Introduction to nonlinear tomography I: applications and basic properties of EIT

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Outline of the nonlinear part of the course

Part I  Electrical Impedance Tomography (EIT) and its applications.

Part II  Tikhonov regularization for the nonlinear EIT problem using iterative Gauss-Newton solver.

Part III  Complex Geometric Optics, non-linear Fourier transform, and direct inversion for EIT.
Outline

Applications of Electrical Impedance Tomography (EIT)

The principle of EIT

Non-uniqueness, ghosts, and ill-posedness

Regularized inversion for EIT
Chest imaging is the standard application example of EIT in this talk

Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
D-bar reconstruction of *in vivo* chest data

[Montoya & Mueller 2012]
EIT can be used for heart imaging
EIT can potentially be used for imaging changes in vocal folds due to excessive voice use.
EIT can perhaps be used for imaging changes in vocal folds due to dehydration.
The most promising use of EIT is detection of breast cancer in combination with mammography ACT4 and mammography devices. Radiolucent electrodes

Cancerous tissue is up to four times more conductive than healthy breast tissue [Jossinet 1998]. The above experiment by David Isaacson’s team measures 3D X-ray mammograms and EIT data at the same time.
Which of these three breasts have cancer?
Spectral EIT can detect cancerous tissue

[Kim, Isaacson, Xia, Kao, Newell & Saulnier 2007]
Imaging stroke with EIT

**Ischemic stroke:** low conductivity.
CT image from Jansen 2008

**Hemorrhagic stroke:** high conductivity.
CT image from Nakano et al. 2001

Same symptoms in both cases!

Difficulties: resistive skull layer and unknown background. However, see
- Holder 1992
- Romsauerova, McEwan, Horesh, Yerworth, Bayford & Holder 2006
- Shi, You, Xu, Wang, Fu, Liu & Dong 2009
- Bayford & Tizzard 2012
- Malone, Jehl, Arridge, Betcke & Holder 2014
The idea would be to equip every ambulance with an EIT device for classifying strokes.
EIT can be used for nondestructive testing: here for crack detection in concrete structures

[Karhunen, Seppänen, Lehikoinen, Monteiro & Kaipio 2010]
[Karhunen, Seppänen, Lehikoinen, Monteiro, Kaipio, Blunt, Hyvönen]
EIT can be used for nondestructive testing: here for water in concrete structures

[Hallaji, Seppänen & Pour-Ghaz 2015]
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Note that EIT data collection involves applying several current patterns.

Saline and agar phantom

Apply current pattern \( \cos \theta \)

Measure the resulting voltages at the 32 electrodes.
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 2\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom  
Apply current pattern $\cos 3\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern \( \cos 4\theta \)

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern \( \cos 5\theta \)

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom

Apply current pattern $\cos 16\theta$

Measure the resulting voltages at the 32 electrodes
The D-bar method works for real EIT data, such as laboratory phantoms and *in vivo* human data.

Saline and agar phantom

Reconstruction ($R = 4$)

[Isaacson, Mueller, Newell & S 2004]

[Montoya 2012]
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let $\Omega \subset \mathbb{R}^2$ be the unit disc and let conductivity $\sigma : \Omega \rightarrow \mathbb{R}$ satisfy

$$0 < M^{-1} \leq \sigma(z) \leq M.$$ 

Applying voltage $f$ at the boundary $\partial \Omega$ leads to the elliptic PDE

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\ u|_{\partial \Omega} = f. \end{cases}$$

Boundary measurements are modelled by the Dirichlet-to-Neumann map

$$\Lambda_{\sigma} : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega}.$$ 

Calderón’s problem is to recover $\sigma$ from the knowledge of $\Lambda_{\sigma}$. It is a nonlinear and ill-posed inverse problem.
Why is the forward map $F : \sigma \mapsto \Lambda_\sigma$ nonlinear?

Define a quadratic form $P_\sigma$ for functions $f : \partial \Omega \to \mathbb{R}$ by

$$P_\sigma(f) = \int_\Omega \sigma |\nabla u|^2 \, dz,$$

where $u$ is the solution of the Dirichlet problem

$$\begin{cases}
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\
u|_{\partial \Omega} = f.
\end{cases}$$

Now the map $\sigma \mapsto P_\sigma$ is nonlinear because $u$ depends on $\sigma$ in (1). Physically, $P_\sigma(f)$ is the power needed for maintaining the voltage potential $f$ on the boundary $\partial \Omega$. Integrate by parts in (1):

$$P_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) \, ds = \int_{\partial \Omega} f \left( \Lambda_\sigma f \right) \, ds.$$

Thus the map $\sigma \mapsto \Lambda_\sigma$ cannot be linear in $\sigma$. 
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Ill-posed inverse problems are defined as opposites of well-posed direct problems

Hadamard (1903): a problem is well-posed if the following conditions hold.

1. A solution exists,
2. The solution is unique,
3. The solution depends continuously on the input.

Well-posed direct problem:
Input $\sigma$, find infinite-precision data $\Lambda_\sigma$.

Ill-posed inverse problem:
Input noisy data $\Lambda^\delta_\sigma$, reconstruct $\sigma$. 
We illustrate the ill-posedness of EIT using a simulated example.
We apply the voltage distribution $f(\theta) = \cos \theta$ at the boundary of the two different phantoms.
The measurement is the distribution of current through the boundary

\[ \sigma_1 \delta u_1 / \partial \vec{n} \]

\[ \sigma_2 \delta u_2 / \partial \vec{n} \]
The current data are very similar, although the conductivities are quite different.

\[ \sigma_1 \frac{\partial u_1}{\partial n}, \quad \sigma_2 \frac{\partial u_2}{\partial n} \]
Let us apply the more oscillatory distribution $f(\theta) = \cos 2\theta$ of voltage at the boundary.
The measurement is again the distribution of current through the boundary

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \]

\[ \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
The current distribution measurements are almost the same.
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data.

\[ \sigma_1 \]

\[ \sigma_2 \]

\[ \cos \theta \]

\[ \cos 2\theta \]

\[ \cos 3\theta \]

\[ \cos 4\theta \]

\[ \cos 5\theta \]

\[ \cos 6\theta \]
The forward map $F : X \supset D(F) \to Y$ does not have a continuous inverse!

Model space $X = L^\infty(\Omega)$

Data space $Y = L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$

Furthermore, the noisy data $\Lambda^\delta_\sigma$ does not belong to the range $F(D(F))$. So Hadamard’s conditions 1 and 3 fail for EIT. How about uniqueness?
Ghosts, or invisible structures, when using point electrodes in electrical impedance tomography

[Chesnel, Hyvönen & Staboulis 2014]
Anisotropic, or matrix-valued, conductivities $\sigma$ lead to non-uniqueness in EIT

Let $\sigma(x) = [\sigma^{ij}(x)]$ be a symmetric and positive-definite $2 \times 2$ matrix. Define anisotropic DN map by

$$\Lambda_\sigma(f) = \nu \cdot \sigma \nabla u \bigg|_{\partial \Omega}.$$  

Let $F : \Omega \to \Omega$ be a diffeomorphism with $F|_{\partial \Omega} = \text{Identity}$. Then

$$\Lambda_{F_* \sigma} = \Lambda_\sigma,$$

where $F_*$ is the push-forward:

$$(F_* \sigma)^{ij}(y) = \frac{1}{\det \left[ \frac{\partial F_i}{\partial x^j} (x) \right]} \sum_{p,q=1}^{2} \frac{\partial F^i}{\partial x^p}(x) \frac{\partial F^j}{\partial x^q}(x) \sigma^{pq}(x) \bigg|_{x=F^{-1}(y)}$$
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The forward map $F : X \supset \mathcal{D}(F) \to Y$ of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.

Model space $X = L^\infty(\Omega)$

Data space $Y = L(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$

Regularization must be based on combining the incomplete measurement data with \textit{a priori} information about the conductivity.
There are three major methodologies for solving nonlinear inverse problems

1. Variational regularization. Write

\[ \Phi(x) = \| \Lambda \tilde{\sigma} - \Lambda^\delta \|^2_Y + \alpha \| \tilde{\sigma} \|^2_X \]

and define regularized solution \( \Gamma_\alpha(\Lambda^\delta) \) by

\[ \Phi(\Gamma_\alpha(\Lambda^\delta)) = \min_{\tilde{\sigma} \in X} \{ \Phi(\tilde{\sigma}) \}. \]

Find the minimum using an iterative optimization method.

[Bissantz, Burger, Engl, Hanke, Hofmann, Hohage, Justen, Kaltenbacher, Kindermann, Lechleiter, Lu, Mathé, Morozov, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Ramm, Resmerita, Rieder, Scherzer, Seidman, Teschke, Vogel, Yagola]
There are three major methodologies for solving nonlinear inverse problems

2. **Direct nonlinear inversion.** Find an analytical nonlinear solution to the inverse problem and use it to design a regularized reconstruction algorithm.

Typically makes heavy use of advanced PDE theory.

There are three major methodologies for solving nonlinear inverse problems

3. **Bayesian inversion.** Write measurement information in the form of a **likelihood** probability distribution and **a priori** knowledge in the form of a **prior** probability distribution. Multiply those two together to yield **posterior probability distribution** and perform well-posed statistical inference from it.

In the measurement equation \( m = A(f) + \epsilon \), model \( f, m, \epsilon \) as random vectors. The solution is the **posterior distribution**
\[
\pi(f|m) = \frac{\pi(f) \pi(m|f)}{\pi(m)}.
\]

See [Kaipio & Somersalo 2004].
There are three major methodologies for solving nonlinear inverse problems

1. Variational regularization.
   + The same code applies to many problems.
   - Repeated solution of direct problem needed.
   - Can get stuck in local minima.

2. Direct nonlinear inversion.
   + Can deal efficiently with a specific nonlinearity.
   - Each formula applies to only one inverse problem.

3. Bayesian inversion (not covered in this course).
   + Very flexible framework.
   + Includes uncertainty quantification.
   - Computationally heavy.