Regularized D-bar method for the inverse conductivity problem

\[
\frac{\partial}{\partial k} \mu_R(x, k) = \frac{t_R^\text{exp}(k)}{4 \pi k} e^{-i(kx + \overline{k}x)} \mu_R(x, k)
\]
This is a joint work with

David Isaacson
Rensselaer Polytechnic Institute, USA

Kim Knudsen
Technical University of Denmark

Matti Lassas
University of Helsinki, Finland

Jennifer Mueller
Colorado State University, USA

Jon Newell
Rensselaer Polytechnic Institute, USA
1. The inverse conductivity problem of Calderón

2. Theory of d-bar imaging: infinite precision data

3. Regularized d-bar imaging for noisy data

4. Numerical aspects

5. Reconstructions
Electrical impedance tomography (EIT) is an emerging medical imaging method

Feed electric currents through electrodes, measure voltages

Reconstruct the image of electric conductivity in a two-dimensional slice

Applications include: monitoring heart and lungs of unconscious patients, detecting pulmonary edema, enhancing ECG and EEG
The inverse conductivity problem of Calderón is the mathematical model of EIT

Problem: given the Dirichlet-to-Neumann map, how to reconstruct the conductivity? The reconstruction problem is nonlinear and ill-posed.

\[ \Lambda_{\gamma} f = \frac{\partial u}{\partial \nu} |_{\partial \Omega}, \]

\[ \nabla \cdot \gamma \nabla u = 0 \quad \text{in} \ \Omega, \]

\[ u = f \quad \text{on} \ \partial \Omega. \]

We assume that \( 0 < c \leq \gamma(x) \leq C \) for all \( x \in \Omega \).
EIT reconstruction algorithms can be divided roughly into the following classes:

**Linearization** (Barber, Bikowski, Brown, Cheney, Isaacson, Mueller, Newell)

**Iterative regularization** (Dobson, Hua, Kindermann, Lechleiter, Neubauer, Rieder, Rondi, Santosa, Tompkins, Webster, Woo)

**Statistical (Bayesian) inversion** (Fox, Kaipio, Kolehmainen, Nicholls, Somersalo, Vauhkonen, Voutilainen)

**Resistor network methods** (Borcea, Druskin, Vasquez)

**Convexification** (Beilina, Klibanov)

**Layer stripping** (Cheney, Isaacson, Isaacson, Somersalo)

**The d-bar method** (Astala, Bikowski, Bowerman, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Saulnier, S, Tamasan)

**Teichmüller space methods** (Kolehmainen, Lassas, Ola)

---

**Methods for partial information** (Alessandrini, Ammari, Bilotta, Brühl, Erhard, Gebauer, Hanke, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kwon, Lechleiter, Lim, Morassi, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others)
This is a brief history of the d-bar method in 2D

<table>
<thead>
<tr>
<th>Theory</th>
<th>Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980 Calderón</td>
<td>2008 Bikowski and Mueller</td>
</tr>
<tr>
<td>1987 Sylvester and Uhlmann</td>
<td>2000 S, Mueller and Isaacson</td>
</tr>
<tr>
<td>1987 R G Novikov</td>
<td>2003 Mueller and S</td>
</tr>
<tr>
<td>1988 Nachman</td>
<td>2004 Isaacson, Mueller, Newell and S</td>
</tr>
<tr>
<td>1996 Nachman</td>
<td>2006 Isaacson, Mueller, Newell and S</td>
</tr>
<tr>
<td>1997 Liu</td>
<td>2007 Murphy</td>
</tr>
<tr>
<td>1997 Brown and Uhlmann</td>
<td>2008 Knudsen, Lassas, Mueller and S</td>
</tr>
<tr>
<td>2000 Francini</td>
<td></td>
</tr>
<tr>
<td><strong>2003 Astala and Päivärinta</strong></td>
<td>2008 Astala, Mueller, Päivärinta and S</td>
</tr>
<tr>
<td>2007 Barceló, Barceló and Ruiz</td>
<td></td>
</tr>
<tr>
<td>2008 Clop, Faraco and Ruiz</td>
<td></td>
</tr>
<tr>
<td>2008 Bukhgeim</td>
<td></td>
</tr>
</tbody>
</table>
1. The inverse conductivity problem of Calderón

2. Theory of d-bar imaging: infinite precision data

3. Regularized d-bar imaging for noisy data

4. Numerical aspects

5. Reconstructions
Nachman’s 1996 proof consists of two steps:

\[ \Lambda_\gamma \rightarrow t \rightarrow \gamma \]

The intermediate object \( t \) is a complex-valued function called *scattering transform* and defined as follows:

\[ t(k) := \int_{\mathbb{R}^2} e^{ikx} q(x) \psi(x, k) \, dx \]

\[ q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}} \]

\[ (-\Delta + q) \psi(\cdot, k) = 0 \]

\[ \psi(x, k) \sim e^{ikx} = e^{i(k_1 + ik_2)(x_1 + ix_2)} \]
Step 1: from DN map to scattering transform

Solve traces of $\psi$ from the boundary integral equation

$$\psi(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k),$$

where the single-layer operator has Faddeev Green’s function as kernel.

Compute the scattering transform as

$$t(k) = \int_{\partial \Omega} e^{i\bar{k}x}(\Lambda_\gamma - \Lambda_1)\psi(x, k)\,d\sigma(x).$$
Let us take a closer look at Faddeev Green’s function and the related single layer operator

The operator

\[(S_k \phi)(x) := \int_{\partial \Omega} G_k(x - y) \phi(y) d\sigma(y)\]

involves the Faddeev Green’s function \(G_k\) for the Laplacian:

\[-\Delta G_k(x) = \delta_0(x).\]

The function \(G_k\) can be written in the form

\[G_k(x) := e^{ikx} g_k(x),\]

where

\[g_k(x) := \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix \cdot \xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} d\xi.\]
Step 2: from scattering transform to $\gamma$

Define $\mu(x, k) = e^{-ikx}\psi(x, k)$

Then the following d-bar equation holds:

$$\frac{\partial}{\partial k} \mu(x, k) = \frac{t(k)}{4\pi k} e^{-i(kx + k\bar{x})} \mu(x, k).$$

Here $\frac{\partial}{\partial k} = \frac{1}{2} (\frac{\partial}{\partial k_1} + i \frac{\partial}{\partial k_2}).$

The d-bar equation has a unique solution for all $x$. The conductivity can be recovered from

$$\gamma^{1/2}(x) = \lim_{k \to 0} \mu(x, k).$$
1. The inverse conductivity problem of Calderón

2. Theory of d-bar imaging: infinite precision data

3. Regularized d-bar imaging for noisy data

4. Numerical aspects

5. Reconstructions
We work within the following assumptions:

Let $\Omega \subset \mathbb{R}^2$ be the open unit disc.

Define the forward map $F$ between the spaces $F : \mathcal{D}(F) \subset L^\infty(\Omega) \to Y$.

Domain $\mathcal{D}(F)$ is defined as follows.

Let $M > 0$ and $0 < \rho < 1$. The set $\mathcal{D}(F)$ contains functions $\gamma : \Omega \to \mathbb{R}$ satisfying

(a) $\|\gamma\|_{C^2(\bar{\Omega})} \leq M$,
(b) $\gamma(x) \geq M^{-1}$ for all $x \in \Omega$,
(c) $\gamma(x) \equiv 1$ for $\rho < |x| < 1$.

Space $Y$ of data is defined as follows.

$Y$ consists of bounded linear operators $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ satisfying $\int_{\partial\Omega} \Lambda(f) d\sigma = 0$ and $\Lambda(1) = 0$. 
Let us emphasize one of the strengths of our new results

Conditional stability results have the form

$$\|\gamma_1 - \gamma_2\|_Z \leq f(\|\Lambda_\gamma \gamma_1 - \Lambda_\gamma \gamma_2\|_Y),$$

where $\gamma_1, \gamma_2$ belong to some function space $Z$ and $f$ is a continuous function with $f(0) = 0$.

The above estimate is not practically relevant. The noisy measurement $\Lambda_\gamma^\varepsilon$ is in general not the DN map of some conductivity.

In contrast, we prove regularization properties for the D-bar method under the practically feasible assumption $\|\Lambda_\gamma^\varepsilon - \Lambda_\gamma\|_Y \leq \varepsilon$. 
Let us define nonlinear regularization strategy 
(following Engl, Hanke & Neubauer and Kirsch)

Recall direct problem: $\gamma \in X$ maps to $\Lambda_\gamma \in Y$.

A family of continuous mappings $\Gamma_\alpha : Y \rightarrow X$ with $0 < \alpha < \infty$ is a regularization strategy if

$$\lim_{\alpha \rightarrow 0} \| \Gamma_\alpha \Lambda_\gamma - \gamma \|_{L^\infty(\Omega)} = 0$$

for each fixed $\gamma \in X$. A regularization strategy with a choice $\alpha = \alpha(\varepsilon)$ is called admissible if

$$\alpha(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

and for any fixed $\gamma \in X$ the following holds:

$$\sup_{\Lambda^\varepsilon_\gamma} \left\{ \| \Gamma_{\alpha(\varepsilon)} \Lambda^\varepsilon_\gamma - \gamma \|_{L^\infty(\Omega)} : \| \Lambda^\varepsilon_\gamma - \Lambda_\gamma \|_Y \leq \varepsilon \right\} \rightarrow 0 \quad \varepsilon \rightarrow 0.$$
This is our regularized D-bar method for EIT

Given the noise level \( \varepsilon > 0 \), solve the equation

\[
\psi^\varepsilon(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial \Omega}
\]

for \( |k| < R(\varepsilon) := -\frac{1}{10} \log(\varepsilon) \).

Introduce nonlinear low-pass filtering

\[
t^\varepsilon_R(k) := \begin{cases} 
\int_{\partial \Omega} e^{i\bar{k}\bar{x}} (\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)d\sigma & \text{for } |k| < R(\varepsilon), \\
0 & \text{for } |k| \geq R(\varepsilon).
\end{cases}
\]

For each \( x \in \Omega \), solve the integral equation

\[
\mu_R(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t^\varepsilon_R(s)}{(k - s)^2} e^{-x(s)\mu_R(x, s)} ds_1 ds_2,
\]

and define \( \alpha(\varepsilon) = \frac{1}{R(\varepsilon)} \) and \( (\Gamma \alpha \Lambda^\varepsilon)(x) := (\mu_R(x, 0))^2 \).
Theorem [Knudsen, Lassas, Mueller & S 2008]

The family $\Gamma_\alpha$ is well-defined for small $\alpha > 0$. It is an admissible regularization strategy with

$$\alpha(\varepsilon) = \left(-\frac{1}{10} \log(\varepsilon)\right)^{-1}.$$ 

Furthermore, we have the explicit estimate

$$\sup_{\Lambda^\varepsilon_\gamma} \left\{ \| \Gamma_\alpha(\varepsilon) \Lambda^\varepsilon_\gamma - \gamma \|_{L^\infty(\Omega)} : \| \Lambda^\varepsilon_\gamma - \Lambda_\gamma \|_Y \leq \varepsilon \right\} \leq C(-\log \varepsilon)^{-1/14}$$ 

$$\rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$
Proof of the main theorem is divided into several lemmata. First a D-bar estimate:

**Lemma 1.** Let $4/3 < r_0 < 2$ and suppose that $\phi_1, \phi_2 \in L^r(\mathbb{R}^2)$ for all $r \geq r_0$. Let $\mu_1, \mu_2$, be the solutions of

$$\mu_j(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\phi_j(k')}{(k - k')} \mu_j(x, k') dk_1' dk_2',$$

$j = 1, 2$. Then for fixed $x \in \overline{\Omega}$ we have

$$\|\mu_1(x, \cdot) - \mu_2(x, \cdot)\|_{C^\alpha(\mathbb{R}^2)} \leq C \|\phi_1 - \phi_2\|_{L^{r_0} \cap L^{r_0'}(\mathbb{R}^2)},$$

where $\alpha < 2/r_0 - 1$ and $1/r_0' = 1 - 1/r_0$.

**Proof.** Combination of well-known results.
Lemma 2. Let $\phi_0 \in H^{-1/2}(\partial \Omega)$ with $\int \phi_0 = 0$. Then we have the estimate
\[
\|S_k \phi_0\|_{H^{1/2}(\partial \Omega)} \leq C e^{2|k|}(1 + |k|) \|\phi_0\|_{H^{-1/2}(\partial \Omega)}.
\]

Lemma 3. For $k \in \mathbb{C}$ we have the estimate
\[
\|[I + S_k(\Lambda_\gamma - \Lambda_1)]^{-1}\|_{L(H^s(\partial \Omega))} \leq C_2 e^{2|k|}(1 + |k|),
\]
where $C_2$ depends only on $M$ and $\rho$. These results follow from careful analysis of Faddeev’s Green function.
Combining previous results, a perturbation argument, and delicate $L^p$ analysis shows

**Lemma 4.** There exists $\varepsilon_0 > 0$, depending only on $M$ and $\rho$, such that equation

$$\psi^\varepsilon(\cdot, k)|_{\partial \Omega} = e^{ikx} - S_k(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial \Omega}$$

is solvable in $H^{1/2}(\partial \Omega)$ for all $0 < \varepsilon \leq \varepsilon_0$ and $|k| < R$ with

$$R = R(\varepsilon) = -\frac{1}{10} \log \varepsilon.$$  

Furthermore, for $p > 1$ we have the estimate

$$\left\| \frac{t(k) - t^\varepsilon_R(k)}{k} \right\|_{L^p(|k| \leq R)} \leq C\varepsilon^{1/10} \left( -\frac{1}{10} \log \varepsilon \right)^{2/p},$$

where $C$ is independent of $p$ and $R$ and $\varepsilon$. 

Sketch of proof of main theorem

(i) \( \lim_{\alpha \to 0} \| \Gamma \alpha \Lambda \gamma - \gamma \|_{L^\infty(\Omega)} = 0 \) for \( \gamma \in X \).

(ii) \( \alpha(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \),

(iii) \( \sup_{\Lambda^\varepsilon_{\gamma}} \left\{ \| \Gamma \alpha(\varepsilon) \Lambda^\varepsilon_{\gamma} - \gamma \|_{L^\infty(\Omega)} : \| \Lambda^\varepsilon_{\gamma} - \Lambda \gamma \|_Y \leq \varepsilon \} \) tends to zero as \( \varepsilon \to 0 \).

Claim (i) follows from [Nachman 1996] (with delicate choices of \( L^p \) spaces) and Lemma 1. Claim (ii) is OK: \( \alpha(\varepsilon) = \frac{1}{R(\varepsilon)} = -10(\log \varepsilon)^{-1} \).
Sketch of proof of main theorem

To prove that
\[
\sup_{\Lambda_{\gamma}} \left\{ \| \Gamma_{\alpha(\varepsilon)}\Lambda_{\gamma}^\varepsilon - \gamma \|_{L^\infty(\Omega)} : \| \Lambda_{\gamma}^\varepsilon - \Lambda_{\gamma} \|_{Y} \leq \varepsilon \right\}
\]
tends to zero as \( \varepsilon \to 0 \) we combine [Nachman 1996] with Lemmata 1 and 4 to estimate

\[
\| \mu(x, \cdot) - \mu_R(x, \cdot) \|_{C^\alpha(\mathbb{R}^2)}
\] 
\[
\leq C \left\| \frac{t(k) - t_{R}^\varepsilon(k)}{k} \right\|_{L^p \cap L^{p'}(\mathbb{R}^2)}
\] 
\[
\leq C \left\| \frac{t(k) - t_{R}^\varepsilon(k)}{k} \right\|_{L^p \cap L^{p'}(|k|<R)} + C \left\| \frac{t(k)}{k} \right\|_{L^p(|k|>R)}
\] 
\[
\leq C \left( - \frac{1}{10} \log \varepsilon \right)^{10/7} \varepsilon^{1/10} + CR(\varepsilon)^{-1/7} + CR(\varepsilon)^{-1/14}
\] 
\[
= C(- \log \varepsilon)^{10/7} \varepsilon^{1/10} + C(- \log \varepsilon)^{-1/7} + C(- \log \varepsilon)^{-1/14}.
\]
One more thing: the regularization strategy is not yet defined on all of data space $Y$

The range $F(\mathcal{D}(F)) \subset Y$ is not known, and its structure may be complicated.
(This is related to the open and notoriously difficult characterization problem.)

The previous results show the claim only for operators $\varepsilon_0$-close to the range $F(\mathcal{D}(F))$.

The problem can be overcome using spectral theoretical arguments.
1. The inverse conductivity problem of Calderón

2. Theory of d-bar imaging: infinite precision data

3. Regularized d-bar imaging for noisy data

4. Numerical aspects

5. Reconstructions
Practical measurements use electrodes. Here is a typical arrangement:

The number of electrodes here is $N=32$. The device is in Rensselaer Polytechnic Institute, New York, USA.
A linearly independent set of current patterns is fed into the domain and voltages measured.

Three examples of current patterns in the case $N=32$:

Conservation of charge dictates that there are $N-1$ linearly independent current patterns.
Discrete current patterns approximate trigonometric basis functions at the boundary

\[ \cos(\theta) \quad \cos(4\theta) \quad \cos(16\theta) \]
We use the truncated Fourier basis \( \{ e^{in\theta} \}_{n=-N}^N \) to express functions defined on the unit circle.

Integral operators \( A : H^s(\partial \Omega) \to H^r(\partial \Omega) \) are represented as finite matrices \( \langle Ae^{in\theta}, e^{im\theta} \rangle \).

- \( \Lambda_1 \) we know analytically,
- \( \Lambda_\gamma \) we compute using Finite Element Method,
- \( S_k \) we evaluate by numerical integration.
This is our practical two-step regularized D-bar method for EIT

1. We solve for $|k| < R$ the matrix version of
   \[
   \psi^\varepsilon(\cdot, k)|_{\partial\Omega} = e^{ikx} - S_k(\Lambda^\varepsilon - \Lambda_1)\psi^\varepsilon(\cdot, k)|_{\partial\Omega}
   \]
   with $R$ as large as numerically stable.

2. The integral equation
   \[
   \mu_R(x, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{t^\varepsilon(s)}{(k - s)s} e^{-x(s)}\overline{\mu_R(x, s)} ds_1 ds_2
   \]
   can be solved by our established D-bar solver. The reconstructed conductivity is $\mu_R(x, 0)$. 
The d-bar equation is written in integral form for easier numerical solution

Write the d-bar equation

\[ \frac{\partial}{\partial k} \mu_R(x, k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + \overline{k}x)} \mu_R(x, k) \]

in integral form using the appropriate Green function:

\[ \mu_R(x, k) = 1 + \frac{1}{\pi k} \ast \left( \frac{t_R(k)}{4\pi k} e^{-i(kx + \overline{k}x)} \mu_R(x, k) \right) \].

This equation of the Lippmann-Schwinger form can be solved numerically as explained below. Then

\[ \gamma^{1/2}_R(x) = \mu_R(x, 0). \]
The d-bar equation is defined on the whole $k$-plane.

However, numerical solution requires a finite computational domain.

To this end, we consider periodic functions. The plane is tiled by the square $S = [-2R - \epsilon, 2R + \epsilon]^2$. 
We introduce an S-periodic version of the d-bar equation.

Green’s functions $g : \mathbb{C} \rightarrow \mathbb{C}$ and $\tilde{g} : S \rightarrow \mathbb{C}$:

$$g(k) = \frac{1}{\pi k}, \quad \tilde{g}(k) = \frac{1}{\pi k} |_{S}$$

Denote the multiplier function as follows:

$$T_R(k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + \bar{k} \bar{x})},$$

and set $\tilde{T}_R(k) = T_R(k)|_S$. The d-bar equations:

$$\mu_R(x, k) = 1 + \int_{\mathbb{C}} g(k - \lambda) T_R(\lambda) \overline{\mu_R(x, \lambda)} d\lambda$$

$$\tilde{\mu}_R(x, k) = 1 + \int_{S} \tilde{g}(k - \lambda) \tilde{T}_R(\lambda) \overline{\tilde{\mu}_R(x, \lambda)} d\lambda$$
Now the d-bar equation can be essentially solved in the square $S$ instead of the $k$-plane.

Green’s functions $g : \mathbb{C} \to \mathbb{C}$ and $\tilde{g} : S \to \mathbb{C}$:

$$g(k) = \frac{1}{\pi k}, \quad \tilde{g}(k) = \frac{1}{\pi k}_{|S}$$

Denote the multiplier function as follows:

$$T_R(k) = \frac{t_R(k)}{4\pi k} e^{-i(kx + k\bar{x})},$$

and set $\tilde{T}_R(k) = T_R(k)|_S$. The d-bar equations:

$$\mu_R(x, k) = 1 + \int_{D(0,R)} g(k - \lambda) T_R(\lambda) \mu_R(x, \lambda) d\lambda$$

$$\tilde{\mu}_R(x, k) = 1 + \int_{D(0,R)} \tilde{g}(k - \lambda) \tilde{T}_R(\lambda) \tilde{\mu}_R(x, \lambda) d\lambda$$

It can be shown that

$$\mu_R(x, k) = \tilde{\mu}_R(x, k) \quad \text{for } |k| < R.$$
We form a grid suitable for FFT (fast Fourier transform).

Here 8x8 grid is shown; in practice we typically use 512x512 points.

Periodic functions are represented by their values at the grid points.
Periodic functions are represented by their values at the grid points, real and imaginary parts separately:

\[
\begin{bmatrix}
\text{Re} \varphi(k_1) \\
\text{Re} \varphi(k_2) \\
\vdots \\
\text{Re} \varphi(k_{64}) \\
\text{Im} \varphi(k_1) \\
\text{Im} \varphi(k_2) \\
\vdots \\
\text{Im} \varphi(k_{64})
\end{bmatrix}
\in \mathbb{R}^{128}
\]
The solution of the periodic equation is reduced to iterative solution of a linear system

Write the periodic integral equation

\[ \tilde{\mu}_R(x, k) = 1 + \int_S \tilde{g}(k - \lambda)\tilde{T}_R(\lambda)\tilde{\mu}_R(x, \lambda) d\lambda \]

in the form

\[ [I - \tilde{g} * (\tilde{T}_R \cdot -)]\tilde{\mu}_R = 1. \]  \quad (1)

We represent the solution \( \tilde{\mu}_R \) as a vector of point values as shown above.

Then we can use an iterative solver, such as GMRES, to solve (1) provided we have a computational routine for the real-linear operation

\[ \varphi \mapsto \varphi - \tilde{g} * (\tilde{T}_R \varphi). \]

See [Vainikko 2000] and [Knudsen, Mueller and S 2004].
Periodic convolution is conveniently implemented using the FFT

\[
\begin{bmatrix}
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\
\frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k}
\end{bmatrix}
\xrightarrow{\text{FFT}}
\begin{bmatrix}
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} \\
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} \\
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} \\
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tilde{\phi} & \tilde{\phi} & \tilde{\phi} & \tilde{\phi} \\
\tilde{\phi} & \tilde{\phi} & \tilde{\phi} & \tilde{\phi} \\
\tilde{\phi} & \tilde{\phi} & \tilde{\phi} & \tilde{\phi} \\
\tilde{\phi} & \tilde{\phi} & \tilde{\phi} & \tilde{\phi}
\end{bmatrix}
\xrightarrow{\text{Multiplication}}
\begin{bmatrix}
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} \\
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} \\
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} \\
T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi} & T_R\tilde{\phi}
\end{bmatrix}
\xrightarrow{\text{IFFT}}
\begin{bmatrix}
\varphi - \tilde{g} \ast (\tilde{T}_R\tilde{\phi})
\end{bmatrix}
\]
1. The inverse conductivity problem of Calderón

2. Theory of d-bar imaging: infinite precision data

3. Regularized d-bar imaging for noisy data

4. Numerical aspects

5. Reconstructions
We construct a simulated human chest phantom for numerical testing.
Here we see the reconstructions corresponding to various levels of measurement noise

\[ \| \mathcal{E} \|_Y \approx 10^{-6} \quad \| \mathcal{E} \|_Y \approx 10^{-5} \quad \| \mathcal{E} \|_Y \approx 10^{-4} \quad \| \mathcal{E} \|_Y \approx 10^{-3} \quad \| \mathcal{E} \|_Y \approx 10^{-2} \]
The numerical results actually improve the exponential behaviour predicted by theory.
We have achieved excellent EIT reconstructions from practical data (phantom and \textit{in vivo} human) using Born approximation to linearize Step 1. However, that approach does not allow regularization analysis.
Thank you!

Preprints available at
www.siltanen-research.net
Knudsen K, Lassas M, Mueller J L and Siltanen S
Reconstructions of Piecewise Constant Conductivities by the D-bar Method for Electrical Impedance Tomography

D-bar method for electrical impedance tomography with discontinuous conductivities

Lassas M, Mueller J L and Siltanen S 2007
Mapping properties of the nonlinear Fourier transform in dimension two
Communications in Partial Differential Equations 32(4), pp. 591-610

Imaging Cardiac Activity by the D-bar Method for Electrical Impedance Tomography
Physiological Measurement 27, pp. S43-S50

Isaacson D, Mueller J L, Newell J and Siltanen S 2004
Reconstructions of chest phantoms by the d-bar method for electrical impedance tomography
IEEE Transactions on Medical Imaging 23(7), pp. 821-828

Knudsen K, Mueller J L and Siltanen S 2004
Numerical solution method for the dbar-equation in the plane
Journal of Computational Physics 198(2), pp. 500-517

Mueller J L and Siltanen S 2003
Direct reconstructions of conductivities from boundary measurements

Siltanen S, Mueller J L and Isaacson D 2000
An implementation of the reconstruction algorithm of A. Nachman for the 2-D inverse conductivity problem
Inverse Problems 16, pp. 681-699; Erratum Inverse problems 17, pp. 1561-1563
This is a brief history of regularization methods for electrical impedance tomography

1991 Hua, Woo, Webster and Tompkins (Tikhonov and smoothness)
1992 Goble, Cheney and Isaacson (truncated Newton method)
1994 Dobson and Santosa (Total variation)
1999 Vauhkonen et al. (Tikhonov in 3D)
2001 Kindermann and Neubauer (surface representation)
2001 Rondi and Santosa (Mumford-Shah-functional)
2003 Lukaschewitsch, Maass and Pidcock (Tikhonov regularization)
2005 Chung, Chan and Tai (level set, total variation)
2005 Eppler and Harbrecht (Newton regularization)
2006 Lechleiter and Rieder (numerical Newton regularization)
2008 Rondi (theory for regularized recovery of discontinuities)
2008 Lechleiter and Rieder (local convergence of Newton regularization)