Classifying stroke from electric boundary data by nonlinear Fourier analysis

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Finland

- University of Eastern Finland
- Oulu University
- University of Jyväskylä
- Tampere University of Technology
- Finnish Meteorological Institute
- Aalto University
Outline

Electrical impedance tomography (EIT)

Complex geometric optics (CGO) solutions, D-bar method

Application of EIT to stroke

Virtual Hybrid Edge Detection (VHED)
   The scattering series
   Filtered back-projection theorem

Combining machine learning with VHED
This section concentrates on applications of EIT to chest imaging

Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos \theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos 2\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes.
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Apply current pattern \( \cos 3\theta \)

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Apply current pattern $\cos 4\theta$

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern \( \cos 5\theta \)

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos 6\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos 7\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 8\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 9\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Apply current pattern $\cos 10\theta$

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 11\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos 12\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern $\cos 13\theta$

 Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes.
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern \( \cos 14\theta \)

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 15\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 16\theta$

Phantom and data: Jon Newell, Rensselaer Polytechnic Institute

Measure the resulting voltages at all 32 electrodes
Here is a reconstruction of the conductivity, computed using a nonlinear Fourier transform.

Saline and agar phantom

[Isaacson, Mueller, Newell & S 2004]
[Montoya 2012]

D-bar reconstruction

Cut-off frequency $R = 4$
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón.

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \to \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$  

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases} 
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = f.
\end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_{\sigma} : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega}.$$  

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_{\sigma}$. It is a nonlinear and ill-posed inverse problem.
Why is the forward map \( F : \sigma \mapsto \Lambda_\sigma \) nonlinear?

Define a quadratic form \( P_\sigma \) for functions \( f : \partial \Omega \to \mathbb{R} \) by

\[
P_\sigma(f) = \int_\Omega \sigma |\nabla u|^2 \, dz, \tag{1}
\]

where \( u \) is the solution of the Dirichlet problem

\[
\begin{cases}
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
u|_{\partial \Omega} &= f.
\end{cases}
\]

Now the map \( \sigma \mapsto P_\sigma \) is nonlinear because \( u \) depends on \( \sigma \) in (1). Physically, \( P_\sigma(f) \) is the power needed for maintaining the voltage potential \( f \) on the boundary \( \partial \Omega \). Integrate by parts in (1):

\[
P_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) \, ds = \int_{\partial \Omega} f \left( \Lambda_\sigma f \right) \, ds.
\]

Thus the map \( \sigma \mapsto \Lambda_\sigma \) cannot be linear in \( \sigma \).
We define a matrix approximation for the Dirichlet-to-Neumann map $\Lambda_{\sigma}$

We define Fourier basis functions $\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}$ on the unit circle and form the DN matrix in this way:

\[
\begin{pmatrix}
\langle \Lambda_{\sigma} \varphi_{-2}, \varphi_{-2} \rangle & \langle \Lambda_{\sigma} \varphi_{-1}, \varphi_{-2} \rangle & 0 & \langle \Lambda_{\sigma} \varphi_{1}, \varphi_{-2} \rangle & \langle \Lambda_{\sigma} \varphi_{2}, \varphi_{-2} \rangle \\
\langle \Lambda_{\sigma} \varphi_{-2}, \varphi_{-1} \rangle & \langle \Lambda_{\sigma} \varphi_{-1}, \varphi_{-1} \rangle & 0 & \langle \Lambda_{\sigma} \varphi_{1}, \varphi_{-1} \rangle & \langle \Lambda_{\sigma} \varphi_{2}, \varphi_{-1} \rangle \\
0 & 0 & 0 & 0 & 0 \\
\langle \Lambda_{\sigma} \varphi_{-2}, \varphi_{1} \rangle & \langle \Lambda_{\sigma} \varphi_{-1}, \varphi_{1} \rangle & 0 & \langle \Lambda_{\sigma} \varphi_{1}, \varphi_{1} \rangle & \langle \Lambda_{\sigma} \varphi_{2}, \varphi_{1} \rangle \\
\langle \Lambda_{\sigma} \varphi_{-2}, \varphi_{2} \rangle & \langle \Lambda_{\sigma} \varphi_{-1}, \varphi_{2} \rangle & 0 & \langle \Lambda_{\sigma} \varphi_{1}, \varphi_{2} \rangle & \langle \Lambda_{\sigma} \varphi_{2}, \varphi_{2} \rangle 
\end{pmatrix}
\]
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data.

\[ \sigma_1 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
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Combining machine learning with VHED
This part is a joint work with

David Isaacson, Rensselaer Polytechnic Institute, USA

Kim Knudsen, Technical University of Denmark

Matti Lassas, University of Helsinki, Finland

Jon Newell, Rensselaer Polytechnic Institute, USA

Jennifer Mueller, Colorado State University, USA
There exists a nonlinear Fourier transform adapted to electrical impedance tomography.
The nonlinear Fourier transform can be recovered from infinite-precision EIT measurements

\[ \Lambda \sigma \]

[BIE] Nonlinear IFFT

[Nachman 1996]
Measurement noise prevents the recovery of the nonlinear Fourier transform at high frequencies.
We truncate away the bad part in the transform; this is a nonlinear low-pass filter.
The D-bar method is a regularization strategy for reconstructing the full conductivity distribution.

Practical measurement

BIE

Lowpass

Nonlinear IFFT

[S, Mueller & Isaacson 2000]
[Knudsen, Lassas, Mueller & S 2009]
Infinite-precision data:

Solve boundary integral equation

\[ \psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]

for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:

\[ t(k) = \int_{\partial \Omega} e^{ikz}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)}\mu(z, k) \]

with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

Practical data:

Solve boundary integral equation

\[ \psi^\varepsilon(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\varepsilon - \Lambda_1)\psi^\varepsilon \]

for all \( 0 < |k| < R = -\frac{1}{10} \log \varepsilon \).

For \( |k| \geq R \) set \( t^\varepsilon_R(k) = 0 \). For \( |k| < R \)

\[ t^\varepsilon_R(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation

\[ \frac{\partial}{\partial k} \mu^\varepsilon_R(z, k) = \frac{t^\varepsilon_R(k)}{4\pi k} e^{-i(kz + \bar{k}z)}\mu^\varepsilon_R(z, k) \]

with \( \mu^\varepsilon_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma_{1/R(\varepsilon)}(\Lambda_\varepsilon) := (\mu^\varepsilon_R(z, 0))^2 \).
We define spaces for our regularization strategy

Model space $X = L^\infty(\Omega)$

Data space $Y$

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ with

- $\|\sigma\|_{C^2(\overline{\Omega})} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators $A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying

- $A(1) = 0$,
- $\int_{\partial\Omega} A(f) \, ds = 0$. 

Nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

Fix a conductivity $\sigma \in D(F)$. Assume given noisy data $\Lambda^\varepsilon_\sigma$ satisfying

$$\|\Lambda^\varepsilon_\sigma - \Lambda_\sigma\|_Y \leq \varepsilon.$$ 

Then $\Gamma_\alpha$ with the choice

$$R(\varepsilon) = -\frac{1}{10} \log \varepsilon, \quad \alpha(\varepsilon) = \frac{1}{R(\varepsilon)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\varepsilon)}(\Lambda^\varepsilon_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \varepsilon)^{-1/14}.$$
Here are the D-bar reconstructions from simulated EIT data using frequency cutoff $R = 4$
The difference image shows clearly where the two patients are not the same.
Medical application of EIT and the D-bar method: quantifying air-trapping in cystic fibrosis patients

All results on this slide are from Jennifer Mueller’s group at Colorado State University.

Images: ventilation-perfusion index maps, computed from three subjects at Children’s Hospital Colorado using EIT and the D-bar method.

Dark blue regions are well-perfused but poorly ventilated.

Radiologist’s report for Subject B: extensive regions of air trapping, regional to the lung areas affected by plugging, approximately 50% of both lungs.

Healthy control
Average index 0.46

CF Subject A
Average index 0.34

CF Subject B
Average index 0.10
D-bar images can be sharpened by Deep Learning

[Hamilton & Hauptmann 2017]
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Motivation of this study: imaging stroke with EIT

**Ischemic stroke:** low conductivity. 
CT image from Jansen 2008

**Hemorrhagic stroke:** high conductivity. 
CT image from Nakano et al. 2001

Same symptoms in both cases!

Difficulties: resistive skull layer and unknown background. However, see
- Romsauerova, McEwan, Horesh, Yerworth, Bayford & Holder 2006
- Shi, You, Xu, Wang, Fu, Liu & Dong 2009
- Bayford & Tizzard 2012
- Malone, Jehl, Arridge, Betcke & Holder 2014
- Boverman, Kao, Wang, Ashe, Davenport & Amm 2016
Brain EIT imaging is based on covering the head partly by electrodes

<table>
<thead>
<tr>
<th>Tissue</th>
<th>$\Omega\text{cm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cortex</td>
<td>229</td>
</tr>
<tr>
<td>White matter</td>
<td>344</td>
</tr>
<tr>
<td>Blood</td>
<td>125</td>
</tr>
<tr>
<td>CS fluid</td>
<td>69</td>
</tr>
<tr>
<td>Scalp</td>
<td>490</td>
</tr>
<tr>
<td>Skull</td>
<td>6500</td>
</tr>
</tbody>
</table>
The idea would be to equip every ambulance with an EIT device for classifying strokes.
Another important application of stroke-EIT is monitoring a patient in an intensive care unit.
We have a collaboration network in place for the stroke-EIT project

Project funded for 2017–2020

- Jari Hyttinen & Antti Paldanius (U Tampere)
- Ville Kolehmainen, Asko Hänninen & Jussi Toivanen (U Eastern Finland)
- S, Matti Lassas, Minh Mach & Rashmi Murthy (U Helsinki)

Finnish collaboration:
Stefan Björkman (U Helsinki)
Valentina Candiani (Aalto U)
Antti Hannukainen (Aalto U)
Nuuutti Hyvönen (Aalto U)

International collaboration:
Juan Pablo Agnelli (U Córdoba)
Melody Alsaker (Gonzaga U)
Aynur Çöl (Sinop U)
Sarah Hamilton (Marquette U)
Andreas Hauptmann (UCL)
Jennifer Mueller (CSU),
Toshiaki Yachimura (Tohoku)
We can test our algorithms with realistic head phantoms

Ville Kolehmainen, Asko Hänninen, Tuomo Savolainen, Jussi Toivanen
University of Eastern Finland
We consider three simulated 2D stroke phantoms: here healthy brain
We consider three simulated 2D stroke phantoms: here ischemic stroke
We consider three simulated 2D stroke phantoms: here hemorrhagic stroke.
New result: inverse scattering methods can transform EIT into “X-ray tomography”

Video:

https://www.youtube.com/watch?v=37yOcfBfRJk

[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]
New result: inverse scattering methods can transform EIT into “X-ray tomography”

[Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018]
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Combining machine learning with VHED
The results in this part are a joint work with

Allan Greenleaf, University of Rochester, NY, USA

Matti Lassas, University of Helsinki, Finland

Matteo Santacesaria, University of Helsinki, Finland

Gunther Uhlmann, University of Washington, USA
We consider exponentially behaving Complex Geometric Optics (CGO) solutions

Denote \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( k = \tau \theta \) where

\[
\theta = \theta_1 + i\theta_2 \in \mathbb{C} \quad \text{with} \ |\theta| = 1.
\]

Let \( z = x_1 + ix_2 \in \mathbb{C} \) and

\[
\eta = \eta_R + i\eta_I = (\theta_1 + i\theta_2, -\theta_2 + i\theta_1) \in \mathbb{C}^2,
\]

so that \( z\theta = x_1\theta_1 - x_2\theta_2 + i(x_1\theta_2 + x_2\theta_1) = x \cdot \eta \). Note that \( \eta \cdot \eta = 0 \). We consider solutions of the conductivity equation

\[
\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \ \Omega,
\]

with a strictly positive conductivity \( \sigma \in L^\infty(\Omega) \), of the form

\[
u(x) = e^{i\tau\theta z} w(x, \tau) = e^{i\tau \eta \cdot x} w(x, \tau).\]
Since \( u(x) = e^{i\tau \eta \cdot x}w(x, \tau) \) satisfies the conductivity equation,

\[
0 = \frac{1}{\sigma(x)} \nabla \cdot (\sigma(x) \nabla u(x))
\]

\[
= (\Delta + \frac{1}{\sigma}(\nabla \sigma) \cdot \nabla)(e^{i\tau \eta \cdot x}w(x, \tau))
\]

\[
= \left( \Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta)w(x, \tau) \right) e^{i\tau \eta \cdot x}.
\]

Hence, we have

\[
\Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta)w(x, \tau) = 0.
\]
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

Let \( \hat{w}(x, t) \) be the Fourier transform of \( w(x, \tau) \) in the \( \tau \) variable:

\[
\hat{w}(x, t) = \mathcal{F}w(x, t) = \int_{\mathbb{R}} e^{-it\tau} w(x, \tau) d\tau.
\]

We call \( t \) the *pseudo-time* corresponding to complex frequency \( \tau \). Then equation

\[
\Delta w(x, \tau) + 2i\tau \eta \cdot \nabla w(x, \tau) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + i\tau \eta)w(x, \tau) = 0
\]

yields

\[
\Delta \hat{w}(x, t) + 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \left( \frac{1}{\sigma} \nabla \sigma \right) \cdot (\nabla + \eta \frac{\partial}{\partial t})\hat{w}(x, t) = 0.
\]
Complex principal type operator leads to singularities propagating along leaves

In the equation

\[ \Delta \hat{w}(x, t) - 2\eta \frac{\partial}{\partial t} \cdot \nabla \hat{w}(x, t) + \frac{1}{\sigma} (\nabla \sigma) \cdot (\nabla - \eta \frac{\partial}{\partial t}) \hat{w}(x, t) = 0 \]

the principal part is

\[ \Delta - 2\eta \frac{\partial}{\partial t} \cdot \nabla, \]

which is a complex principal type operator in the sense of Duistermaat and Hörmander.

For a real principal type operator the characteristic singularities propagate along one-dimensional rays. For instance, for the wave equation the light-like singularities propagate along light rays.

For a complex principal type operator the characteristic singularities propagate along two-dimensional surfaces called leaves.
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
We use the Beltrami-type complex geometric optics (CGO) solutions

Set $\mu := (1 - \sigma)/(1 + \sigma)$. Write $f = u + iv$ and note that

$$\bar{\partial}_z f_\mu = \mu \bar{\partial}_z f_\mu \iff \nabla \cdot \sigma \nabla u = 0 \text{ and } \nabla \cdot \sigma^{-1} \nabla v = 0.$$ 

The CGO solutions of [Astala-Päivärinta 2006] have the form

$$f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k)),$$
$$f_{-\mu}(z, k) = e^{ikz}(1 + \omega^-(z, k)),$$

with the asymptotic condition

$$\omega^\pm(z, k) = \mathcal{O}\left(\frac{1}{|z|}\right) \text{ as } |z| \to \infty.$$ 

Here $ikz = i(k_1 + ik_2)(x + iy)$ and $\bar{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$. 
This is a brief history of computational solution methods for the Beltrami CGO solutions

1987 Sylvester and Uhlmann: Introduction of CGO solutions
2000 S, Mueller and Isaacson: Numerical CGOs
2006 Astala and Päivärinta: Original Beltrami-type construction

2010 Astala, Mueller, Päivärinta and S: First numerical solution method

2011 Astala, Mueller, Päivärinta, Perämäki and S: Novel EIT reconstruction method

2012 Huhtanen and Perämäki: Preconditioned Krylov subspace method for real-linear systems

2014 Astala, Päivärinta, Reyes and S: Computational high-frequency experiments
New trick: apply one-dimensional Fourier transform to the complex spectral parameter

In $f_\mu(z, k) = e^{ikz}(1 + \omega^+(z, k))$, write the complex parameter in the form $k = \tau e^{i\varphi}$ with $\tau \in \mathbb{R}$. Denote $\omega^+(z, \tau, e^{i\varphi}) = \omega^+(z, k)$.

Fourier transform $\omega^+(z, \tau, e^{i\varphi})$ in the $\tau$ variable:

$$\hat{\omega}^+(z, t, e^{i\varphi}) = \mathcal{F}_{\tau \to t}(\omega^+(z, \tau, e^{i\varphi})) = \int_{-\infty}^{\infty} e^{-it\tau} \omega^+(z, \tau, e^{i\varphi}) d\tau.$$

We call $t$ the pseudo-time.
We define an averaging operator

**Definition** (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018)

Define operator $T^\pm$ by complex contour integral:

$$T^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \hat{\omega}^\pm(e^{i\gamma}, t, e^{i\varphi}) e^{i\gamma} d\gamma.$$
\[ \hat{\omega}^+(1, 2t, 1) \quad [\hat{\omega}^+ - \hat{\omega}^-](1, 2t, 1) \quad [T^+ - T^-]_\mu(2t, 1) \]

Simulation by Rashmi Murthy
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Cauchy and Beurling transforms

Define the Cauchy and Beurling transforms by

\[ Pf(z) = \partial^{-1} f(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\lambda)}{\lambda - z} d\lambda_1 d\lambda_2, \]
\[ Sg(z) = \partial \bar{\partial}^{-1} g(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\lambda - z| > \varepsilon} \frac{g(\lambda)}{(\lambda - z)^2} d\lambda_1 d\lambda_2. \]

Also, set

\[ e_k(z) = \exp(i(kz + \bar{k}\bar{z})), \]
\[ \alpha(z, k) = -i\bar{k}e_{-k}(z)\mu(z), \]
\[ \nu(z, k) = e_{-k}(z)\mu(z), \]

and define the operator \( A \) by

\[ A := (-\nu S - \bar{\alpha} P). \]
We introduce a new scattering series

Huhtanen and Perämäki (2012) modified the original construction of Astala and Päivärinta (2006) for computational purposes. We use the 2012 technique for the construction of a novel scattering series

\[ \omega = \sum_{n=1}^{\infty} \omega_n, \]

where \( A := (-\nu S - \alpha P) \) and

\[ \omega_n = -\overline{\partial}_z^{-1} u_n, \quad u_n = -Au_{n-1}, \quad u_1 = -\alpha. \]

The single scattering term \( \omega_1 = \overline{\partial}_z^{-1} \alpha \) determines singularities of \( \mu \).

The terms \( \omega_n \) with \( n > 1 \) arise from multiple scattering.
Let us consider a rotationally symmetric conductivity with jump along the circle $|z| = 0.2$. 
We use propagation and reflection of singularities along leaves for detecting inclusions.

Here the magenta plane wave hits the blue surface, producing light blue reflected waves.
Here are the two lowest-order odd terms in the scattering series, with subtraction

\[ [T_3^+-T_3^-] \mu \quad [T_1^+-T_1^-] \mu \]
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Recovery by complex “filtered back-projection”

**Theorem.** (Greenleaf, Lassas, Santacesaria, S and Uhlmann 2018)
Define averaged operators $T_j^\pm$ for $j = 1, 2, 3, \ldots$ by the complex contour integral:

$$T_j^\pm \mu(t, e^{i\varphi}) = \frac{1}{2\pi i} \int_{\partial \Omega} \hat{\omega}_j^\pm(z, t, e^{i\varphi}) dz,$$

Then we have a filtered back-projection formula

$$(-\Delta)^{-1/2}(T_1^\pm)^* T_1^\pm \mu = \mu.$$

(Compare to X-ray FBP formula $(-\Delta)^{1/2}(\mathcal{R})^* \mathcal{R}f = f$.)
Simple example of tomographic imaging with a double-disc target

https://youtu.be/5DUGTXd26nA
We can back-project the measured data into the image, integrating over all directions
Final FBP reconstruction involves filtering on top of the back-projection.

\[ \text{Multiplication by } |\xi| \]
(Calderón’s operator)
FBP-type reconstruction algorithm for EIT

Step 1. Given the measurement $\Lambda_{\sigma}$, follow [Astala, Mueller, Päivärinta, Perämäki & S 2011] to compute both $\omega^+(x, k)$ and $\omega^-(x, k)$ for $x \in \partial \Omega$ by solving the boundary integral equation derived in [Astala & Päivärinta 2006].

Note: In practice this can only be done in a disc $|k| < R$ with $R$ depending on measurement noise amplitude.

Step 2. Write $k = \tau e^{i\varphi}$ and compute the partial Fourier transform to get $\hat{\omega}^\pm(z, t, e^{i\varphi})$.

Note: In practice the Fourier transform needs to be windowed.

Step 3. Reconstruct $\sigma = (\mu - 1)/(\mu + 1)$ approximately as $(\tilde{\mu} - 1)/(\tilde{\mu} + 1)$ using formula $\tilde{\mu} = (\tilde{\mu}^+ - \tilde{\mu}^-)/2$ with

$$\tilde{\mu}^\pm := \Delta^{-1/2}(T_1^\pm)^* T^\pm \mu.$$
Conductivity Filtered back-projection “Λ-tomography”
Outline

Electrical impedance tomography (EIT)

Complex geometric optics (CGO) solutions, D-bar method

Application of EIT to stroke

Virtual Hybrid Edge Detection (VHED)
  The scattering series
  Filtered back-projection theorem

Combining machine learning with VHED
The results in this part are a joint work with

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Aynur Çöl, Sinop University, Turkey
Allan Greenleaf, University of Rochester, NY, USA
Matti Lassas, University of Helsinki, Finland
Minh Mach, University of Helsinki, Finland
Rashmi Murthy, University of Helsinki, Finland
Matteo Santacesaria, University of Helsinki, Finland
Gunther Uhlmann, University of Washington, USA
Toshiaki Yachimura, Tohoku University, Japan
We can see the difference in conductivity reflected in the VHED projections (blue and red graphs).
Given unrealistic-precision EIT measurements on full boundary we can classify the stroke easily.
Practical EIT measurements blur the information due to heavily windowed Fourier transform
Remember the noise-induced stable and unstable parts of the nonlinear frequency domain
We need to window the 1D Fourier transform. Window width depends on noise amplitude.

\[ \Re(\omega^+(1, \tau, 1)) \]

- low-noise \( \Lambda_\sigma \)
- high-noise \( \Lambda_\sigma \)
Practical EIT measurements blur the information due to heavily windowed Fourier transform

\[
\frac{1}{2\pi} \int_{0}^{2\pi} (\hat{\omega}_a^+ (e^{i\gamma}, t, 1) - \hat{\omega}_a^- (e^{i\gamma}, t, 1)) e^{i\gamma} d\gamma
\]

\[
\hat{\omega}_a^\pm (z, t, 1) := \int_{-4}^{4} e^{-a\tau^2} e^{-it\tau} \omega^\pm (z, \tau, 1) d\tau
\]
Perhaps machine learning will help us?
We simulate a set of 5000 conductivities with ischemic/hemorrhagic stroke on right hemisphere
We simulate a set of 5000 conductivities with ischemic/hemorrhagic stroke on right hemisphere.
We simulate a set of 5000 conductivities with ischemic/hemorrhagic stroke on right hemisphere.
The conductivities have random parameters

- **Boundary**
  \[ \sigma(x) = 1 \]

- **Skull**
  \[ \sigma(x) \in [0.45, 0.55] \]

- **Stroke**
  \[ \sigma(x) \in [3, 4] \]

- **Background**
  \[ \sigma(x) \in [0.8, 1.1] \]
Results with Fully Connected Neural Networks

The training data consists of 4000 circular inclusion and is then tested on 2000 elliptic inclusions using FCNN.

<table>
<thead>
<tr>
<th>Noise $\delta$</th>
<th>DN maps</th>
<th>VHED profiles</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$10^{-3}$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>0.8563</td>
<td>0.8560</td>
</tr>
<tr>
<td></td>
<td>0.7750</td>
<td>1</td>
</tr>
<tr>
<td>Specificity</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Accuracy</td>
<td>0.9252</td>
<td>0.9250</td>
</tr>
<tr>
<td></td>
<td>0.8829</td>
<td>1</td>
</tr>
</tbody>
</table>

[Agnelli, Çöl, Murthy and Siltanen, unpublished results]
Thank you for your attention!
Links to open computational resources

Open EIT datasets:

- Finnish Inverse Problems Society (FIPS) dataset page

Reconstruction algorithms: FIPS Computational Blog

- The D-bar Method for EIT—Simulated Data
- The D-bar Method for EIT—Experimental Data

For these slides see: http://www.siltanen-research.net/talks.html
Practical challenges in applying VHED

VHED works with ideal simulated data and simple digital phantoms. However, these issues must be solved before it can be applied to stroke classification:

**Data is noisy.** We know the Fourier transform of the desired function only in an interval $[-R, R]$ with $R \approx 5$.

**Anatomy is complicated.** Need to be tested with realistic phantoms.

**We can only measure on a part of the boundary.** Some progress is reported in [Hauptmann, Santacesaria and S 2017].

**Measurements are done using a finite number of electrodes.** Recovering CGO solutions from electrode data needs new research.

**People are three-dimensional.** VHED needs to be extended to 3D.