Electrical impedance imaging using nonlinear Fourier transform

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Outline

Electrical impedance tomography

Regularization of nonlinear inverse problems in general

The D-bar method for EIT with infinite-precision data

Regularization of EIT using non-linear low-pass filtering
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Regularization of EIT using non-linear low-pass filtering
Electrical impedance tomography (EIT) is an emerging medical imaging technique

**Feed** electric currents through electrodes. **Measure** the resulting voltages. Repeat the measurement for several current patterns.

**Reconstruct** distribution of electric conductivity inside the patient. Different tissues have different conductivities, so EIT gives an image of the patient’s inner structure.

EIT is a harmless and painless imaging method suitable for long-term monitoring.
This talk concentrates on applications of EIT to chest imaging

Applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.
The mathematical model of EIT is the inverse conductivity problem introduced by Calderón

Let \( \Omega \subset \mathbb{R}^2 \) be the unit disc and let conductivity \( \sigma : \Omega \rightarrow \mathbb{R} \) satisfy

\[
0 < M^{-1} \leq \sigma(z) \leq M.
\]

Applying voltage \( f \) at the boundary \( \partial \Omega \) leads to the elliptic PDE

\[
\begin{cases}
\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \\

u|_{\partial \Omega} = f.
\end{cases}
\]

Boundary measurements are modelled by the Dirichlet-to-Neumann map

\[ \Lambda_{\sigma} : f \mapsto \sigma \frac{\partial u}{\partial \vec{n}}|_{\partial \Omega}. \]

Calderón’s problem is to recover \( \sigma \) from the knowledge of \( \Lambda_{\sigma} \). It is a nonlinear and ill-posed inverse problem.
Why is Calderón’s problem nonlinear?

Define a quadratic form $P_\sigma$ for functions $f : \partial \Omega \rightarrow \mathbb{R}$ by

$$P_\sigma(f) = \int_\Omega \sigma |\nabla u|^2 \, dz,$$

(1)

where $u$ is the solution of the Dirichlet problem

$$\left\{ \begin{array}{ll}
\nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega, \\
u |_{\partial \Omega} &= f.
\end{array} \right.$$

Now the map $\sigma \mapsto P_\sigma$ is nonlinear because $u$ depends on $\sigma$ in (1). Physically, $P_\sigma(f)$ is the power needed for maintaining the voltage potential $f$ on the boundary $\partial \Omega$. Integrate by parts in (1):

$$P_\sigma(f) = \int_{\partial \Omega} f \left( \sigma \frac{\partial u}{\partial \vec{n}} \right) \, ds = \int_{\partial \Omega} f \left( \Lambda_\sigma f \right) \, ds.$$

Thus the map $\sigma \mapsto \Lambda_\sigma$ cannot be linear in $\sigma$. 
We illustrate the ill-posedness of Calderón’s problem using a simulated example.
We apply the voltage distribution \( f(\theta) = \cos \theta \) at the boundary of the two different phantoms.
The measurement is the distribution of current through the boundary

\[ \sigma_1 \nabla u_1 \quad \sigma_2 \nabla u_2 \]
The measurements are very similar, although the conductivities are quite different.
Let us apply the more oscillatory distribution $f(\theta) = \cos 2\theta$ of voltage at the boundary.
The measurement is again the distribution of current through the boundary

\[ \sigma_1 \frac{\partial u_1}{\partial \vec{n}} \]

\[ \sigma_2 \frac{\partial u_2}{\partial \vec{n}} \]
The current distribution measurements are again very similar.
EIT is an ill-posed problem: big differences in conductivity cause only small effect in data

\[ \sigma_1 \quad \cos \theta \quad \cos 4\theta \\
\sigma_2 \quad \cos 2\theta \quad \cos 5\theta \\
\quad \cos 3\theta \quad \cos 6\theta \]
EIT is an ill-posed problem: noise in data causes serious difficulties in interpreting the data.
Many different types of reconstruction methods have been suggested for EIT in the literature

- **Linearization:** Barber, Bikowski, Brown, Calderón, Cheney, Isaacson, Mueller, Newell
- **Iterative regularization:** Dobson, Gehre, Hua, Jin, Kaipio, Kindermann, Kluth, Leitão, Lechleiter, Lipponen, Maass, Neubauer, Rieder, Rondi, Santosa, Seppänen, Tompkins, Webster, Woo
- **Bayesian inversion:** Fox, Kaipio, Kolehmainen, Nicholls, Pikkarainen, Ronkanen, Somersalo, Vauhkonen, Voutilainen
- **Resistor network methods:** Borcea, Druskin, Mamonov, Vasquez
- **Layer stripping:** Cheney, Isaacson, Isaacson, Somersalo
- **D-bar methods:** Astala, Bikowski, Bowerman, Delbary, Hansen, Isaacson, Kao, Knudsen, Lassas, Mueller, Murphy, Nachman, Newell, Päivärinta, Perämäki, Saulnier, S, Tamasan
- **Teichmüller space methods:** Kolehmainen, Lassas, Ola, S
- **Methods for partial information:** Alessandrini, Ammari, Bilotta, Brühl, Eckel, Erhard, Gebauer, Hanke, Harrach, Hyvönen, Ide, Ikehata, Isozaki, Kang, Kim, Kress, Kwon, Lechleiter, Lim, Morassi, Nakamura, Nakata, Potthast, Rossetand, Seo, Sheen, S, Turco, Uhlmann, Wang, and others
## History of CGO-based methods for real 2D EIT

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Outline

Electrical impedance tomography

Regularization of nonlinear inverse problems in general

The D-bar method for EIT with infinite-precision data

Regularization of EIT using non-linear low-pass filtering
The forward map $F : X \supset \mathcal{D}(F) \rightarrow Y$ of an ill-posed problem does not have a continuous inverse.
Regularization means constructing a continuous map $\Gamma_\alpha : Y \rightarrow X$ that inverts $F$ approximately.
A regularization strategy needs to be constructed so that the assumptions below are satisfied

A family $\Gamma_\alpha : Y \to X$ of continuous mappings parameterized by $0 < \alpha < \infty$ is a regularization strategy for $F$ if

$$\lim_{\alpha \to 0} \| \Gamma_\alpha(y) - x \|_X = 0$$

for each fixed $x \in \mathcal{D}(F)$.

Further, a regularization strategy with a choice $\alpha = \alpha(\delta)$ of regularization parameter is called admissible if

$$\alpha(\delta) \to 0 \text{ as } \delta \to 0,$$

and for any fixed $x \in \mathcal{D}(F)$ the following holds:

$$\sup_{y^\delta} \left\{ \| \Gamma_{\alpha(\delta)}(y^\delta) - x \|_X : \| y^\delta - y \|_Y \leq \delta \right\} \to 0 \text{ as } \delta \to 0.$$
1. Tikhonov regularization: write a penalty functional
\[ \Phi(x) = \|F(x) - y^\delta\|_Y^2 + \alpha \|x\|_X^2, \]
and \( \Gamma_\alpha(y^\delta) \) is defined by \( \Phi(\Gamma_\alpha(y^\delta)) = \min_{x \in X} \{ \Phi(x) \} \).

**Pro:** The same code applies to many problems.

**Con:** Repeated solution of direct problem needed.

**Con:** Prone to get stuck in local minima.

See [Bissantz, Burger, Engl, Hanke, Hofmann, Hohage, Justen, Kaltenbacher, Kindermann, Lechleiter, Lu, Mathé, Morozov, Munk, Neubauer, Pereverzev, Pöschl, Pricop, Ramlau, Ramm, Resmerita, Rieder, Scherzer, Seidman, Teschke, Vogel, Yagola]

2. Problem-specific regularization

**Pro:** Can deal efficiently with a specific nonlinearity.

**Con:** Each code applies to only one problem.

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Nachman's 1996 uniqueness proof in 2D uses complex geometric optics (CGO) solutions

Define a potential $q$ by setting $q(z) \equiv 0$ for $z$ outside $\Omega$ and

$$q(z) = \frac{\Delta \sqrt{\sigma(z)}}{\sqrt{\sigma(z)}} \quad \text{for } z \in \Omega.$$

Then $q \in C_0(\Omega)$. We look for solutions of the Schrödinger equation

$$(−\Delta + q)\psi(\cdot, k) = 0 \quad \text{in } \mathbb{R}^2$$

parametrized by $k \in \mathbb{C} \setminus 0$ and satisfying the asymptotic condition

$$e^{-ikz}\psi(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2),$$

where $\tilde{p} > 2$ and $ikz = i(k_1 + ik_2)(x + iy)$. 
The CGO solutions are constructed using a generalized Lippmann-Schwinger equation

Define $\mu(z, k) = e^{-ikz}\psi(z, k)$. Then $(-\Delta + q)\psi = 0$ implies

$$(-\Delta - 4ik\bar{\partial}_z + q)\mu(\cdot, k) = 0,$$

where the D-bar operator is defined by $\bar{\partial}_z = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

A solution of (2) satisfying $\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ can be constructed using the Lippmann-Schwinger type equation

$$\mu = 1 - g_k * (q\mu),$$

where $g_k$ satisfies $(-\Delta - 4ik\bar{\partial}_z)g_k = \delta$ and is defined by

$$g_k(z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz\cdot\xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} \, d\xi_1 \, d\xi_2.$$
The Faddeev fundamental solution $g_1(z)$ has a logarithmic singularity at $z = 0$.

It is enough to know $g_1(z)$ because of the relation $g_k(z) = g_1(kz)$. 

Real part of $g_1(z)$

Imaginary part of $g_1(z)$
One of the breakthroughs in Nachman’s 1996 article is showing uniqueness of $\mu$

A solution of $(-\Delta - 4ik\bar{\partial}_z + q)\mu(\cdot, k) = 0$ satisfying $\mu(z, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ can be constructed using the formula

$$\mu - 1 = [I + g_k \ast (q \cdot)]^{-1}(g_k \ast q),$$

provided that the inverse operator exists.

Now $q \in L^p(\mathbb{R}^2)$ with $1 < p < 2$ and $1/\tilde{p} = 1/p - 1/2$, and

$$q \cdot : W^{1,\tilde{p}}(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \text{ is compact,}$$
$$g_k \ast : L^p(\mathbb{R}^2) \to W^{1,\tilde{p}}(\mathbb{R}^2) \text{ is bounded.}$$

Thus $I + g_k \ast (q \cdot) : W^{1,\tilde{p}}(\mathbb{R}^2) \to W^{1,\tilde{p}}(\mathbb{R}^2)$ is Fredholm of index zero, and Nachman proved injectivity for all $k \neq 0$. 
The conductivity $\sigma$ can be recovered from the functions $\mu(z, k)$ at $k = 0$

Recall that

$$(-\Delta - 4ik\overline{\partial}z + q)\mu(\cdot, k) = 0$$

with the asymptotics

$$\mu(z, k) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2).$$

Substituting $k = 0$ gives

$$(-\Delta + \frac{\Delta\sqrt{\sigma}}{\sqrt{\sigma}})\mu(\cdot, 0) = 0,$$  \hspace{1cm} (3)

and setting $\mu(z, 0) = \sqrt{\sigma(z)}$ gives the unique solution of (3) satisfying $\mu(z, 0) - 1 \in W^{1, \tilde{p}}(\mathbb{R}^2)$. 
The crucial intermediate object in the proof is the non-physical scattering transform \( t(k) \)

We denote \( z = x + iy \in \mathbb{C} \) or \( z = (x, y) \in \mathbb{R}^2 \) whenever needed.

The scattering transform \( t : \mathbb{C} \rightarrow \mathbb{C} \) is defined by

\[
t(k) := \int_{\mathbb{R}^2} e^{i\bar{k}z} q(z) \psi(z, k) \, dx \, dy.
\]

(4)

Sometimes (4) is called the nonlinear Fourier transform of \( q \).

This is because asymptotically \( \psi(z, k) \sim e^{ikz} \) as \( |z| \rightarrow \infty \),

and substituting \( e^{ikz} \) in place of \( \psi(z, k) \) into (4) results in

\[
\int_{\mathbb{R}^2} e^{i(kz + \bar{k}z)} q(z) \, dx \, dy = \int_{\mathbb{R}^2} e^{-i(-2k_1, 2k_2) \cdot (x, y)} q(z) \, dx \, dy
\]

\[
= \hat{q}(-2k_1, 2k_2).
\]
Alessandrini’s equation gives a way to write $t$ in terms of $\Lambda_\sigma$ and traces of the CGO solutions:

The following boundary integral equation is a Fredholm equation of the second kind and uniquely solvable in the space $H^{1/2}(\partial \Omega)$:

$$
\psi(\cdot, k)|_{\partial \Omega} = e^{ikz}|_{\partial \Omega} - S_k(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k).
$$

Here $S_k$ is the single-layer operator with Faddeev Green’s function:

$$(S_k \phi)(z) := \int_{\partial \Omega} G_k(z - \zeta)\phi(\zeta) \, ds(\zeta),$$

where $G_k(z) := e^{ikz}g_k(z)$ satisfies $-\Delta G_k = \delta$.

The scattering transform can be evaluated by

$$
t(k) = \int_{\partial \Omega} e^{i\bar{k}\bar{z}}(\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds.
$$
The difference between the usual Green’s function and Faddeev Green’s function is exponential

Usual: \( G_0(z) = -\frac{1}{2\pi} \log |z| \)

Faddeev: \( G_1(z) = e^{iz} g_1(z) \)
The functions $\mu$ can be recovered from the scattering transform $t$ using a D-bar equation

It is natural to ask whether $\mu(z, k)$ depends analytically on the parameter $k$. If it does, the D-bar operator

$$\frac{\partial}{\partial k} = \frac{1}{2}(\frac{\partial}{\partial k_1} + i \frac{\partial}{\partial k_2})$$

will give zero when applied to $\mu(z, k)$.

It turns out that the $\bar{k}$-differential of $g_k^* \ast$ is a rank-one operator, and differentiating $\mu = 1 - g_k \ast (q\mu)$ yields

$$\frac{\partial}{\partial k}\mu(z, k) = \frac{1}{4\pi k}t(k) e_{-k}(z) \mu(z, k).$$

Thus the dependence of $\mu(z, k)$ on $k$ is not analytic. The D-bar equation was discovered by Beals and Coifman in the 1980’s.
These are the steps of Nachman’s 1996 proof:

Solve boundary integral equation
\[ \psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_{\sigma} - \Lambda_1)\psi \]
for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:
\[ t(k) = \int_{\partial \Omega} e^{i\bar{k}z}(\Lambda_{\sigma} - \Lambda_1)\psi(\cdot, k) \, ds. \]

Fix \( z \in \Omega \). Solve D-bar equation
\[ \frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \mu(z, k) \]
with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

Fredholm equation of 2nd kind, ill-posedness shows up here.

Simple integration.

Well-posed problem, can be analyzed by scattering theory.

Trivial step.
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### Infinite-precision data:

Solve boundary integral equation
\[
\psi(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1)\psi
\]
for every complex number \( k \in \mathbb{C} \setminus 0 \).

Evaluate the scattering transform:
\[
t(k) = \int_{\partial \Omega} e^{ikz} (\Lambda_\sigma - \Lambda_1)\psi(\cdot, k) \, ds.
\]

Fix \( z \in \Omega \). Solve D-bar equation
\[
\frac{\partial}{\partial k} \mu(z, k) = \frac{t(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \frac{1}{\mu(z, k)}
\]
with \( \mu(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Reconstruct: \( \sigma(z) = (\mu(z, 0))^2 \).

### Practical data:

Solve boundary integral equation
\[
\psi^\delta(\cdot, k)|_{\partial \Omega} = e^{ikz} - S_k(\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta
\]
for all \( 0 < |k| < R = -\frac{1}{10} \log \delta \).

For \( |k| \geq R \) set \( t^\delta_R(k) = 0 \). For \( |k| < R \)
\[
t^\delta_R(k) = \int_{\partial \Omega} e^{ikz} (\Lambda_\sigma^\delta - \Lambda_1)\psi^\delta(\cdot, k) \, ds.
\]

Fix \( z \in \Omega \). Solve D-bar equation
\[
\frac{\partial}{\partial k} \mu^\delta_R(z, k) = \frac{t^\delta_R(k)}{4\pi k} e^{-i(kz + \bar{k}z)} \frac{1}{\mu^\delta_R(z, k)}
\]
with \( \mu^\delta_R(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C}) \).

Set \( \Gamma_{1/R(\delta)}(\Lambda_\sigma^\delta) := (\mu^\delta_R(z, 0))^2 \).
We define spaces for our regularization strategy

Let $M > 0$ and $0 < \rho < 1$. The domain $\mathcal{D}(F)$ consists of functions $\sigma : \Omega \rightarrow \mathbb{R}$ with

- $\|\sigma\|_{C^2(\Omega)} \leq M$,
- $\sigma(z) \geq M^{-1}$,
- $\sigma(z) \equiv 1$ for $\rho < |z| < 1$.

Bounded linear operators $A : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ satisfying

- $A(1) = 0$,
- $\int_{\partial\Omega} A(f) \, ds = 0$. 
Main result: nonlinear low-pass filtering yields a regularization strategy with convergence speed

**Theorem (Knudsen, Lassas, Mueller & S 2009)**

There exists a constant $0 < \delta_0 < 1$, depending only on $M$ and $\rho$, with the following properties. Let $\sigma \in \mathcal{D}(F)$ be arbitrary and assume given noisy data $\Lambda^\delta_\sigma$ satisfying

$$\|\Lambda^\delta_\sigma - \Lambda_\sigma\|_Y \leq \delta < \delta_0.$$ 

Then $\Gamma_\alpha$ with the choice

$$R(\delta) = -\frac{1}{10} \log \delta, \quad \alpha(\delta) = \frac{1}{R(\delta)},$$

is well-defined, admissible and satisfies the estimate

$$\|\Gamma_{\alpha(\delta)}(\Lambda^\delta_\sigma) - \sigma\|_{L^\infty(\Omega)} \leq C(-\log \delta)^{-1/14}.$$
The proof of the main theorem is divided into several steps. First a D-bar estimate:

Lemma 1. Let $4/3 < r_0 < 2$ and suppose that $\phi_1, \phi_2 \in L^r(\mathbb{R}^2)$ for all $r \geq r_0$. Let $\mu_1, \mu_2$ be the solutions of

$$
\mu_j(z, k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\phi_j(k')}{(k - k')} \mu_j(z, k') \, dk_1' \, dk_2',
$$

$j = 1, 2$. Then for fixed $z \in \Omega$ we have

$$
\|\mu_1(z, \cdot) - \mu_2(z, \cdot)\|_{C^\beta(\mathbb{R}^2)} \leq C \|\phi_1 - \phi_2\|_{L^{r_0} \cap L^{r_0'}(\mathbb{R}^2)},
$$

where $\beta < 2/r_0 - 1$ and $1/r_0' = 1 - 1/r_0$.

Proof. Combination of well-known results.
These results follow from careful analysis of Faddeev Green’s function.

**Lemma 2.** Let $\phi_0 \in H^{-1/2}(\partial \Omega)$ with $\int_{\partial \Omega} \phi_0 ds = 0$. Then

$$\|S_k \phi_0\|_{H^{1/2}(\partial \Omega)} \leq Ce^{2|k|}(1 + |k|) \|\phi_0\|_{H^{-1/2}(\partial \Omega)}.$$ 

**Lemma 3.** For $k \in \mathbb{C}$ we have the estimate

$$\|[l + S_k(\Lambda_\sigma - \Lambda_1)]^{-1}\|_{L(H^{1/2}(\partial \Omega))} \leq Ce^{2|k|}(1 + |k|),$$

where $C$ depends only on $M$ and $\rho$. 
Combining previous results, a perturbation argument, and delicate $L^p$ analysis shows

**Lemma 4.** There exists $\delta_0 > 0$, depending only on $M$ and $\rho$, such that the equation

$$\psi^\delta(\cdot, k)|_{\partial\Omega} = e^{ikz} - S_k(\Lambda^\delta_\sigma - \Lambda_1)\psi^\delta(\cdot, k)|_{\partial\Omega}$$

is solvable in $H^{1/2}(\partial\Omega)$ for all $0 < \delta < \delta_0$ and $|k| < R$ with

$$R = R(\delta) = -\frac{1}{10} \log \delta.$$  

Furthermore, for $p > 1$ we have the estimate

$$\left\| \frac{t(k) - t^\delta_R(k)}{k} \right\|_{L^p(|k| < R)} \leq C \delta^{1/10} \left( -\frac{1}{10} \log \delta \right)^{2/p},$$

where $C$ is independent of $p$ and $R$ and $\delta$. 
Sketch of the proof that nonlinear low-pass filtering gives a regularization strategy for EIT

We need to show the following:

(i) \( \lim_{\alpha \to 0} \| \Gamma_{\alpha}(\Lambda_\sigma) - \sigma \|_X = 0 \),

(ii) \( \alpha(\delta) \to 0 \) as \( \delta \to 0 \),

(iii) \( \sup_{\Lambda_\delta} \{ \| \Gamma_{\alpha(\delta)}(\Lambda_\delta) - \sigma \|_X : \| \Lambda_\delta - \Lambda_\sigma \|_Y \leq \delta \} \to 0 \) as \( \delta \to 0 \).

Claim (i) follows from Lemma 1 and careful choices of Lebesgue exponents in Nachman’s original proof. Claim (ii) holds by the definition

\[
\alpha(\delta) = \frac{1}{R(\delta)} = -10(\log \delta)^{-1}.
\]
Proof of claim (iii)

To prove that the worst-case reconstruction error

$$\sup_{\Lambda_{\delta}} \left\{ \| \Gamma_{\alpha(\delta)}(\Lambda_{\delta}) - \sigma \|_X : \| \Lambda_{\delta} - \Lambda_{\sigma} \|_Y \leq \delta \right\}$$

tends to zero as $\delta \to 0$ we combine Nachman’s results with Lemmas 1 and 4 to estimate

$$\| \mu(z, \cdot) - \mu_R(z, \cdot) \|_{C^\beta(\mathbb{R}^2)} \leq C \left\| \frac{t(k) - t^\delta_R(k)}{k} \right\|_{L^p \cap L^{p'}(\mathbb{C})} \leq C \left\| \frac{t(k) - t^\delta_R(k)}{k} \right\|_{L^p \cap L^{p'}(|k|<R)} + C \left\| \frac{t(k)}{k} \right\|_{L^p(|k|>R)} \leq C \left( -\frac{1}{10} \log \delta \right)^{10/7} \delta^{1/10} + R(\delta)^{-1/7} + R(\delta)^{-1/14} \right) \leq C \left( -\log \delta \right)^{10/7} \delta^{1/10} + (-\log \delta)^{-1/7} + (-\log \delta)^{-1/14} \right).$$
We still need to define the regularization strategy on all of the data space $Y$, not only near $F(D(F))$.

The previous results show the claim only for operators $\delta_0$-close to the range $F(D(F)) \subset Y$.

The structure of the set $F(D(F))$ is not understood at the moment.

However, the proof can be extended to the whole data space $Y$ using spectral-theoretic arguments.
Let us analyze how the regularization works using a simulated heart-and-lungs phantom.
Numerical solution of traces of CGO solutions from the boundary integral equation

Define Fourier basis functions

\[ \varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}. \]

We invert the linear operator appearing in the equation

\[ \psi^\delta(\cdot, k)|_{\partial\Omega} = [I + S_k(\Lambda^\delta - \Lambda_1)]^{-1} e^{ikz}|_{\partial\Omega} \]

as a matrix in \( \text{span}(\{\varphi_n\}^{N}_{n=-N}) \).

The single-layer operator

\[ (S_k \phi)(z) = \int_{\partial\Omega} G_k(z-w)\phi(w) \, ds(w) \]

uses Faddeev’s Green’s function.
This is how the actual scattering transform looks like in the disc $|k| < 10$, computed by knowing $\sigma$.
Scattering transform in the disc $|k| < 10$, here computed from noisy measurement $\Lambda_\sigma^\delta$
Numerical solution of the D-bar equation is based on the periodization approach of G. Vainikko

The generalization of Vainikko’s method for the D-bar equation is described in [Knudsen, Mueller & S 2004].

The D-bar equation

$$\frac{\partial}{\partial k} \mu_R^\delta = \frac{1}{4\pi k} t_R^\delta(k) e^{-z(k)\mu_R^\delta}$$

together with the asymptotics

$$\mu_R^\delta(z, \cdot) - 1 \in L^r \cap L^\infty(\mathbb{C})$$

can be combined in a generalized Lippmann-Schwinger equation:

$$\mu_R^\delta(z, k) = 1 - \frac{1}{4\pi^2} \int_{\mathbb{C}} \frac{t_R^\delta(k')}{(k - k') k'} e^{-z(k')\mu_R^\delta(z, k')} dk_1 dk_2.$$
The D-bar equation is not complex-linear, so real and imaginary parts must be written separately.

The grid points are numbered with one index as shown. Any function \( \phi : Q \to \mathbb{C} \) is represented by a vector of its values at the grid points, organized as follows:

\[
\begin{bmatrix}
\text{Re}\phi(z_1) \\
\text{Re}\phi(z_2) \\
\vdots \\
\text{Re}\phi(z_{64}) \\
\text{Im}\phi(z_1) \\
\text{Im}\phi(z_2) \\
\vdots \\
\text{Im}\phi(z_{64})
\end{bmatrix} \in \mathbb{R}^{128}
\]
This is the real-linear operation given to GMRES:

\[ \begin{bmatrix} \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\ \frac{1}{\pi k} & \frac{1}{\pi k} & 0 & \frac{1}{\pi k} \\ \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \\ \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} & \frac{1}{\pi k} \end{bmatrix} \]

\[ \rightarrow \text{FFT} \]

\LeadsTo

\[ \begin{bmatrix} \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} \\ \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} \\ \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} \\ \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} & \mathbf{T}_R \overline{\phi} \end{bmatrix} \]

\[ \rightarrow \text{Element-wise multiplication} \]

\[ \rightarrow \text{IFFT} \]

\[ \phi - \frac{1}{\pi k} \ast (\mathbf{T}_R \overline{\phi}) \]
Regularized reconstructions from simulated data with noise amplitude \( \delta = \| \Lambda_\delta - \Lambda_{\sigma} \|_Y \)

\[
\begin{align*}
\delta &\approx 10^{-6} \\
\delta &\approx 10^{-5} \\
\delta &\approx 10^{-4} \\
\delta &\approx 10^{-3} \\
\delta &\approx 10^{-2}
\end{align*}
\]

The percentages are the relative square norm errors in the reconstructions.
The observed radii are better (=larger) than those given by the theoretical formula $R(\delta) = -\frac{1}{10} \log \delta$.
The method works for real data as well, including laboratory phantoms and *in vivo* human data.

Saline and agar phantom

Reconstruction ($R = 4$)

[Isaacson, Mueller, Newell & S 2006]
[Montoya, Mueller & S 2012]
Unknown boundary shape can be estimated from EIT data using Teichmüller space methods

[Kolehmainen, Lassas, Ola & S 2012]
Forthcoming book:

*Linear and Nonlinear Inverse Problems with Practical Applications*

by Jennifer L. Mueller and S.S.
Publisher: SIAM.


