Electrical impedance imaging using nonlinear Fourier transform

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International Conference on Scientific Computing
S. Margherita di Pula, Italy, October 14, 2011
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Outline

Electrical impedance tomography

Regularization of nonlinear inverse problems

D-bar method for infinite-precision data

Regularization using non-linear low-pass filtering
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**Electrical impedance tomography (EIT)** is an emerging medical imaging technique

**Feed** electric currents through electrodes. **Measure** the resulting voltages. Repeat the measurement for several current patterns.

**Reconstruct** distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient’s inner structure.

EIT is a harmless and painless imaging method suitable for long-term monitoring.
The most promising use of EIT is detection of breast cancer in combination with mammography. ACT4 and mammography devices Radiolucent electrodes

Cancerous tissue is up to four times more conductive than healthy breast tissue [Jossinet 1998]. The above setup of David Isaacson’s team measures 3D X-ray mammograms and EIT data at the same time.
Which of these three breasts have cancer?
Spectral EIT can detect cancerous tissue

[Kim, Isaacson, Xia, Kao, Newell & Saulnier 2007]
This talk concentrates on applications of EIT to chest imaging

Applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases} 
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
\text{u}|_{\partial \Omega} = f.
\end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_{\sigma} : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_{\sigma}$. It is a nonlinear and ill-posed inverse problem.
Many different types of reconstruction methods have been suggested for EIT in the literature

- **Linearization:** Barber, Bikowski, Brown, Cheney, Isaacson, Mueller, Newell
- **Iterative regularization:** Dobson, Hua, Kindermann, Leitão, Lechleiter, Neubauer, Rieder, Rondi, Santosa, Tompkins, Webster, Woo
- **Bayesian inversion:** Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen
- **Resistor network methods:** Borcea, Druskin, Mamonov, Vasquez
- **Layer stripping:** Cheney, Isaacson, Isaacson, Somersalo
- **D-bar methods:** Astala, Bikowski, Bowerman, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Saulnier, S, Tamasan
- **Teichmüller space methods:** Kolehmainen, Lassas, Ola
- **Methods for partial information:** Alessandrini, Ammari, Bilotta, Brühl, Erhard, Gebauer, Hanke, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kwon, Lechleiter, Lim, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others
## History of CGO-based methods for real 2D EIT

<table>
<thead>
<tr>
<th>(3D)</th>
<th>Infinite-precision data</th>
<th>Practical data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1980 Calderón</td>
<td>2008 Bikowski &amp; Mueller</td>
</tr>
<tr>
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<td>1987 Sylvester &amp; Uhlmann</td>
<td>2008 Boverman, Isaacson, Kao, Saulnier &amp; Newell</td>
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<td></td>
<td>1988 Nachman</td>
<td>2010 Bikowski, Knudsen &amp; Mueller</td>
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<td></td>
<td>1988 R G Novikov</td>
<td>2008 Boverman, Isaacson, Kao, Saulnier &amp; Newell</td>
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| $C^2(\Omega)$ | 1996 Nachman | 2000 S, Mueller & Isaacson |
|               | 1997 Liu     | 2003 Mueller & S |

| $C^1(\Omega)$ (complex) | 1997 Brown & Uhlmann | 2000 Francisco |
|                          | 2001 Barceló, Barceló & Ruiz | 2003 Knudsen |

| $L^\infty(\Omega)$ | 2003 Astala & Päivärinta | 2009 Astala, Mueller, Päivärinta & S |
|                    | 2005 Astala, Lassas & Päivärinta | 2011 Astala, Mueller, Päivärinta, Perämäki & S |
|                    | 2007 Barceló, Faraco & Ruiz | 2008 Clop, Faraco & Ruiz |
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Regularization using non-linear low-pass filtering
The forward map $F : X \supset \mathcal{D}(F) \rightarrow Y$ of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.
The regularization strategy need to be constructed so that these assumptions are satisfied

A family $\Gamma_\alpha : Y \to X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a regularization strategy for $F$ if

$$\lim_{\alpha \to 0} \| \Gamma_\alpha(\Lambda_\sigma) - \sigma \|_X = 0$$

for each fixed $\sigma \in \mathcal{D}(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called admissible if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $\sigma \in \mathcal{D}(F)$ the following holds:

$$\sup_{\Lambda_\sigma} \left\{ \| \Gamma_\alpha(\delta)(\Lambda_\sigma) - \sigma \|_X : \| \Lambda_\delta - \Lambda_\sigma \|_Y \leq \delta \right\} \to 0 \text{ as } \delta \to 0.$$
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Nachman’s 1996 uniqueness proof for 2D inverse conductivity problem relies on CGO solutions

Define a potential $q$ by setting $q(z) \equiv 0$ for $z$ outside $\Omega$ and

$$q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for } z \in \Omega.$$  

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(-\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz}\psi(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2), \quad \tilde{p} > 2,$$

where $ikz = i(k_1 + i k_2)(x + iy)$. By [Nachman 1996] we know that there exists a unique solution $\psi(\cdot, k)$ for any fixed $k \neq 0$. 
The crucial intermediate object in the proof is the non-physical scattering transform \( t(k) \)

We denote \( z = x + iy \in \mathbb{C} \) or \( z = (x, y) \in \mathbb{R}^2 \) whenever needed.

The scattering transform \( t : \mathbb{C} \to \mathbb{C} \) is defined by

\[
t(k) := \int_{\mathbb{R}^2} e^{i\overline{k}z} q(z) \psi(z, k) \, dx \, dy.
\]

(1)

Sometimes (1) is called the nonlinear Fourier transform of \( q \).

This is because asymptotically \( \psi(z, k) \sim e^{ikz} \) as \( |z| \to \infty \), and substituting \( e^{ikz} \) in place of \( \psi(z, k) \) into (1) results in

\[
\int_{\mathbb{R}^2} e^{i(kz + \overline{k}z)} q(z) \, dx \, dy = \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx \, dy
\]

\[
= \hat{q}(-2k_1, 2k_2).
\]
Another convenient trick in the proof is to make use of the functions $\mu(z, k) = e^{-ikz}\psi(z, k)$

Define $\mu(z, k) = e^{-ikz}\psi(z, k)$. Then $(-\Delta + q)\psi = 0$ implies

$$(-\Delta - 4ik\overline{\partial}_z + q)\mu(\cdot, k) = 0,$$  \hspace{1cm} (2)

where the D-bar operator is defined by $\overline{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

The asymptotic properties of $\psi$ imply that

$$\mu(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2), \quad \tilde{p} > 2.$$  \hspace{1cm} (3)

Substituting $k = 0$ into (2) gives

$$(-\Delta + \Delta\sqrt{\sigma})\mu(\cdot, 0) = 0,$$  \hspace{1cm} (4)

and $\mu(z, 0) = \sqrt{\sigma(z)}$ gives the unique solution of (3) and (4).
These are the steps of Nachman’s 1996 proof:

Solve boundary integral equation
\[ \psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]
for every complex number \( k \in \mathbb{C} \).

Evaluate the scattering transform:
\[ t(k) = \int_{\partial\Omega} e^{i\bar{k}z}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k)\, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial \bar{k}} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu(z, k) \]
with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

Fredholm equation of 2nd kind, ill-posedness shows up here.

Simple integration.

Well-posed problem, can be analyzed by scattering theory.

Trivial step.
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Let us analyze how the regularization works using a simulated heart-and-lungs phantom.
This is how the actual scattering transform looks like in the disc $|k| < 10$, computed by knowing $\sigma$.
Scattering transform in the disc $|k| < 10$, here computed from noisy measurement $\Lambda^\delta_{\sigma}$.
### Infinite-precision data:

Solve boundary integral equation
\[ \psi(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi \]
for every complex number \( k \in \mathbb{C} \).

Evaluate the scattering transform:
\[ t(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial \bar{k}} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}\bar{z})} \mu(z, k) \]
with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

### Practical data:

Solve boundary integral equation
\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta \]
for all \( |k| < R = R(\delta) \).

For \( |k| \geq R \) set \( t_R^\delta(k) = 0 \). For \( |k| < R \)
\[ t_R^\delta(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{z}}(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial \bar{k}} \mu_R^\delta(z, k) = \frac{t_R^\delta(k)}{4\pi k} e^{-i(kz + \bar{k}\bar{z})} \mu_R^\delta(z, k) \]
with \( \mu_R^\delta(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma_{\alpha(\delta)}(\Lambda_\sigma^\delta) := (\mu_R^\delta(z, 0))^2 \).
We define spaces for our regularization strategy

Consider $F : X \supset D(F) \to Y$ with $X = L^\infty(\Omega)$. Let $M > 0$ and $0 < \rho < 1$. Now $D(F)$ consists of functions $\sigma : \Omega \to \mathbb{R}$ satisfying

$$\|\sigma\|_{C^2(\Omega)} \leq M, \quad \sigma(z) \geq M^{-1}, \quad \text{and} \quad \sigma(z) \equiv 1 \text{ for } \rho < |z| < 1.$$

$Y$ comprises bounded linear operators $A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying $A(1) = 0$ and $\int_{\partial\Omega} A(f) \, d\sigma = 0$. 
Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

Theorem (Knudsen, Lassas, Mueller & S 2009)

There exists a constant $0 < \delta_0 < 1$, depending only on $M$ and $\rho$, with the following properties. Let $\sigma \in D(F)$ be arbitrary and assume given noisy data $\Lambda^\delta_\sigma$ satisfying

$$\|\Lambda^\delta_\sigma - \Lambda_\sigma\|_Y \leq \delta < \delta_0.$$

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_\alpha(\delta)(\Lambda^\delta_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \delta)^{-1/14}.$$
Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

\[ \varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}. \]

We invert the linear operator appearing in the equation

\[ \psi^\delta (\cdot, k)|_{\partial \Omega} = [I + S_k (\Lambda^\delta - \Lambda_1)] e^{ikz} |_{\partial \Omega} \]

as a matrix in \( \text{span}(\{ \varphi_n \}_{n=-N}^N) \).

The single-layer operator

\( (S_k \phi)(z) = \int_{\partial \Omega} G_k(z-w) \phi(w) \, ds(w) \)

uses Faddeev’s Green’s function.
Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

The generalization of Vainikko’s method for the D-bar equation is described in [Knudsen, Mueller & S 2004]. The D-bar equation

$$\frac{\partial}{\partial k} \mu^\delta_R = \frac{1}{4\pi k} t^\delta_R(k) \overline{t^\delta_R(k)} e^{-z(k)\mu^\delta_R}$$

together with the asymptotics

$$\mu^\delta_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$$

can be combined in a generalized Lippmann-Schwinger equation:

$$\mu^\delta_R(z, k) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{t^\delta_R(k')}{(k - k')^2} e^{-z(k')\mu^\delta_R(z, k')} \mu^\delta_R(z, k') \, dk_1 \, dk_2.$$
This is the real-linear operation given to GMRES:

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\]

\[
\begin{bmatrix}
T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi} \\
T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi} \\
T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi} \\
T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi} & T_R \overline{\phi}
\end{bmatrix}
\]

Element-wise multiplication

IFTT

\[
\begin{bmatrix}
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi
\end{bmatrix}
\]

\[
\phi - \frac{1}{\pi k} \ast (T_R \overline{\phi})
\]
Regularized reconstructions from simulated data with noise amplitude $\|\delta\| = \|\Lambda^\delta_\sigma - \Lambda_\sigma\|_Y$

$\|\delta\| \approx 10^{-6}$  $\|\delta\| \approx 10^{-5}$  $\|\delta\| \approx 10^{-4}$  $\|\delta\| \approx 10^{-3}$  $\|\delta\| \approx 10^{-2}$

The percentages are the relative square norm errors in the reconstructions.
The observed radii are better (\(\approx\) larger) than those given by the theoretical formula \(R(\delta) = -\frac{1}{10} \log \delta\).
Conclusion

We have constructed the first direct (non-iterative) regularization strategy for a global nonlinear PDE coefficient recovery problem.

Efficient implementation available, based on Vainikko’s method.

The nonlinear low-pass filter regularization approach has an explicit speed of convergence in a Banach space setting.

The method works with real data as well:

[Isaacson, Mueller, Newell & S 2006]
Thank you for your attention!

Preprints available at www.siltanen-research.net.
Imaging cardiac activity by the d-bar method for electrical impedance tomography.

Regularized d-bar method for the inverse conductivity problem.

A. I. Nachman.
Global uniqueness for a two-dimensional inverse boundary value problem.

S. Siltanen, J. Mueller, and D. Isaacson.
An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem.