Introduction to nonlinear tomography II: variational regularization

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Outline

Nonlinear variational regularization

Forward map of EIT
The penalty functional

Consider the Tikhonov regularization problem of minimizing

$$\Phi(x) := \frac{1}{2} \left\{ \| F(x) - m \|_2^2 + \alpha \| L(x - x^*) \|_2^2 \right\},$$

where

- $x \in \mathbb{R}^n$ is the unknown,
- $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the forward map modelling the measurement,
- $m \in \mathbb{R}^k$ is the measurement,
- $x^* \in \mathbb{R}^n$ is known \textit{a priori} to be close to the true object,
- $\alpha > 0$ is a regularization parameter,
- $L$ is an $n \times n$ regularization matrix, for example a finite-difference approximation of a derivative.
We apply linearization around a point $\tilde{x}$

Use a linear approximation to the forward map $F$:

$$F(x) \approx F(\tilde{x}) + J(\tilde{x})(x - \tilde{x}),$$

where $J$ is the $k \times n$ Jacobian matrix

$$J = \begin{bmatrix}
\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n}
\end{bmatrix}.$$ 

Then we wish to minimize the functional

$$\Phi_0(x) = \frac{1}{2} \left\{ \| F(\tilde{x}) + J(\tilde{x})(x - \tilde{x}) - m \|_2^2 + \alpha \| L(x - x^*) \|_2^2 \right\}.$$
The minimum of a quadratic form is located at the zero of the gradient

So we need to compute

$$\nabla \Phi_0(x) = \frac{1}{2} \nabla \| J x - (J \tilde{x} - F(\tilde{x}) + m) \|_2^2 + \frac{1}{2} \alpha \nabla \| L x - L x^* \|_2^2.$$  

Let $A$ be a $k \times n$ matrix and $b \in \mathbb{R}^k$. Then

$$\nabla \| Ax - b \|_2^2 = 2 A^T A x - 2 A^T b.$$
Let’s find the zero of the gradient

Denote $J(\tilde{x}) = J$ and calculate the gradient of $\Phi_0(x)$:

$$\nabla \Phi_0(x) = \frac{1}{2} \nabla \|Jx - (J\tilde{x} - F(\tilde{x}) + m)\|^2 + \frac{1}{2} \alpha \nabla \|Lx - Lx^*\|^2$$

$$= J^T J x - J^T (J\tilde{x} - F(\tilde{x}) + m) + \alpha L^T L x - \alpha L^T L x^*$$

$$= J^T J (x - \tilde{x}) + J^T (F(\tilde{x}) - m) +$$

$$+ \alpha L^T L (x - \tilde{x}) + \alpha L^T L (\tilde{x} - x^*)$$

$$= (J^T J + \alpha L^T L)(x - \tilde{x}) + J^T (F(\tilde{x}) - m) + \alpha L^T L(\tilde{x} - x^*).$$

Now $\nabla \Phi_0(x_{\text{min}}) = 0$. Assuming the invertibility of $J^T J + \alpha L^T L$,

$$x_{\text{min}} - \tilde{x} = \left(J^T J + \alpha L^T L\right)^{-1} \left[ J^T (F(\tilde{x}) - m) + \alpha L^T L(\tilde{x} - x^*) \right].$$
We can define an iteration with the formula

\[ x^{(\nu+1)} = x^{(\nu)} - \left( J_\nu^T J_\nu + \alpha L^T L \right)^{-1} \left[ J_\nu^T (F(x^{(\nu)}) - m) + \alpha L^T L (x^{(\nu)} - x^*) \right], \]

where we denote \( J_\nu = J(x^{(\nu)}) \).

If the forward map \( F \) is linear, then the above iteration leads to the minimum in one step. For nonlinear maps \( F \) we can expect convergence if the initial guess \( x^{(0)} \) is close to the global minimum.

With a bad initial guess the iteration converges to a local minimum.
Nonlinear variational regularization

Forward map of EIT
Note that EIT data collection involves applying several current patterns

Saline and agar phantom (RPI)  Apply current pattern \( \cos \theta \)

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom (RPI)  Apply current pattern $\cos 2\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom (RPI)  Apply current pattern $\cos 3\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns.

*Saline and agar phantom (RPI)*  

*Apply current pattern* \( \cos 4\theta \)

Measure the resulting voltages at the 32 electrodes.
Note that EIT data collection involves applying several current patterns

Saline and agar phantom (RPI)  Apply current pattern $\cos 5\theta$

Measure the resulting voltages at the 32 electrodes
Note that EIT data collection involves applying several current patterns

Saline and agar phantom (RPI)  Apply current pattern \( \cos 16\theta \)

Measure the resulting voltages at the 32 electrodes
Discretization of the conductivity

In finite element method (FEM), the conductivity values are constant in each triangle in the mesh.

Our degrees of freedom are $\sigma_1, \ldots, \sigma_n$, where $n$ is the total number of triangles:

$$\sigma(x) = \sum_{j=1}^{n} \sigma_j \chi_j(x).$$

In other words, the discrete conductivity is $\sigma \in \mathbb{R}^n$. 
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Definition of the forward map of EIT

Given discrete conductivity is $\sigma \in \mathbb{R}^n$, define the vector $F(\sigma) \in \mathbb{R}^k$ using the following steps.

1. Denote by $L$ the number of electrodes.

2. EIT data collection used a maximal linearly independent set of current patterns, denoted by $I^{(j)} \in \mathbb{R}^L$ with $j = 1, 2, \ldots, L - 1$.

3. For each $j = 1, 2, \ldots, L - 1$,
   - plug the current pattern $I^{(j)}$ and the conductivity $\sigma \in \mathbb{R}^n$ into a FEM discretization of the Complete Electrode Model (CEM),
   - record the vector $U^{(j)} = [U_1^{(j)}, U_2^{(j)}, \ldots, U_{L-1}^{(j)}]^T \in \mathbb{R}^{L-1}$ containing simulated voltage differences to a ground electrode.

4. Collect all the simulated measurements $U^{(j)}$ into a vector $F(\sigma) \in \mathbb{R}^k$ with $k = (L - 1)^2$. 

Complete electrode model: electrodes

$\Omega$
Complete electrode model: elliptic boundary-value problem

Consider a weak solution \( u \in H^1(\Omega) \) of the conductivity equation

\[
\nabla \cdot \sigma \nabla u = 0 \quad \text{in} \ \Omega,
\]

with a positive \( \sigma \in L^\infty(\Omega) \), satisfying these boundary conditions:

- Disjoint sets \( e_\ell \subset \partial \Omega, \ \ell = 1, \ldots, L \) model the electrodes.
- Outside electrodes, there is no current through the boundary:
  \[
  \sigma \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega \setminus \bigcup_{\ell=1}^{L} e_\ell.
  \]
- At electrodes, Robin condition with contact impedances \( \zeta_\ell \):
  \[
  u + \zeta_\ell \sigma \frac{\partial u}{\partial \nu} = U_\ell \quad \text{on} \ e_\ell, \ \ell = 1, 2, \ldots, L.
  \]
- For a current pattern \([I_1, I_2, \ldots, I_L]\) it holds
  \[
  \int_{e_\ell} \sigma \frac{\partial u}{\partial \nu} \, ds = I_\ell.
  \]
Complete electrode model: accuracy

[Somersalo, Cheney & Isaacson 1992]
By conservation of charge we have $\sum_{\ell=1}^{L} I_\ell = 0$, and the reference potential is chosen so that $\sum_{\ell=1}^{L} U_\ell = 0$.

Now a pair $(u, U) \in H^1(\Omega) \oplus \mathbb{R}^L$ is a solution to the complete electrode model if the identity

$$B((u, U), (v, V)) = \sum_{\ell=1}^{L} I_\ell V_\ell$$

holds for all $(v, V) \in H^1(\Omega) \oplus \mathbb{R}^L$. Here $B$ is the sesquilinear form

$$B((u, U), (v, V)) = \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx + \sum_{\ell=1}^{L} \frac{1}{\zeta_\ell} \int_{e_\ell} (u - U_\ell)(v - V_\ell) \, dS.$$  

[Somersalo, Cheney & Isaacson 1992]
Complete electrode model: finite element method

We approximate the space $H^1(\Omega)$ with a finite-dimensional subspace $\mathcal{V} \subset H^1(\Omega)$. For example, $\mathcal{V}$ can be spanned by piecewise linear functions $\varphi_q(x)$ defined on a triangular mesh.
Complete electrode model: finite element method

Now a pair \( (u, U) \in V \oplus \mathbb{R}^L \) is a solution to the complete electrode model if the identity

\[
\mathcal{B}((u, U), (v, V)) = \sum_{\ell=1}^{L} I_\ell V_\ell
\]

holds for all \( (v, V) \in V \oplus \mathbb{R}^L \). Here \( \mathcal{B} \) is the sesquilinear form

\[
\mathcal{B}((u, U), (v, V)) = \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx + \sum_{\ell=1}^{L} \frac{1}{\zeta_\ell} \int_{e_\ell} (u - U_\ell)(v - V_\ell) \, dS,
\]

where the conductivity \( \sigma \) is represented by its constant values at each of the \( n \) triangle in the mesh:

\[
\sigma(x) = \sum_{j=1}^{n} \sigma_j \chi_j(x).
\]
Complete electrode model: finite element method

The electric potential inside the domain is represented by

\[ u(x) = \sum_{q=1}^{Q} a_q \varphi_q(x). \]

Also, we use a basis for vectors describing boundary information:

\[ U = \sum_{\ell=1}^{L-1} b_\ell \theta_\ell, \]

where

\[ \theta_1 = [1, -1, 0, 0, \ldots, 0]^T \in \mathbb{R}^L, \]
\[ \theta_2 = [1, 0, -1, 0, \ldots, 0]^T \in \mathbb{R}^L, \]
\[ \vdots \]
\[ \theta_{L-1} = [1, 0, 0, 0, \ldots, -1]^T \in \mathbb{R}^L. \]
Construction of the FEM system matrix

Finite element method (FEM) reduces to a matrix equation

\[ A \mathbf{x} = \mathbf{b}, \text{ where } A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix} \]

with

\[ B_{qq'} = \mathcal{B}((\varphi_q, 0), (\varphi_{q'}, 0)) \]
\[ = \int_\Omega \sigma \nabla \varphi_q \cdot \nabla \varphi_{q'} \, dx + \sum_{\ell=1}^{L} \frac{1}{\zeta_\ell} \int_{e_\ell} \varphi_q \varphi_{q'} \, dS, \]
\[ C_{q\ell} = \mathcal{B}((\varphi_q, 0), (0, \theta_\ell)) = -\frac{1}{\zeta_1} \int_{e_1} \varphi_q \, dS + \frac{1}{\zeta_{\ell+1}} \int_{e_{\ell+1}} \varphi_q \, dS, \]
\[ D_{\ell\ell'} = \mathcal{B}((0, \theta_\ell), (0, \theta_{\ell'})) = \sum_{\nu=1}^{L} \frac{1}{\zeta_\nu} \int_{e_\nu} (\theta_\ell)_\nu (\theta_{\ell'})_\nu \, dS. \]
Construction of the FEM system matrix

The integrals of the form
\[ \int_{\Omega} \sigma \nabla \varphi_q \cdot \nabla \varphi_{q'} \, dx \]
can be computed by mapping the relevant triangles to a standard simplex.
We denote by $\chi_j(x)$ the characteristic function of a triangle in the FEM mesh, and the conductivity is

$$\sigma(x) = \sum_{j=1}^{n} \sigma_j \chi_j(x).$$

The computation of the Jacobian matrix is reduced to the evaluation of the integrals in

$$\frac{1}{\sigma_j} \int_{\text{supp}(\chi_j)} \nabla \varphi_q(x) \cdot \nabla \varphi_{q'}(x) \, dx.$$
Gauss-Newton iteration

Starting with an initial guess $\sigma^{(0)}$ for the conductivity, the iteration proceeds as follows:

$$\sigma^{(\nu+1)} = \sigma^{(\nu)} - \left( J_{\nu}^T J_{\nu} + \alpha L^T L \right)^{-1} \left[ J_{\nu}^T (F(\sigma^{(\nu)}) - U_{\text{meas}}) + \alpha L^T L (\sigma^{(\nu)} - \sigma^*) \right]$$

Here $J_{\nu} = J(\sigma^{(\nu)})$ is the Jacobian matrix.
Gauss-Newton iteration: original conductivity

[Vauhkonen, Lionheart, Heikkinen, Vauhkonen & Kaipio 2001]

Computation: thanks to Janne Tamminen

Your resistivity distribution
Gauss-Newton iteration: reconstruction
[Vauhkonen, Lionheart, Heikkinen, Vauhkonen & Kaipio 2001]

5. step

Computation: thanks to Janne Tamminen