Total variation regularized X-ray tomography with a multiresolution parameter choice method

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http://wiki.helsinki.fi/display/inverse/Home
This is a joint work with

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Matti Lassas, University of Helsinki, Finland

Esa Niemi, University of Helsinki, Finland

Kati Niinimäki, Royal Institute of Technology (KTH), Sweden
Outline

Total variation regularized tomography

A multiresolution parameter choice method for TV

How about theory?
Construction of the sinogram

Angle of X-rays: 3.0 degrees
Construction of the sinogram

Angle of X-rays: 12.2 degrees
Construction of the sinogram

Angle of X-rays: 21.5 degrees
Construction of the sinogram

Angle of X-rays: 30.7 degrees
Construction of the sinogram

Angle of X-rays: 39.9 degrees
Construction of the sinogram

Angle of X-rays: 49.2 degrees
Construction of the sinogram

Angle of X-rays: 58.4 degrees
Construction of the sinogram

Angle of X-rays: 67.6 degrees
Construction of the sinogram

Angle of X-rays: 76.8 degrees
Construction of the sinogram

Angle of X-rays: 86.1 degrees
Construction of the sinogram

Angle of X-rays: 95.3 degrees
Construction of the sinogram

Angle of X-rays: 104.5 degrees
Construction of the sinogram

Angle of X-rays: 113.8 degrees
Construction of the sinogram

Angle of X-rays: 123.0 degrees
Construction of the sinogram

Angle of X-rays: 132.2 degrees
Construction of the sinogram

Angle of X-rays: 141.5 degrees
Construction of the sinogram

Angle of X-rays: 150.7 degrees
Construction of the sinogram

Angle of X-rays: 159.9 degrees
Construction of the sinogram

Angle of X-rays: 169.2 degrees
Construction of the sinogram

Angle of X-rays: 178.4 degrees
Construction of the sinogram

Angle of X-rays: 187.6 degrees
Construction of the sinogram

Angle of X-rays: 196.8 degrees
Construction of the sinogram

Angle of X-rays: 206.1 degrees
Construction of the sinogram

Angle of X-rays: 215.3 degrees
Construction of the sinogram

Angle of X-rays: 224.5 degrees
Construction of the sinogram

Angle of X-rays: 233.8 degrees
Construction of the sinogram

Angle of X-rays: 243.0 degrees
Construction of the sinogram

Angle of X-rays: 252.2 degrees
Construction of the sinogram

Angle of X-rays: 261.5 degrees
Construction of the sinogram

Angle of X-rays: 270.7 degrees
Construction of the sinogram

Angle of X-rays: 279.9 degrees
Construction of the sinogram

Angle of X-rays: 289.2 degrees
Construction of the sinogram

Angle of X-rays: 298.4 degrees
Construction of the sinogram

Angle of X-rays: 307.6 degrees
Construction of the sinogram

Angle of X-rays: 316.8 degrees
Construction of the sinogram

Angle of X-rays: 326.1 degrees
Construction of the sinogram

Angle of X-rays: 335.3 degrees
Construction of the sinogram

Angle of X-rays: 344.5 degrees
Construction of the sinogram

Angle of X-rays: 353.8 degrees
We have object and data for the inverse problem

\[ f \in \mathbb{R}^{32 \times 32} \]

\[ Af \in \mathbb{R}^{49 \times 39} \]
Illustration of the ill-posedness of tomography

Difference 0.00254
Illustration of the ill-posedness of tomography

Difference 0.00124
Naive reconstruction using the Moore-Penrose pseudoinverse; data has 0.1% relative noise

Original phantom, values between zero (black) and one (white)

Naive reconstruction with minimum $-14.9$ and maximum $18.5$
Standard Tikhonov regularization

\[
\arg \min_{f \in \mathbb{R}^n} \left\{ \| Af - m \|_2^2 + \alpha \| f \|_2^2 \right\}
\]

Original phantom

Reconstruction

Relative square norm error 35%
Constrained Tikhonov regularization

\[
\arg \min_{f \in \mathbb{R}^n_+} \left\{ \|Af - m\|_2^2 + \alpha \|f\|_2^2 \right\}
\]

Original phantom

Reconstruction
Relative square norm error 13%
Constrained TV regularization

$$\arg \min_{f \in \mathbb{R}_+^n} \left\{ \| Af - m \|_2^2 + \alpha \| \nabla f \|_1 \right\}$$

Original phantom

TV regularized reconstruction
Relative square norm error 3\%
This is Professor Keijo Hämäläinen’s X-ray lab
We collected X-ray projection data of a walnut from 1200 directions

Laboratory and data collection by Keijo Hämäläinen and Aki Kallonen, University of Helsinki.

The data is openly available at http://fips.fi/dataset.php, thanks to Esa Niemi and Antti Kujanpää
Reconstructions of a 2D slice through the walnut using filtered back-projection (FBP)

FBP with comprehensive data (1200 projections)

FBP with sparse data (20 projections)
Sparse-data reconstruction of the walnut using non-negative Tikhonov regularization

\[ \arg\min_{f \in \mathbb{R}_+^n} \left\{ \| Af - m \|_2^2 + \alpha \| f \|_2^2 \right\} \]
Sparse-data reconstruction of the walnut using non-negative total variation regularization

Filtered back-projection

Constrained TV regularization
\[
\arg\min_{f \in \mathbb{R}_+^n} \left\{ \| Af - m \|_2^2 + \alpha \| \nabla f \|_1 \right\}
\]
In variational regularization, the penalty term expresses \textit{a priori} knowledge about the unknown

**Standard Tikhonov regularization:**

$$\arg\min_{f \in \mathbb{R}^n} \left\{ ||Af - m||_2^2 + \alpha ||f||_2^2 \right\}$$

**Non-negativity constrained Tikhonov regularization:**

$$\arg\min_{f \in \mathbb{R}_+^n} \left\{ ||Af - m||_2^2 + \alpha ||f||_2^2 \right\}$$

**Non-negativity constrained Total Variation (TV) regularization:**

$$\arg\min_{f \in \mathbb{R}_+^n} \left\{ ||Af - m||_2^2 + \alpha ||\nabla f||_1 \right\}$$
TV tomography: \( \arg \min_{f} \{ \|Af - m\|_2^2 + \alpha \|\nabla f\|_1 \} \)

1992 Rudin, Osher & Fatemi: denoise images by taking \( A = I \)
1998 Delaney & Bresler
2001 Persson, Bone & Elmqvist
2003 Kolehmainen, S, Järvenpää, Kaipio, Koistinen, Lassas, Pirttilä & Somersalo (first TV work with measured X-ray data)
2006 Kolehmainen, Vanne, S, Järvenpää, Kaipio, Lassas & Kalke
2006 Sidky, Kao & Pan
2008 Liao & Sapiro
2008 Sidky & Pan
2008 Herman & Davidi
2009 Tang, Nett & Chen
2009 Duan, Zhang, Xing, Chen & Cheng
2010 Bian, Han, Sidky, Cao, Lu, Zhou & Pan
2011 Jensen, Jørgensen, Hansen & Jensen
2011 Tian, Jia, Yuan, Pan & Jiang
2012–present: dozens of articles indicated by Google Scholar
V. Kolehmainen and I in 2002 after 11 hours of measurements at Instrumentarium Imaging lab
Outline

Total variation regularized tomography

A multiresolution parameter choice method for TV

How about theory?
How to choose the regularization parameter in the total variation (TV) functional?

Heuristics: Rullgård 2008

Balancing $\ell^1$ and TV: Clason, Jin & Kunisch 2010

Local variance: Dong, Hintermüller & Rincon-Camacho 2011

Discrepancy principle: Wen & Chan 2012

S-curve method: Kolehmainen, Lassas, Niinimäki & S 2012

Dantzig estimation: Frick, Mar- nitz & Munk 2012

Quasi-optimality principle and Hanke-Raus rules: Kindermann, Mutimbu & Resmerita 2013

KKT system: Chen, Loli Piccolomini & Zama 2014

Discrepancy principle: Toma, Sixou & Peyrin 2015

Cross validation, Stein’s unbiased risk estimates, L-curve method, ...

No single choice rule works perfectly for all applications. Therefore, it is good to have a collection of rules.
The continuous tomographic model needs to be approximated using a discrete model

Continuous model:

Discrete model:

In this schematic setup we have 5 projection directions and a 10-pixel detector. Therefore, the number of data points is 50.
The resolution of the discrete model can be freely chosen according to computational resources.

**Continuous model:**

In this schematic setup we have 5 projection directions and a 10-pixel detector. Therefore, the number of data points is **50**.

The number of degrees of freedom in the three discrete models below are **16**, **64** and **256**, respectively.
Intuition: discrete TV reconstructions at different resolutions should converge to a continuous limit.
**Intuition:** discrete TV reconstructions at different resolutions should converge to a continuous limit.
**Intuition:** discrete TV reconstructions at different resolutions should converge to a continuous limit.
We define the total variation (TV) norm consistently for continuous and discrete cases

Continuous anisotropic TV norm for attenuation coefficient $f : \Omega \rightarrow \mathbb{R}$:

$$\int_{\Omega} \left( \left| \frac{\partial f}{\partial x_1} \right| + \left| \frac{\partial f}{\partial x_2} \right| \right) \, dx.$$  

Discrete anisotropic TV norm for an image matrix of size $n \times n$:

$$\frac{1}{n} \sum |f_\kappa - f_{\kappa'}|,$$

where the sum is over horizontally and vertically neighboring pixel values $f_\kappa$ and $f_{\kappa'}$.

The above is based on this approximate two-dimensional computation:

$$\int_{\Omega} \left| \frac{f(x_1 + \frac{1}{n}, x_2) - f(x_1, x_2)}{1/n} \right| \, dx \approx \left( \frac{1}{n} \right)^2 \sum \left| \frac{f_\kappa - f_{\kappa'}}{1/n} \right|,$$

where the sum is over horizontally neighboring pixel values $f_\kappa$ and $f_{\kappa'}$. 
Low-noise TV reconstructions of a walnut using several regularization parameters

\( \alpha = 0.001 \quad \alpha = 1 \quad \alpha = 1000 \)

Too small \( \alpha \) \quad Just right \( \alpha \) \quad Too large \( \alpha \)

Computations by Kati Niinimäki using a primal-dual interior point method.
Low-noise TV reconstructions of a walnut using several regularization parameters

\( \alpha = 0.001 \) \hspace{1cm} \( \alpha = 1 \) \hspace{1cm} \( \alpha = 1000 \)

Too small \( \alpha \hspace{1cm} \) Just right \( \alpha \hspace{1cm} \) Too large \( \alpha \)

What happens when we compare reconstructions at different resolutions?
Low-noise TV reconstructions of a walnut at many resolutions using $\alpha = 0.001$
Low-noise TV reconstructions of a walnut at many resolutions using $\alpha = 1$

128 x 128

192 x 192

256 x 256
Low-noise TV reconstructions of a walnut at many resolutions using $\alpha = 1000$
TV norms of low-noise reconstructions with various resolutions and parameters $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Resolution</th>
<th>128 × 128</th>
<th>192 × 192</th>
<th>256 × 256</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>1.51</td>
<td>2.29</td>
<td>3.64</td>
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<tr>
<td>$10^{-3}$</td>
<td>1.51</td>
<td>2.29</td>
<td>3.46</td>
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</tr>
<tr>
<td>$10^{-2}$</td>
<td>1.50</td>
<td>2.23</td>
<td>2.97</td>
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</tr>
<tr>
<td>$10^{-1}$</td>
<td>1.43</td>
<td>1.85</td>
<td>1.93</td>
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</tr>
<tr>
<td>$10^{0}$</td>
<td>1.08</td>
<td>1.11</td>
<td>1.11</td>
<td></td>
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<tr>
<td>$10^{1}$</td>
<td>0.78</td>
<td>0.78</td>
<td>0.77</td>
<td></td>
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<tr>
<td>$10^{2}$</td>
<td>0.48</td>
<td>0.48</td>
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<tr>
<td>$10^{3}$</td>
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<tr>
<td>$10^{4}$</td>
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<td>$10^{5}$</td>
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<td>$10^{6}$</td>
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</tbody>
</table>
TV norms of low-noise reconstructions with various resolutions and parameters $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Resolution</th>
<th>$128 \times 128$</th>
<th>$192 \times 192$</th>
<th>$256 \times 256$</th>
</tr>
</thead>
<tbody>
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<td>$10^{-4}$</td>
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<td>$10^{6}$</td>
<td>0</td>
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What happens when we add noise to the data?
5% noise TV reconstructions of a walnut at many resolutions using $\alpha = 0.001$
5% noise TV reconstructions of a walnut at many resolutions using $\alpha = 10$

128 × 128  |  192 × 192  |  256 × 256
5% noise TV reconstructions of a walnut at many resolutions using $\alpha = 10000$
**TV norms of reconstructions using various noise levels, resolutions and parameters** $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Low noise</th>
<th>5% noise</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$128^2$</td>
<td>$192^2$</td>
</tr>
<tr>
<td>$10^{-4}$</td>
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<td>2.29</td>
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<td>1.51</td>
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<tr>
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<td>2.23</td>
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<tr>
<td>$10^{-1}$</td>
<td>1.43</td>
<td>1.85</td>
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<tr>
<td>$10^0$</td>
<td>1.08</td>
<td>1.11</td>
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<tr>
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<td>0.78</td>
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<td>$10^2$</td>
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<td>0.48</td>
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<tr>
<td>$10^3$</td>
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<td>$10^4$</td>
<td>0.04</td>
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<tr>
<td>$10^6$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We present another real-data example involving an arrangement of 10 sugarcubes.

We use 120 projections. Shown above is a reconstruction using FBP.
### TV norms of low-noise reconstructions with various resolutions and parameters $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Resolution</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$128 \times 128$</td>
<td>$256 \times 256$</td>
<td>$512 \times 512$</td>
<td></td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>0.85</td>
<td>2.30</td>
<td>7.40</td>
<td></td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>0.84</td>
<td>2.30</td>
<td>6.80</td>
<td></td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>0.93</td>
<td>2.20</td>
<td>7.00</td>
<td></td>
</tr>
<tr>
<td>$10^{-1}$</td>
<td>0.91</td>
<td>1.80</td>
<td>3.10</td>
<td></td>
</tr>
<tr>
<td>$10^0$</td>
<td>0.76</td>
<td>0.91</td>
<td>1.00</td>
<td></td>
</tr>
<tr>
<td>$10^1$</td>
<td><strong>0.53</strong></td>
<td><strong>0.54</strong></td>
<td><strong>0.55</strong></td>
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<tr>
<td>$10^2$</td>
<td><strong>0.37</strong></td>
<td><strong>0.37</strong></td>
<td><strong>0.36</strong></td>
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</tr>
<tr>
<td>$10^3$</td>
<td>0.27</td>
<td>0.30</td>
<td>0.51</td>
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<tr>
<td>$10^4$</td>
<td>0.17</td>
<td>0.14</td>
<td>0.09</td>
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<tr>
<td>$10^5$</td>
<td>0.11</td>
<td>0.33</td>
<td>1.10</td>
<td></td>
</tr>
<tr>
<td>$10^6$</td>
<td>0.14</td>
<td>0.28</td>
<td>1.90</td>
<td></td>
</tr>
</tbody>
</table>
TV reconstruction using rounded absolute-value function and projected Barzilai-Borwein

We used 120 projections, parameter $\alpha = 10$, and resolution $512 \times 512$. Data collected by Aki Kallonen, computations by Esa Niemi.
Outline

Total variation regularized tomography

A multiresolution parameter choice method for TV

How about theory?
Assumptions on the linear forward map $\mathcal{A}$

Assume either (A) or (B) about the linear operator $\mathcal{A}$:

(A) $\mathcal{A} : L^2(D) \to L^2(\Omega)$ is compact and $\mathcal{A} : L^1(D) \to D'(\Omega)$ is continuous with some open and bounded set $\Omega \subset \mathbb{R}^2$.

This covers the case of classical Radon transform with image domain $D$ and sinogram domain $\Omega$. We denote the set of distributions by $D'(\Omega)$.

(B) $\mathcal{A} : L^1(D) \to \mathbb{R}^M$ is bounded.

This covers the practically important discrete pencil beam model of tomographic measurements.
Definition of discrete and continuous regularization functionals

Let $D$ be the square $[0, 1]^2 \subset \mathbb{R}^2$. Use anisotropic $BV(D)$ norm

$$\|u\|_{BV} = \|u\|_{L^1} + V(u) = \|u\|_{L^1} + \int_D \left( \left| \frac{\partial u(x)}{\partial x_1} \right| + \left| \frac{\partial u(x)}{\partial x_2} \right| \right) dx.$$ 

Define $S_\infty : BV(D) \to \mathbb{R}$ and $S_j : BV(D) \to \mathbb{R} \cup \{\infty\}$ by

$$S_\infty(u) = \|Au - m\|_{L^2(\Omega)}^2 + \alpha_1 \|u\|_{L^1(D)} + \alpha V(u)$$

with positive regularization parameters $\alpha_1 > 0$ and $\alpha > 0$, and

$$S_j(u) = \begin{cases} 
S_\infty(u), & \text{for } u \in \text{Range}(T_j), \\
\infty, & \text{for } u \notin \text{Range}(T_j).
\end{cases}$$

Linear operator $T_j$ projects to functions that are piecewise constant on a regular $2^j \times 2^j$ square pixel grid.
Our main theorem ensures the convergence of regularized solutions as resolution grows

- There exists a minimizer $\tilde{u}_j \in \arg\min(S_j)$ for all $j = 1, 2, 3, \ldots$
- There exists a minimizer $\tilde{u}_\infty \in \arg\min(S_{\infty})$.
- Any sequence $\tilde{u}_j \in \arg\min(S_j)$ of minimizers has a subsequence $\tilde{u}_{j(\ell)}$ that converges weakly in $BV(D)$ to some limit $w \in BV(D)$. Furthermore, $\lim_{\ell \to \infty} V(\tilde{u}_{j(\ell)}) = V(w)$.
- The limit $w$ is a minimizer: $w \in \arg\min(S_{\infty})$.

There are some related results in the literature

1992 Vainikko: *On the discretization and regularization of ill-posed problems with noncompact operators*

We use geometric arguments similar to those here:


These works consider TV functionals and Γ-convergence when discretization is refined, but without a measurement operator:

2009 Chambolle, Giacomini & Lussardi
2012 Gennip & Bertozzi
2013 Bellettini, Chambolle & Goldman
2013 Trillos & Slepčev

This paper achieves a result analogous with ours, using wavelet frames in the finite-dimensional functionals:

2012 Cai, Dong, Osher & Shen
How to prove the main theorem?

The proof is an analysis of $\Gamma$-convergence of functionals $S_j$ to $S_\infty$. However, the choice of topologies is very delicate. For details, see http://arxiv.org/abs/1407.2386v2.

The approximation lemma on the right serves as the foundation of the proof. Generalizations of the result to higher dimension or to other TV norms would require modifying this key lemma.

**Lemma.** For all $u \in BV(D)$ and $\varepsilon > 0$ there exists $j > 0$ and a function $u'$, piecewise constant in the dyadic $2^j \times 2^j$ grid, such that

$$\|u - u'\|_{L^1(D)} + |V(u) - V(u')| < \varepsilon.$$ 

Recall that

$$V(u) = \int_D \left( \left| \frac{\partial u(x)}{\partial x_1} \right| + \left| \frac{\partial u(x)}{\partial x_2} \right| \right) dx.$$
We need to move from triangulation-based to pixel-based approximation

[Bělík and Luskin 2003]: the desired inequality holds with PW constant functions in a fine triangularization. However, we need to work with dyadic $2^j \times 2^j$ pixel grids.
We surround any triangle vertex (blue dot) with a “pixel cluster” neighborhood (gray box)
Refine the grid outside clusters so that pixel-wise polygonal chains (on pink) connect the clusters.
Using the anisotropic BV norm reduces the approximation to estimating small intervals

\[ v_1 = a \]

\[ v_1 = b \]

The difference between the BV norms of the piecewise constant functions \( v_1 \) and \( v_2 \) comes entirely from jumps over the two red vertical intervals below.
What can we say about the proposed method?

Benefits of our multiresolution TV parameter choice method:

- simple definition,
- easy implementation, and
- no need of *a priori* information about the noise amplitude.

Also, it seems to perform well for real tomographic data.

Downside: several reconstructions need to be computed. Also, it is still unclear why the method works so nicely: if there is convergence for any $\alpha$ in theory, what is the instability we are observing?

The method can be tried out with 3D tomography (it works!) and with other inverse problems and regularizers.
Thank you for your attention!

http://www.siltanen-research.net
We took photographs of walnuts cut in half. These photos are used for estimating the expected total variation norm (number of nonzero gradient elements) in a two-dimensional tomographic reconstruction. We compute the discrete TV norm for these images downsampled to 256 × 256 pixel resolution.

[Hämäläinen, Kallonen, Kolehmainen, Lassas, Niinimäki & S 2013]

Special thanks to Esa Niemi for his careful job in sawing the walnuts.
The S-curve method applied to the real data case with no added noise.  (Multiresolution gave $\alpha = 1$)

[Niinemäki, Lassas, Hämäläinen, Kallonen, Kolehmainen, Niemi & S, submitted]
The S-curve method applied to the real data case with 5% noise.  
(Multiresolution gave $\alpha = 10$)

[256 × 256]

$0.76$

$10^{-2}$ $\alpha = 40$ $10^4$

[Niinimäki, Lassas, Hämäläinen, Kallonen, Kolehmainen, Niemi & S, submitted]
All Matlab codes freely available at this site!

Part I: Linear Inverse Problems
1 Introduction
2 Naïve reconstructions and inverse crimes
3 Ill-Posedness in Inverse Problems
4 Truncated singular value decomposition
5 Tikhonov regularization
6 Total variation regularization
7 Besov space regularization using wavelets
8 Discretization-invariance
9 Practical X-ray tomography with limited data
10 Projects

Part II: Nonlinear Inverse Problems
11 Nonlinear inversion
12 Electrical impedance tomography
13 Simulation of noisy EIT data
14 Complex geometrical optics solutions
15 A regularized D-bar method for direct EIT
16 Other direct solution methods for EIT
17 Projects