016
Brain EIT imaging is based on covering the head partly by electrodes.

<table>
<thead>
<tr>
<th>Tissue</th>
<th>Ω cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cortex</td>
<td>229</td>
</tr>
<tr>
<td>White matter</td>
<td>344</td>
</tr>
<tr>
<td>Blood</td>
<td>125</td>
</tr>
<tr>
<td>CS fluid</td>
<td>69</td>
</tr>
<tr>
<td>Scalp</td>
<td>490</td>
</tr>
<tr>
<td>Skull</td>
<td>6500</td>
</tr>
</tbody>
</table>

The current activity was initiated by Alex Ross from GE. He is a former student of David Isaacson.
The idea would be to equip every ambulance with an EIT device for classifying strokes.

In David Holder’s lab at UCL
We have a collaboration network in place for the stroke-EIT project.

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S (U Helsinki)

Finnish collaboration:
Stefan Björkman (U Helsinki)
Antti Hannukainen (Aalto U)
Nuutti Hyvönen (Aalto U)

International collaboration:
Melody Alsaker (Gonzaga U)
Sarah Hamilton (Marquette U)
Andreas Hauptmann (UCL)
Jennifer Mueller (CSU)
This is a joint work with

Allan Greenleaf, University of Rochester, NY, USA

Andreas Hauptmann, University College London

Matti Lassas, University of Helsinki, Finland

Minh Mach, University of Helsinki, Finland

Matteo Santacesaria, University of Helsinki, Finland

Gunther Uhlmann, University of Washington, USA

Toshiaki Yachimura, Tohoku University, Japan
We consider three simulated 2D stroke phantoms: here healthy brain
We consider three simulated 2D stroke phantoms: here ischemic stroke.
We consider three simulated 2D stroke phantoms: here hemorrhagic stroke.
New result: inverse scattering methods can transform EIT into “X-ray tomography”

Video:

https://www.youtube.com/watch?v=37yOCfBfRJk

[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]
New result: inverse scattering methods can transform EIT into “X-ray tomography”

[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]
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Why machine learning?
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón.

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{align*}
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
\left. u \right|_{\partial \Omega} &= f.
\end{align*}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_{\sigma} : f \mapsto \sigma \frac{\partial u}{\partial n}\big|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_{\sigma}$. It is a nonlinear and ill-posed inverse problem.
We use the Beltrami-type complex geometric optics (CGO) solutions

Set $\mu := (1 - \sigma)/(1 + \sigma)$. Write $f = u + iv$ and note that

$$\overline{\partial_z f_\mu} = \mu \overline{\partial_z f_\mu} \iff \nabla \cdot \sigma \nabla u = 0 \text{ and } \nabla \cdot \sigma^{-1} \nabla v = 0.$$  

The CGO solutions of [Astala-Päivärinta 2006] have the form

$$f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)),$$
$$f_{-\mu}(z, k) = e^{ikz}(1 + \omega^-(z, k)),$$

with the asymptotic condition

$$\omega^\pm(z, k) = O\left(\frac{1}{z}\right) \text{ as } |z| \to \infty.$$  

Here $ikz = i(k_1 + ik_2)(x + iy)$ and $\overline{\partial_z} = \frac{1}{2}(\partial_x + i\partial_y)$. 
This is a brief history of computational solution methods for the Beltrami CGO solutions

1987 Sylvester and Uhlmann: Introduction of CGO solutions
2000 S, Mueller and Isaacson: Numerical CGOs
2006 Astala and Päivärinta: Original Beltrami-type construction

2010 Astala, Mueller, Päivärinta and S: First numerical solution method

2011 Astala, Mueller, Päivärinta, Perämäki and S: Novel EIT reconstruction method

2012 Huhtanen and Perämäki: Preconditioned Krylov subspace method for real-linear systems

2014 Astala, Päivärinta, Reyes and S: Computational high-frequency experiments
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Why machine learning?
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

\[ f_\mu(z, k) = e^{i k z} (1 + \omega^+(z, k)), \]
write the complex parameter in the form \( k = \tau e^{i \phi} \) with \( \tau \in \mathbb{R} \). Denote \( \omega^+(z, \tau, e^{i \phi}) = \omega^+(z, k) \).

Fourier transform \( \omega^+(z, \tau, e^{i \phi}) \) in the \( \tau \) variable:

\[
\hat{\omega}^+(z, t, e^{i \phi}) = \mathcal{F}_{\tau \to t}(\omega^+(z, \tau, e^{i \phi})) = \int_{-\infty}^{\infty} e^{-it \tau} \omega^+(z, \tau, e^{i \phi}) \, d\tau.
\]

We call \( t \) the pseudo-time.
\[ \hat{\omega}^{+}(-1, 2t, 1) \]
\( \hat{\omega}^+(-1, 2t, 1) \) 

\[ [\hat{\omega}^+ - \hat{\omega}^-](-1, 2t, 1) \]
We can define an averaging operator

**Definition** (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018)
Define operator $T^\pm$ by complex contour integral:

$$T^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial\Omega} \hat{\omega}^\pm(z, t, e^{i\varphi}) dz.$$ 

In the case of $\Omega$ being the unit disc, we get

$$T^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\omega}^\pm(e^{i\gamma}, t, e^{i\varphi}) e^{i\gamma} d\gamma.$$
\( \hat{\omega}^+(-1, 2t, 1) \)  
\( [\hat{\omega}^+ - \hat{\omega}^-](-1, 2t, 1) \)  
\( [T^+ - T^-]_\mu(2t, 1) \)
Cauchy and Beurling transforms

Define the Cauchy and Beurling transforms by

\[ Pf(z) = \overline{\partial}^{-1} f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\lambda)}{\lambda - z} d\lambda_1 d\lambda_2, \]

\[ Sg(z) = \partial \overline{\partial}^{-1} g(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\lambda - z| > \varepsilon} \frac{g(\lambda)}{(\lambda - z)^2} d\lambda_1 d\lambda_2. \]

Also, set

\[ e_k(z) = \exp(i(kz + \overline{k}\bar{z})), \]

\[ \alpha(z, k) = -i\overline{k}e_{-k}(z)\mu(z), \]

\[ \nu(z, k) = e_{-k}(z)\mu(z), \]

and define the operator \( A \) by

\[ A := ( -\overline{\nu}S - \overline{\alpha}P ). \]
We introduce a new scattering series

Huhtanen and Perämäki (2012) modified the original construction of Astala and Päivärinta (2006) for computational purposes. We use the 2012 technique for the construction of a novel scattering series

\[ \omega = \sum_{n=1}^{\infty} \omega_n, \]

where \( A := (-\bar{\nu}S - \bar{\alpha}P) \) and

\[ \omega_n = -\bar{\partial}_z^{-1} u_n, \quad u_n = -A\bar{u}_{n-1}, \quad u_1 = -\bar{\alpha}. \]

The single scattering term \( \omega_1 = \bar{\partial}_z^{-1} \alpha \) determines singularities of \( \mu \).

The terms \( \omega_n \) with \( n > 1 \) arise from multiple scattering.
Let us consider a rotationally symmetric conductivity with jump along the circle $|z| = 0.2$. 

$k = \tau e^{i\varphi}$, $\varphi = 0$.
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
Here are the two lowest-order odd terms in the scattering series, with subtraction

\[ [T_{1+} - T_{1-}] \mu \quad [T_{3+} - T_{3-}] \mu \]
Detail from the previous slide, with 70-fold magnification of the function inside green circle.

$T_1 \quad T_3$
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Why machine learning?
Recovery by “filtered back-projection”

**Theorem.** (Greenleaf, Lassas, Santacesaria, S and Uhlmann) Define averaged operators $T_j^\pm$ for $j = 1, 2, 3, \ldots$ by the complex contour integral:

$$T_j^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial \Omega} \hat{\omega}_j^\pm(z, t, e^{i\varphi}) dz,$$

Then we have a filtered back-projection formula

$$(−\Delta)^{-1/2}(T_1^\pm)^* T_1^\pm \mu = \mu.$$
Simple example of tomographic imaging with a double-disc target

https://youtu.be/5DUGTXd26nA
Inverse problem: recover the unknown target from X-ray data collected all around it

https://youtu.be/YhClb0MaB70
We can back-project the measured data into the image, integrating over all directions

https://youtu.be/5DUGTXd26nA
Final FBP reconstruction involves filtering on top of the back-projection.

\[ \text{Multiplication by } |\xi| \] (Calderón’s operator)

\[ \text{FFT} \quad \rightarrow \quad \text{IFFT} \]
FBP-type reconstruction algorithm for EIT

**Step 1.** Given the measurement $\Lambda_\sigma$, follow [Astala, Mueller, Päivärinta, Perämäki & S 2011] to compute both $\omega^+(x, k)$ and $\omega^-(x, k)$ for $x \in \partial \Omega$ by solving the boundary integral equation derived in [Astala & Päivärinta 2006].

*Note:* In practice this can only be done in a disc $|k| < R$ with $R$ depending on measurement noise amplitude.

**Step 2.** Write $k = \tau e^{i\varphi}$ and compute the partial Fourier transform to get $\hat{\omega}^\pm(z, t, e^{i\varphi})$.

*Note:* In practice the Fourier transform needs to be windowed.

**Step 3.** Reconstruct $\sigma = (\mu - 1)/(\mu + 1)$ approximately as $(\tilde{\mu} - 1)/(\tilde{\mu} + 1)$ using formula $\tilde{\mu} = (\tilde{\mu}^+ - \tilde{\mu}^-)/2$ with

$$
\tilde{\mu}^\pm := \Delta^{-1/2}(T_1^\pm)^* T^\pm \mu.
$$
Conductivity Filtered back-projection
Conductivity Filtered back-projection Λ-tomography
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Why machine learning?
We can see the difference in conductivity reflected in the VHED projections (blue and red graphs)
Given unrealistic-precision EIT measurements on full boundary we can classify the stroke easily.
Practical EIT measurements blur the information due to heavily windowed Fourier transform
Hemorrhagic

Real part of $d(t)$

Real part of $d''(t)$

Real part of $v(t)$

Simulations by Antti Hannukainen and Minh Mach
Simulations by Antti Hannukainen and Minh Mach
Thank you for your attention!
We consider exponentially behaving Complex Geometric Optics (CGO) solutions

Denote \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( k = i\tau \theta \) where

\[
\theta = \theta_1 + i\theta_2 \in \mathbb{C} \quad \text{with} \quad |\theta| = 1.
\]

Let \( z = x_1 + ix_2 \in \mathbb{C} \) and

\[
\eta = \eta_R + i\eta_I = (\theta_1 + i\theta_2, -\theta_2 + i\theta_1) \in \mathbb{C}^2,
\]

so that \( z\theta = x_1\theta_1 - x_2\theta_2 + i(x_1\theta_2 + x_2\theta_1) = x \cdot \eta \). Note that \( \eta \cdot \eta = 0 \). We consider solutions of the conductivity equation

\[
\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \ \Omega,
\]

with a strictly positive conductivity \( \sigma \in L^\infty(\Omega) \), of the form

\[
u(x) = e^{i\tau \theta z} w(x, \tau) = e^{i\tau \eta \cdot x} w(x, \tau).
\]
Since \( u(x) = e^{i\tau \eta \cdot x} w(x, \tau) \) satisfies the conductivity equation,

\[
0 = \frac{1}{\sigma(x)} \nabla \cdot (\sigma(x) \nabla u(x)) \\
= (\Delta + \frac{1}{\sigma}(\nabla \sigma) \cdot \nabla)(e^{i\tau \eta \cdot x} w(x, \tau)) \\
= \left( \Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta) w(x, \tau) \right) e^{i\tau \eta \cdot x}.
\]

Hence, we have

\[
\Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta) w(x, \tau) = 0.
\]
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

Let $\hat{w}(x, t)$ be the Fourier transform of $w(x, \tau)$ in the $\tau$ variable:

$$\hat{w}(x, t) = \mathcal{F}w(x, t) = \int_{\mathbb{R}} e^{-it\tau} w(x, \tau) \, d\tau.$$ 

We call $t$ the *pseudo-time* corresponding to complex frequency $\tau$. Then equation

$$\Delta w(x, \tau) + 2i\tau\eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot \left( \nabla + i\tau\eta \right) w(x, \tau) = 0$$

yields

$$\Delta \hat{w}(x, t) + 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot \left( \nabla + \eta \frac{\partial}{\partial t} \right) \hat{w}(x, t) = 0.$$
Complex principal type operator leads to singularities propagating along leaves

In the equation

\[ \Delta \hat{w}(x, t) - 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \frac{1}{\sigma}(\nabla\sigma) \cdot (\nabla - \eta \frac{\partial}{\partial t}) \hat{w}(x, t) = 0 \]

the principal part is

\[ \Delta - 2\eta \frac{\partial}{\partial t} \cdot \nabla, \]

which is a complex principal type operator in the sense of Duistermaat and Hörmander.

For a real principal type operator the characteristic singularities propagate along one-dimensional rays. For instance, for the wave equation the light-like singularities propagate along light rays.

For a complex principal type operator the characteristic singularities propagate along two-dimensional surfaces called leaves.
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
Practical problems in applying VHED

VHED works with ideal simulated data and simple digital phantoms. However, these issues must be solved before it can be applied to stroke classification:

Data is noisy. We know the Fourier transform of the desired function only in an interval $[-R, R]$ with $R \approx 4$.

Anatomy is complicated. Need to be test with realistic phantoms.

We can only measure on a part of the boundary. Some progress is reported in [Hauptmann, Santacesaria and S 2017].

Measurements are done using a finite number of electrodes. Recovering CGO solutions from electrode data needs new research.

People are three-dimensional. VHED needs to be extended to 3D.